Topology II (MTH 516/616), Assignment-4

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Problem 1. Let M be a manifold and x be a point in M. Show that for all i,

$$H_i(M, M - \{x\}) = \begin{cases} 0 & \text{if } i \neq 0 \\ \mathbb{Z} & \text{if } i = n \end{cases}$$

Problem 2. Define Orientability of a manifold at least three ways and compare all definition with definition of orientable Homolofy n-manifold.

(1) Show that any n-manifold is homology n-manifold.
(2) Show that any complex manifold is orientable.
(3) Show that $\mathbb{R}P^{2n}$ is not orientable and $\mathbb{R}P^{2n+1}$ is orientable.
(4) Definition: A fundamental class for a n-manifold M with coefficients in R (ring) is an element of $H_n(M; R)$ whose image in $H_n(M, M - \{x\}; R)$ is a generator for all $x \in M$.

    Show that fundamental class exists for all orientable manifolds. In class we have denoted it with μ_M.

Problem 3. The Poincare duality theorem implies that for a compact, connected and oriented n-manifold without boundary M, the homomorphism

$$H^p(M) \to H_{n-p}(M), \alpha \to \alpha \cap \mu_M,$$

is an isomorphism. Verify theorem for S^n and all compact orientable surfaces. Find an example where Duality theorem fails.

Problem 4. Let M be a manifold which satisfies the Poincare Duality theorem. Show that If dimension of M is odd then $\chi(M) = 0$. If $\dim(M) = 4k + 2$ where k is an positive integer, then $\chi(M)$ is even.

Problem 5. Show that, there is no continuous map $f : S^{n+1} \to S^n$ such that $f(-x) = -f(x)$.

Hint: First show that, if σ be a path connecting an antipodal on S^n. Then under the quotient map $\pi S^n \to \mathbb{R}P^n$, it becomes a singular cycle $\pi_*(\sigma)$ representing a nonzero element in $H_1(\mathbb{R}P^n; \mathbb{Z}_2)$.

Problem 6. Let $A_1, \cdots A_m$ be m measurable sets in $\mathbb{R}^m$. Then we have a hyperplane P which bisects each A_i.

Hints: Borsuk-UlamTheorem.

Problem 7. Compute the Cohomology ring of lens space, $S^m \wedge S^n, \mathbb{R}P^n$ and S^n.

Problem 8. Show that $\mathbb{R}P^2$ is not a boundary of any 3 dimensional compact manifold. Also show that, if L be a non-orientable homology n-manifold. Then L can not be a sub complex of a triangulation of S^{n+1}.

Problem 9. Let K be a triangulation of a homology n-manifold. If $H_1(K;\mathbb{Z}_2) = 0$, then $|K|$ is orientable.

Problem 10. Let X be a closed orientable homology n-manifold. Show that $H_{n-1}(M)$ is a free abelian group.

Problem 11. Find an example of homology n-manifold which is not n-manifold.

See: Maunder Algebraic topology book.