Topology II (MTH 516/616), Assignment-3

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Problem 1. Given a short exact sequence of R modules,

$$0 \to A \to B \to C \to 0,$$

we get a long exact sequence

$$0 \to Hom(C,G) \to Hom(B,G) \to Hom(A,G) \to Ext^1(C,G) \to Ext^1(B,G) \to$$
$$Ext^1(A,G) \to Ext^2(C,G) \to Ext^2(B,G) \to Ext^2(A,G) \to \cdots\cdots$$

Show that we get only 6 terms in long exact sequence if R is PID then.

Problem 2. Let A and B be modules and let $i : Torsion(A) \subset A$ and $j : Torsion(B) \subset B$. Then $i * j : Torsion(A) * Torsion(B) \cong A * B$. Also show that $A * b \cong B * A$.

Problem 3. If A, B, C are R-module. Prove that

$$A \otimes (B * C) \oplus A * (B \otimes C)$$

is symmetric in A, B, C.

Problem 4. Define the tensor product of two chain complex.

Problem 5. Show that,

- Let A be a free abelain group. Then $A * G = 0$.
- $\mathbb{Z}_n * G \cong \{g \in G : ng = 0\} \subset G$.
- If G is torsion free, then $\mathbb{Z}_n * G = 0$.
- $\mathbb{Z}_n * \mathbb{Z}_m = \mathbb{Z}_{gcd(m,n)}$.
- $(A_1 \oplus A_2) * B \cong (A_1 * B) \oplus (A_2 * B)$.

Problem 6. Show that,

- Let A be a free abelain group. Then $Ext^1(A, B) = 0$.
- $\mathbb{Z}_n * G \cong \{g \in G : ng = 0\} \subset G$.
- If G is torsion free, then $\mathbb{Z}_n * G = 0$.
- $\mathbb{Z}_n * \mathbb{Z}_m = \mathbb{Z}_{gcd(m,n)}$.
- $(A_1 \oplus A_2) * B \cong (A_1 * B) \oplus (A_2 * B)$.

Problem 7. Let X be an topological space. Show that,

$$\chi(X) = \sum_i (-1)^i dim(H^i(X, \mathbb{Q}).$$

Problem 8. Show that the singular cohomology module of a space is the direct product of the singular cohomology modules of its path-connected components..

Problem 9. Any two free resolutions of a given module are canonically chain equivalent chain complexes.

Problem 10. Let $\tau : C \to D$ be a chain map between two torsion free chain complexes C and D over R. Suppose $H_i(\tau) : H_i(C) \to H_i(D)$ is an isomorphism for all i. Then show that for any R-module G,

$$H_i(\tau \times 1_G) : H_i(C \otimes G) \to H_i(D \otimes G)$$

is an isomorphism for all i.

Problem 11. Let $\tau : C \to D$ be a chain map between chain complexes of free abelian groups. Suppose that for **each field** k the map $\tau \otimes 1_k$ induces isomorphisms of homology groups. Then τ is a chain equivalence.