MTH 516/616

TOPOLOGY II

Assignment-1

Submission Date: 30/01/2019

Problem 1. Let X be a topological space. Let $\{X_\alpha | \alpha \in \mathcal{A}\}$ denote the path components of X. Prove that for every i one has

$$H_i(X) \cong \oplus_{\alpha \in \mathcal{A}} H_i(X_\alpha).$$

Problem 2. Solve problem 1, 2, 3 from section 4 of chapter 4 from Bredon Topology and geometry.

Problem 3. Let X be a one point space or contractible space or star shaped space. Prove that $H_i(X) = 0$ for all positive i.

Problem 4. Prove that short exact sequence of chain complexes gives a long exact sequence of Homoloy.

Problem 5. Let (X, A) be a pair such that A is a retract of X. Then

$$H_i(X) \cong H_i(A) \oplus H_i(X, A).$$

Problem 6. Show that if $m \neq n$, then $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$.

Problem 7. Let D^n denote closed unit disk in $\mathbb{R}^n$. If $f : D^n \to D^n$ is continuous map, then there is some point $x \in D^n$ such that $f(x) = x$. Note: Solve it using the fact $H_{n-1}(S^{n-1}) = \mathbb{Z}$.

Problem 8. Let A be a subspace of X with inclusion $i : A \to X$. Then for every $n \geq 0$, the map $i_* =: \Delta(i) : \Delta(A) \to \Delta(X)$ is an injective map of chian complexes.

Problem 9. Suppose X is path connected space and A is a non emapty subspace. Then show that $H_0(X, A) = 0$.

Problem 10. Let $f : (X, A) \to (Y, B)$ be a map of pairs (i.e. $f : X \to Y$ and $f(A) \subset B$). So that f induces a map between relative homology of (X, A) and (Y, B).

Definition: Let $f, g : (X, A) \to (Y, B)$ are maps of pairs, we say $f \cong g$ mod(A) if there exist a homotopy map $F : (X \times I, A \times I) \to (Y, B)$ with $F(x, 0) = f(x)$, $F(x, 1) = g(x)$ and $F(A \times I) \subset B$.

Let $f, g : (X, A) \to (Y, B)$ are maps of pairs such that $f \cong g$ mod(A) then foe all $i \geq 0$,

$$H_i(f) = H_j(g) : H_i(X, A) \to H_i(Y, B).$$

You need not to submit the following questions.

Problem 11.

(1) Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of abelain groups.
- Prove that the sequence splits if and only if there exists a map $k : B \to A$ such that $kf = id_A$.
- Give an example of such a short exact sequence where $B \cong A \oplus C$ but such that the sequence does not split.
- Prove that if C is free abelain then the sequence always splits.