Commutative Algebra (MTH-518/618)

Final Semester Examination (18/11/2017)

Time: 180 minutes Maximum Marks: 50

Attempt all questions. Use separate page for each answer.

Problem 1: Solve any two. Each question carries 7 marks.

(1) Let M be a Noetherian A module and N be a finitely generated A module. Show that $M \otimes N$ is a Noetherian A module.
(2) Let I be an ideal of a ring A and let $S = 1 + I$. Show that $S^{-1}I$ is contained in the Jacobson radical of $S^{-1}A$.
(3) Let p be a prime ideal of A. Show that the fields A_p/pA_p and the quotient field of A/p are isomorphic.
Note: Here pA_p denote the ideal generated by image of p in A_p.

Problem 2: Solve any two. Each question carries 8 marks.

(1) Define valuation ring and discrete valuation ring. Show that a valuation ring is Noetherian if and only if it is a discrete valuation ring.
(2) State Ostrowski's theorem. Let $|.|_p$ denote the non-archemidean absolute value for every prime interger p and $|.|_\infty$ denote the usual archemidean absolute value on $\mathbb{Q}$. Show that for any $x \in \mathbb{Q}$,

$$(\prod_p |x|_p).|x|_\infty = 1.$$

(3) Let A be a UFD. Show that a fractional ideal M is invertible if and only if M is principal and nonzero.

Problem 3: Solve any two. Each question carries 10 marks.

(1) State Noether Normalisation lemma. Show that if a finitely generated ring K is a field then it is a finite field.
Note: Every ring can be considered as algebra over $\mathbb{Z}$. Here finitely generated means finitely generated algebra over $\mathbb{Z}$.
(2) Let A be a Dedekind domain and $I \neq 0$ an ideal in A.
Show that every ideal in A/I is principal. Deduce that every ideal in A can be generated by at most 2 elements.
(3) Let A be a non-Noetherian ring and let $\sum$ be the set of ideals in A which are not finitely generated. Show that $\sum$ has maximal elements and that the maximal elements of $\sum$ are prime ideals.