Indian Institute of Science Education and Research, (IISER) Bhopal

Department of Mathematics

An intorduction to Riemannian Geometry (MTH 613)

Mid Sem, Date: Sept 18, 2023

Timing: 10.00 AM to 12.00 PM | 2023-24, Semester-I | Max Mark: 30

Attempt all questions. Write each step. If you use any fact done in class, write it explicitly. For each question, use a new page.

1. Solve all questions.

   (a) Let M be a Riemannian manifold. Show that any geodesic in M has a constant speed. **(2)**

   (b) Let M be a Riemannian manifold and $p \in M$. Assume that every maximal geodesic starting at p has $\mathbb{R}$ as its domain. Show that exp_p is defined on all $T_p(M)$. **(2)**

   (c) Let (M, g) be a Riemannian manifold and R its curvature tensor. Prove that

   $$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0,$$

   forX, Y, Z vector fields on M. **(3)**

   (d) Write down a Riemannian metric on S^1. Compute the connection and exp map for S^1 (Circle). **(4)**

   (e) Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds. Consider the product Riemannian manifold $(M1 \times M2, \pi^* g_1 + \pi^* g_2)$, where

   $$\pi_i : M_1 \times M_2 \to M_i$$

   be the cannonical projections. For any $(p, q) \in M_1 \times M_2$, we let

   $$i_1^q : M_1 \to M_1 \times M_2, \qquad p \mapsto (p, q)$$

   be the embedding of M_1 into $M_1 \times M_2$ as $M_1 \times \{q\}$. Similarly one defines $i_2^p : M_2 \to M_1 \times M_2$. Denote by ∇^i and R_{m_i} the Levi-Civita connection and Riemann curvature tensor on M_i.

Prove that The Levi-Civita connection ∇ on $M_1 \times M_2$ is given by

$$\nabla_X Y(p,q) = di_1^q(\nabla^1_{d\pi_1(X)} d\pi_1(Y)) + di_2^p(\nabla^2_{d\pi_2(X)} d\pi_2(Y))$$

and the Riemann curvature tensor of $M_1 \times M_2$ is

$$R_m = \pi_1^* R_{m_1} + \pi_2^* R_{m_2},$$

where X, Y are vector fields on $M_1 \times M_2$ and $(p,q) \in M_1 \times M_2$. **(4)**

2. Let $\mathbb{H} = \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$ be the upper half plane. Define

$$< X, Y >_{\mathbb{H}} (x,y) = \frac{< X, Y >_E}{y^2} = g_{\mathbb{H}}(X,Y)(x,y),$$

where X, Y are vector fields on $\mathbb{H}$ and $<, >_E$ denote the standard Euclidean Metric on $\mathbb{R}^2$.

- Show that $(\mathbb{H}, <, >_{\mathbb{H}})$ is a Riemannian manifold.
- Compute the Levi-Civita Connection on $\mathbb{H}$.
- Compute the Curvature of $\mathbb{H}$.
- Compute the geodesic on $\mathbb{H}$. **(1+3+2+3)**

3. Solve all questions. (**Take Home**) Submission time 12 PM Tuesday 19 Sept 2023.

   (a) Let (M, g) be a connected Riemannian manifold of dimension $n \geq 3$ with the following property: There is a function $f : M \to \mathbb{R}$ such that, for every $p \in M$, the sectional curvature of all 2-planes $\sigma \subset T_pM$ satisfies

   $$K(\Sigma) = f(p).$$

   Show that f is a constant function. **(3)**

   (b) Let $M = \mathbb{R}^n$ with its standard manifold structure. Give an example of a complete Riemannian metric on M and an example of one non-complete Riemannian metric on M. **(3)**

Best wishes