Indian Institute of Science Education and Research,
(IISER) Bhopal
Department of Mathematics

An introduction to Riemannian Geometry (MTH 613)

End Semester Exam, Date: Nov 22, 2023

Timing: 9.15 AM to 12.15 PM | 2023-24, Semester-I | Max Mark: 40

Attempt all questions. Write each step. If you use any fact done in class, write it explicitly. For each question, use a new page.

1. Solve all questions.

   (a) Write down the statement of the Hopf Rinow Theorem. **(1)**

   (b) State the Gauss Theorem. Use it to prove the famous Theorem Egregium of Gauss. **(3)**

   (c) Compute the sectional curvature of $S^n \subset \mathbb{R}^{n+1}$ **(3)**

   (d) Let $f : (M_1, g_1) \to (M_2, g_2)$ be an isometry between two Riemannian manifolds. Show that

   $$df(\nabla^1_X Y) = \nabla^2_{df(X)} df(Y), \forall X, Y \in \mathcal{X}(M)$$

   where ∇^1, ∇^2 are Riemann connections of M_1, M_2 respectively. **(3)**

2. Solve all questions.

   (a) Let (M, g) be a Riemannian manifold. For a tensor T let ∇T denote its covariant derivative. T is called a parallel tensor if we have $\nabla T = 0$. Assume that $T_1, T_2 : X \times X \to C^\infty M$ are parallel tensors. Show that then the tensor

   $$T : X \times X \times X \times X \to C^\infty(M),$$

   defined as

   $$T(X_1, X_2, X_3, X_4) = T_1(X_1, X_2)T_2(X_3, X_4),$$

   is also parallel. **(4)**

(b) Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, c an arbitrary parallel of latitude of S^2 and V_0 a tangent vector to S^2 at a point of c. Describe geometrically the parallel transport of V_0 along c. **(4)**

(c) Let M be a Riemannian manifold with non-positive sectional curvature. Define a conjugate point and conjugate locus. Prove that, for all p, the conjugate locus $C(p)$ is empty. **(4)**

3. Solve all questions.

(a) Show that $\phi : \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$\phi(\theta, \psi) = \frac{1}{\sqrt{2}}(cos\theta, sin\theta, cos\psi, sin\psi), \qquad (\theta, \psi) \in \mathbb{R}^2$$

is an immersion of $\mathbb{R}^2$ into the unit sphere $S^3 \subset \mathbb{R}^4$, whose image $\phi(\mathbb{R}^2)$ is a torus T^2 with sectional curvature zero in the induced metric. **(6)**

(b) Let (M, g) be a Riemannian manifold with the following property: given any two points $p, q \in M$, the parallel transport from p to q does not depends on the curve that joins p to q. Prove that the curvature of M is identically zero, that is for all $X, Y, Z \in \mathcal{X}(M)$,

$$R(X, Y)Z = 0.$$

(6)

(c)) For $r > 0$, let $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = r^2\}$ and

$$X : [-\pi/2, \pi/2] \to TS$$

be vector field along

$$c : [-\pi/2, \pi/2] \to S \quad with \ \ c(t) = (rcost, 0, rsint),$$

defined by

$$X(t) = (0, cost, 0).$$

Let $\frac{D}{dt}$ denote the covariant derivative on S along c.

- Calculate $\frac{D}{dt}X(t)$ and $\frac{D^2}{dt^2}X(t)$.
- Show that X satisfies the Jacobi equation. **(6)**

Best wishes