Introduction to Algebraic Topology (MTH-507)

End Semester Examination (23/04/2018)

Time: 180 minutes   Maximum Marks: 50

Attempt all questions. Use separate page for each answer.

Problem 1. Solve any two questions. (5+5)

(1) Show that the fundamental group of S^1 is isomorphic to the additive group of integers.
(2) Show that "infinite earring" in the plane have no universal covering.
(3) State and prove Brouwer fixed-point theorem for the disc.

Problem 2. Solve any five questions. (4+4+4+4+4)

(1) Show that every continuous map $f : \mathbb{P}^2 \to S^1$ is nullhomotopic.
(2) Show that $\mathbb{R}^2$ and $\mathbb{R}^n$ are not homeomorphic if $n > 2$.
(3) Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.
(4) Show that there is no retraction of B^2 onto S^1.
(5) Show that the idendity map $i : S^1 \to S^1$ is not nulhomotopic.
(6) Show that a retract of a contractible space is contractible.

Problem 3. Solve any four questions. (5+5+5+5)

(1) Find a two-sheeted covering of the Klein bottle by the Torus. Is it regular?
(2) Compute the fundamental group of $\mathbb{R}^3 - X$, where X is the union of x, y and z axis.
(3) Find all connected pointed coverings (up to isomorphism) of the circle. Show that there is no connected covering space of S^1 with a non-trivial finite fundamental group.
(4) Consider a topological space $X = \{a, b, c, d\}$ with topology determined by the base $\{\{a\}, \{c\}, \{a, b, c\}, \{c, d, a\}\}$. Prove that X is path-connected, but not simply connected. Calculate $\pi_1(X)$.
(5) Consider the covering $\mathbb{C} \to \mathbb{C} - \{0\}, z \to e^z$. Find liftings of the paths $u(t) = 2 - t$ and $v(t) = (1 + t)e^{2\pi i t}$ and their products uv and vu.

Problem 4. (Bonus) Let X be a path connected topological space. Show that there exists a bijective correspondence between the n-sheeted connected coverings space over X and the transitive representations of the fundamental group of X in the symmetric group (permutation group) S_n.