FIELDS AND GALOIS THEORY (MTH 401)

QUIZ-3 (21/11/2020)

Time: 75 minutes Maximum Marks: 23

Notation:

(1) $Aut(E/F) = \{\sigma : E \to E : \sigma \text{ is an automorphism of } E \text{ and } \sigma|_F = Id_F\}$.
(2) $\mathbb{F}_q$ denotes the field of order q, where $q = p^n$ for some prime p.
(3) $\bar{F}$ denotes the algebraic closure of the field F.
(4) D_6 denotes the Dihedral group pf order 12.

Write down your Name and Roll number on the front page. Attempt all questions. Each steps should be very clear. If you are using any fact then mention it very clearly.

Problem-A:. Let G be an arbitrary finite group. Show that there is a field k and a polynomial $g(x) \in k[x]$ such that the Galois group of $g(x)$ is isomorphic to G. (5)

Problem-B:. Let E/F be a finite Galois extension. Let H be a subgroup of $Gal(E/F)$. Show that there is an element $\alpha \in E$ such that $H = \{\sigma \in Gal(E/F) : \sigma(\alpha) = \alpha\}$. (6)

Problem-C:. For the correct answer you will get 2 mark and for the wrong answer you will get -1 mark. (2+2+2)

(1) Let $h(x) = h_1(x)h_2(x)$ be a product of two irreducible polynomials over a finite field $\mathbb{F}_p$. Let 20 and 24 are degree of $h_1(x)$ and $h_2(x)$ respectively. Then the degree of the splitting field of $h(x)$ over $\mathbb{F}_p$ is equal to
A) 480
B) 240
C) 60
D) 120

(2) Let E be a finite extension over a finite field k and $[E : k] = 110$. Then the number of intermediate proper subfields (proper means E is not included) of E/k is
A) 7
B) 8
C) 9
D) 6

(3) Let $q = 3^3$. The number of elements of finite order in $\text{Aut}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ is
A) 9
B) 3
C) 1
D) 27

Problem-D:. Choose all correct answers. Multiple answer can be true. **There is no minus marking.** You will get marks only if you choose all correct answer. (2+2+2)

(1) Let F be a field, $\bar{F}$ an algebraic closure of F and $f \in F[x]$ a non-constant polynomial. Let $E \subset \bar{F}$ denote the splitting field of f in $\bar{F}$. Which of the following assertions are correct:
   A) The extension E/F is a normal extension.
   B) If $\alpha \in \bar{F}$ is a root of f, then $E = F(\alpha)$.
   C) The extension E/F is a Galois extension.
   D) If the polynomial f is irreducible, then E/F is a Galois extension.
   E) If the characteristic of F is zero, then E/F is a Galois extension.

(2) Let k be a field, $\bar{k}$ an algebraic closure of k and $E \subset \bar{k}$ a finite extension of k such that E/k is a Galois extension, and let $G = Gal(E/k)$ be its Galois group. Which of the following assertions are correct:
   A) For any subgroup H of G, the intermediate extension $F = E^H$ is normal extension of E.
   B) Two H_1 and H_2 of G are equal if and only if $E^{H_1} = E^{H_2}$.
   C) Any subgroup H of G is the Galois group of some extension F/k for some $F \subset E$.
   D) Let $k = \mathbb{Q}$ and E be a finite extension over $\mathbb{Q}$. Then the $Gal(E/\mathbb{Q})$ is abelian.

(3) Let E be a Galois extension of F with $|Gal(E/F)| = 12$ and $Gal(E/F) \ncong D_6$. Which of the following assertions are correct:
   A) there always exists a subfield K of E containing F with $[K : F] = 3$.
   B) there always exists a subfield K of E containing F with $[K : F] = 2$.
   C) there always exists a subfield K of E containing F with $[K : F] = 6$.
   D) there always exists a subfield K of E containing F with $[K : F] = 4$.