FIELDS AND GALOIS THEORY (MTH 401)
ASSIGMENT-2

Submission date: 13/09/2020

Problem-A. Submit an answer of all problems.

(1) Let F be a finite field with characteristic p. Prove that the number of elements in F is p^n for some positive integer n.

(2) Determine irreducible monic polynomials over $\mathbb{Q}$ for $1+i, 2+\sqrt{3}$, and $1+\sqrt[3]{2}+\sqrt[3]{4}$. For any two prime numbers p and q, show that $\sqrt{p}+\sqrt{q}$ is algebraic over $\mathbb{Q}$.

(3) Let F be an extension field of K. If $\alpha \in F$ is transcendental over K, then show that every element of $K(\alpha)$ that is not in K is also transcendental over K. Here $K(\alpha)$ is the smallest fields containing K ans α.

(4) Let K/F be an algebraic field extension and R be a ring such that $F \subset R \subset K$. Show that R is a field. What happens if we do not assume that F is algebraic over K?

(5) Let $\beta \in \mathbb{C}$ be a root of the equation $X^2+2=0$. Show that

$$\mathbb{Q}[\beta] := \{a+\beta b : a, b \in \mathbb{Q}\}$$

is a field.

(6) Let E/K be a field extension and $\alpha \in E$ is algebraic over K. Show that for any polynomial $f(x) \in K[x], f(\alpha) \in E$ is algebraic over K.

(7) Consider the subfield $F(X^2) \subset F(X)$. Is X algebraic over $F(X^2)$?

(8) Show that the degree of the field extension $\mathbb{C}/\mathbb{Q}(i)$ is infinite.

(9) Let k be a field and x be an indeterminate. Let $y = \frac{x^3}{(x+1)}$. Find the minimal polynomial of x over $k(y)$.

(10) Let S denote the elements in $\mathbb{C}$ which are algebraic over $\mathbb{Q}$. Then S is a countable set.

(11) Construct finite fields of order 9 and 27. Write down multiplication tables for the fields with 9 elements and verify that the multiplicative groups of this field is cyclic. Can you find two non isomorphic fields of order 9?

Date: 30-08-2020.

Problem:B. You don't need to submit it.

(1) For any prime number p, show that the polynomial $x^2 - p$ has no solution in $\mathbb{Q}$.

(2) Let F be an extension field of K, and let u be a nonzero element of F that is algebraic over K. Show that u^{-1} is also algebraic over K.

(3) Let $\alpha \in \mathbb{C}$ and define,

$$\mathbb{Q}[\alpha] = \{f(\alpha) : f \text{ is a polynomial over } \mathbb{Q}, f \in \mathbb{Q}[x]\}.$$

Show that $\mathbb{Q}[\alpha]$ is the smallest ring containing $\mathbb{Q}$ an α. Prove that $\mathbb{Q}[\alpha]$ is a finite dimensional vector space over $\mathbb{Q}$ if and only if α is algebraic over $\mathbb{Q}$. Also show that $\mathbb{Q}[x]$ is a field if x is algebraic over $\mathbb{Q}$.

(4) Let $f : [0\ 1] \to \mathbb{R}$ be a non-constant continuous function. Show that the set $f([0\ 1]) := \{f(a) : a \in [0\ 1]\}$ contains a number transcendental over $\mathbb{Q}$.

(5) More useful version of Gauss Lemma:

Let $f(x) \in \mathbb{Z}[x]$ be a primitive polynomial. Then show that $f(x)$ is irreducible in $\mathbb{Z}[x]$ if and only if $f(x)$ is irreducible in $\mathbb{Q}[x]$.

Note that for a polynimal $f(x) = \sum_{i=0}^{n} a_i x^i$, The content of $f(x)$ is defined to be the integers,

$$\text{cont}(f) := \gcd(a_0, \ldots, a_n).$$

If $\text{cont}(f) = 1$, we say that $f(x)$ is a primitive polynomial.

(6) Show that $[\mathbb{C} : \mathbb{R}] = 2$.