INTRODUCTION TO GROUPS AND SYMMETRY (MTH 203)

Quiz-1

Total Marks: 10 **Time: 55 Minutes**

Solve all Problems.

Problem-A. Are the following statements **True** or **False**? **DO NOT** write any justification or a proof. (3 Marks)

(1) Union of two subgroups of a group is always a subgroup.
(2) Let G be a finite group with an element of order 2. The order of G cannot be odd.
(3) Any equivalence relation on a set X gives a partition of X.
(4) Every subset of a group G is a subgroup of G.
(5) If G has no nontrivial subgroups then the order of G can be infinite.
(6) There exists a group homomorphism $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(3) = 6$ and $f(133) = 399$.

Answer:

(1) False
(2) True
(3) True
(4) False
(5) False
(6) False

Problem-B. Let S_4 be the group of permutations on the set $\{1, 2, 3, 4\}$.

(1) Compute the product $(3, 4, 1, 2)(1, 3, 4, 2)$ in S_4.
(2) Find the inverse of $(3, 1, 4, 2)$ in S_4. Hint: write $(3, 1, 4, 2)$ as a map and find inverse of that map and then write cycle notation.
(3) Find the order of $(3, 1, 4, 2)$ in S_4. $(1 + 1 + 2$ Marks$)$

Answer:

(1) $(4, 2, 1, 3)$.
(2) Now let $\sigma = (3, 1, 4, 2)$ means that

$$\sigma(1) = 3,$$
$$\sigma(2) = 1,$$
$$\sigma(3) = 4,$$
$$\sigma(4) = 2.$$

Now The inverse of σ as a set map is,

$$\sigma^{-1}(1) = 2,$$
$$\sigma^{-1}(2) = 4,$$

$$\sigma^{-1}(3) = 1,$$
$$\sigma^{-1}(4) = 3.$$

Hence the inverse of $(3, 1, 4, 2)$ is $(2, 4, 1, 3)$.

(3) To compute the order of $(3, 1, 4, 2)$ we have to find smallest n such that $(3, 1, 4, 2)^n = (1, 2, 3, 4)$.

$$(3, 1, 4, 2)(3, 1, 4, 2) = (4, 3, 2, 1),$$
$$(3, 1, 4, 2)(3, 1, 4, 2)(3, 1, 4, 2) = (4, 3, 2, 1)(3, 1, 4, 2) = (2, 4, 1, 3),$$
$$(3, 1, 4, 2)(3, 1, 4, 2)(3, 1, 4, 2)(3, 1, 4, 2) = (2, 4, 1, 3)(3, 1, 4, 2) = (1, 2, 3, 4).$$

Hence $o((3, 1, 4, 2)) = 4$.

Problem-C. Let $f : G \to G$ be a group homomorphism. Let $a \in G$ be a non-identity element. Show that order of $f(a)$ divides the order of a.

Show that there are no non-zero (non-trivial) group homomorphism from $S_3 \to G$, where G is a group of order 217. (2 + 1 Marks)

Answer: Let $o(a) = n$ and $o(f(a) = m$.

$$(f(a))^n = f(a^n) = f(e) = e,$$

implies $o(f(a) \leq o(a)$. Also by division algorithm we get q, r such that

$$n = mq + r$$

where $0 \leq r < m$ (why $r \geq 0$?). So

$$(f(a))^n = f(a)^{qm+r} = (f(a)^m)^q (f(a))^r = (f(a))^r,$$

which implies that $r = 0$. Hence $m|n$.

Note that order of S_3 is 6 and order of G is 217. So by corolllary of Lagrange's theorem possible order of any element in S_3 can be $\{1, 2, 3\}$ and in G can be $\{1, 7, 31\}$.

Note that the above result can be proved for any $f : G_1 \to G_2$ using the same method.

Suppose there exists any non trivial homomorphism so we get $a \in S_3$ such that $f(a) \neq e$ is not identity. By above result $o(f(a)$ should divide the order of a. Which only possible if $o(f(a)) = 1$ i.e. $f(a) = e$.