INTRODUCTION TO GROUPS AND SYMMETRY (MTH 203)

Mid Semester Examination

Total Marks: 98 Time: 11.15 AM to 1.15 PM Date: 02/10/2019

Instructions: Solve all problems. Please write all steps properly. If you are using any theorem, fact, formula then write a complete statement.

Problem-A. For each of the following statements indicate whether it is **TRUE** or **FALSE**. You **DO NOT** need to provide justification for your answers in this problem. (30)

(1) Subtraction $-$ is a binary operation on $\mathbb{Z}$.

(2) There exists no nonzero group homomorphism from $\mathbb{Z}_5$ to $\mathbb{Z}_{13}$.

(3) There is no isomorphism from $(\mathbb{Q}, +) \to (\mathbb{R}, +)$.

(4) If $m, n, x, y \in \mathbb{Z}$ are such that $mx + ny = 1$ then $gcd(m, n) = 1$.

(5) Any subgroup of an abelian group is always normal subgroup.

(6) Any permutation of order ≥ 12 can be written as product of disjoint transpositions.

(7) Two groups are isomorphic if both have same number of element.

(8) Let G_1 and G_2 are two groups and G_1 has no proper nontrivial subgroup. There exists a non zero group homomorphism $\phi : G_1 \to G_2$ which is not one-one.

(9) $4^{15} - 1$ is divisible by 31.

(10) There is a group homomorphism $\phi : (\mathbb{Z}, +) \to (\mathbb{Q}^*, \cdot)$ such that $\phi(2) = \frac{1}{9}$.

Problem-B. Solve all questions.

(1) Let G be a group and H, N are subgroups of G such that N is normal subgroup. Show that HN is a subgroup of G.
State the second isomorphism theorem. (5+3)

(2) Let G be a group and H be a subgroup of G. Let

$$N = \bigcap_{x \in G} xHx^{-1}.$$

Prove that N is a normal subgroup of G. (10)

(3) What is the largest possible order of a permutation in S_7? Write down an explicit element of this order. (8)

(4) Suppose that H is a subgroup of G such that whenever $Ha \neq Hb$ then $aH \neq bH$, where $a, b \in G$. Prove that $gHg^{-1} \subset H$ for all $g \in G$. (10)

(5) Prove or disprove: $U_{20} \simeq U_{15}$. (10)

(6) Show that any cyclic group is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_m$. (10)

(7) Let $\phi : G_1 \to G_2$ be an isomorphism.

- Show that $\phi^{-1} : G_2 \to G_1$ is an isomorphism.
- Argue that $o(G_1) = o(G_2)$.
- Show that $o(\phi(a)) = o(a) \ \forall a \in G_1$. $(6+3+3)$