INTRODUCTION TO GROUPS AND SYMMETRY

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MTH 203

Practice Problem Set for revision

Problems:

(1) Let $(G, \circ)$ be a group. Define an operation $*$ on G by

$$g * h = h \circ g$$

Show that $(G, *)$ is a group and

$$(G, \circ) \cong (G, *).$$

Hint: Define a map $g \mapsto g^{-1}$ from $(G, \circ)$ to $(G, *)$.

(2) Let G be a group and $g_1, g_2 \in G$. Show that $o(g_1g_2) = o(g_2g_1)$.

(3) If G and H are groups with coprime finite orders, then there exists no non zero group homomorphism from G to H.

(4) Find all group homomorphism from
- $\mathbb{Z}_2 \to \mathbb{Z}_4$.
- $\mathbb{Z}_2 \to \mathbb{Z}_5$.
- $\mathbb{Z}_4 \to \mathbb{Z}_4$.

(5) Let G be the cyclic group of order 12. How many subgroups of G have order 3? Explain.

(6) Using method defined in Cayley's theorem, write down an explicit injective homomorphism $S_3 \to S_6$.

(7) Let G be a finite abelian group whose order is divisible by the prime number p. Show G has an element of order p.

Hint: Use induction on the order of G. Let $g(\neq e) \in G$, then p divides the order of $< g >$ or $G/ < g >$.

(8) Let $G = G_1 \times G_2$ be a finite group with $gcd(|G_1|, |G_2|)) = 1$. Then every subgroup H of G is of the form $H = H_1 \times H_2$ where H_1, H_2 are subgroups of G.

Show by example if $gcd(|G_1|, |G_2|)) \neq 1$ then above result need not be true.

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(9) Prove or disprove: the set $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a subgroup of $(\mathbb{C}, +)$.

(10) Let $\phi : G \to G$ be an isomorphism. Show that the set
$$H = \{g \in G : \phi(g) = g\}$$
is a subgroup of G.

(11) If an automorphism fixes more than half of the elements of a finite group, then it is the identity automorphism.

(12) Let $x, y \in G$ and let $xy = z$ if $z \in Z(G)$, then show that x and y commute.

(13) Let $G =$ the set of all bijective maps from $\mathbb{R}$ to $\mathbb{R}$ which fixes almost all elements except finite. $\phi \in G$ means $\phi : \mathbb{R} \to \mathbb{R}$ is a bijective map and the set $\{x \in \mathbb{R} : \phi(x) = x\}$ is a finite subset set of $\mathbb{R}$. Prove or disprove: G is a group under composition.

(14) Prove that if K is a normal subgroup of the group G, then the centre $Z(K)$ of K is a normal subgroup of G. Show by an example that $Z(K)$ need not be contained in $Z(G)$.
Hint: $Z(A_3) = A_3$ but $Z(S_3) = 1$.

(15) Let G be a group of order n and let m be an integer relatively prime to n. Show that if $x^m = y^m$, then $x = y$. Hence show that for each $z \in G$ there is a unique $x \in G$ such that $x^m = z$.

(16) Let H and K are normal subgroups of G. Then $H \cap K$ is also normal subgroup of G.
Define a map,
$$\phi : G/(H \cap K) \to G/H \times G/K$$
$$g(H \cap K) \mapsto (gH, gK)$$
for all $g \in G$. Show that ϕ is an injective homomoprphism. Using it deduce that if G/H and G/K both are abelian, then $G/H \cap K$ also abelian.

(17) Fix a set X and write $P(X)$ for the set of all subsets of X. This is called the power set of X. Define a product on $P(X)$ by the rule
$$A \bullet B := A \cup B - A \cap B.$$
Show that $P(X)$ is a group with respect to $\bullet$.

(18) Let $(G, \bullet)$ be a group and X any set. Let S be the set of all set maps from X to G. Define a binary operation $*$ on S by
$$(f * g)(x) := f(x) \bullet g(x).$$
Is $(S, *)$ a group? If so, prove that it is. If not, give an axiom which is violated and prove that this is so.

(19) Let $g \in G$ with $o(g) = nm$ where n, m are co-prime positive integers. Show that there are elements $g_1; g_2 \in G$ such that $g = g_1g_2 = g_2g_1$ and $o(g_1) = n, o(g_2) = m$.

(20) Determine the order of each of the following quotient groups:
- $\mathbb{Z}_8/ < \bar{4} >$
- $\mathbb{Z}_8/ < \bar{3} >$

Are these groups cyclic?

(21) Let G_1 and G_2 are two groups. Define an operation on $G_1 \times G_2$ by,

$$(g_1, g_2)(h_1, h_2) = (g_1h_1, g_2h_2),$$

where $g_1, h_1 \in G_1$ and $g_2, h_2 \in G_2$. Show that $G_1 \times G_2$ is a group with respect to the above defined operation.
Let G be a group. Define $H = \{(g, g) \in G \times G : g \in G\}$. Show that H is a subgroup of $G \times G$. Show also that, H is normal in $G \times G$ iff G is abelian.

(22) Let H be a normal subgroup of the group G. Show that if $x, y \in G$ such that $xy \in H$, then $yx \in H$.

(23) Let H and K be subgroups of G and $x, y \in G$ with $Hx = Ky$. Then show that $H = K$.

(24) Let H be a subgroup of the group G, $g \in G$ such that $o(g) = n$ and $g^m \in H$ where m, n are coprime intergers. Then show that $g \in H$.

(25) Find the subgroup of S_5 generated by $(3, 1)$ and $(1, 5)$. Is it isomorphic to S_3? Explain your answer.

(26) For each of the following statements either prove the statement or give a counter-example.
- If H is a proper subgroup of G then $H \cong G$.
- If G is a group such that every element of G has finite order, then the oder of G is finite.
- If G is a finite group and n devides the order of the group then G has an element of order n.
- If G is an infinite group then G has at least two distinct normal subgroups.
- If H and K are conjugate subgroups of G then $H \cong K$.
- Any two cycles of length 4 are conjugate in A_5.
- A_4 is simple group.
- Let G be a group of order 15 then G contains an element of order 3.