INTRODUCTION TO GROUPS AND SYMMETRY

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MTH 203

Mid Semester Practice Problem

Problem Set-1

(1) Prove that if G is any group with identity e then $G/\{e\} \simeq G$.

(2) Let G be a finite group and let H and K be subgroups of G with K normal in G. Using the second isomorphism theorem prove that,

$$o(HK) = \frac{o(H)o(K)}{o(H \cap K)}.$$

(3) Let m, n be nonzero integers show that

$$m\mathbb{Z} \cap n\mathbb{Z} = lcm(m, n)\mathbb{Z}.$$

(4) Let m, n be nonzero integers show that

$$m\mathbb{Z} + n\mathbb{Z} = gcd(m, n)\mathbb{Z}.$$

Note that $m\mathbb{Z} + n\mathbb{Z}$ is product of $m\mathbb{Z}$ and $n\mathbb{Z}$ in $\mathbb{Z}$ with respect + operation. It is subgroup becuase any subgroup of $\mathbb{Z}$ is normal.

(5) Let G be a group and let H and K be normal subgroups of G with $K \subset H$. Suppose that G/K is cyclic. Prove that G/H and H/K are cyclic.

(6) Let $\bar{7} \in \mathbb{Z}_{28}$. Define $H =< \bar{7} >=$ the subgroup of $\mathbb{Z}_{28}$ generated by $\bar{7}$. Prove that

$$\frac{\mathbb{Z}_{28}}{H} \simeq \mathbb{Z}_7.$$

(7) Prove that,

$$\frac{6\mathbb{Z}}{12\mathbb{Z}} \simeq \frac{3\mathbb{Z}}{6\mathbb{Z}}.$$

(8) Let m, n be nonzero integers such that $m|n$, show that

$$m\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}_{n/m}.$$

Also show that,

$$|m\mathbb{Z}/n\mathbb{Z}| = |\mathbb{Z}_{n/m}| = n/m,$$

where $|m\mathbb{Z}/n\mathbb{Z}| =$ the number of elements in $m\mathbb{Z}/n\mathbb{Z}$.

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Hint: Define a map

$$\phi : \mathbb{Z} \to m\mathbb{Z}/n\mathbb{Z},\ x \mapsto [mx] = mx + n\mathbb{Z}.$$

and use it to prove the result. You can also try any other method.

(9) Let G be a group and $a \in G$. Then show that

$$< a^m > \cap < a^n > = < a^{lcm(m,n)} >$$

where m, n nonzero integers.

(10) Prove that if a group G contains a subgroup H of finite index, then G contains a normal subgroup of finite index.

(11) Let N be a subgroup in the centre $Z(G)$ of G. Show that N is normal in G. Prove that if the factor group G/N is cyclic, then G is abelian. Is converse true ?

(12) Prove that if a group G has no non-trivial automorphisms, then G is abelian and $g^2 = e$ for all $g \in G$.

(13) Let G be a cyclic group. Compute all subgroups of G.
Hints: Any cyclic group is homomorphic image of $\mathbb{Z}$.

(14) Prove or disprove:
- U_{20} and U_{24} are isomorphic.
- U_{20} and U_{15} are isomorphic.
- U_{20} and D_4 are isomorphic.

(15) Prove that if m and n are relatively prime then $\mathbb{Z}_{mn} \simeq \mathbb{Z}_m \times \mathbb{Z}_n$.

(16) Let $\phi : \mathbb{Z}_{36} \to \mathbb{Z}_{20}$ be a map defined by $\phi(\bar{n}) = \overline{5n}$. Find the Kernal of ϕ.

(17) Let k and n be positive integers. For a fixed $m \in \mathbb{Z}$, define

$$f ; \mathbb{Z}_n \to \mathbb{Z}_k$$

by $f(\bar{a}) =: \overline{ma}$, for $\bar{a} \in \mathbb{Z}_n$. Find all values of m such that f is a well defined function.

(18) Let $\phi : \mathbb{Z}_{20} \to \mathbb{Z}_{45}$ be a group homomorphism. What are all possible orders of $\phi(\bar{1})$.

(19) Describe all group homomorphisms from $\mathbb{Z}_{15}$ to $\mathbb{Z}_{25}$.

(20) Describe all group isomorphism from $\mathbb{Z}_{25}$ to $\mathbb{Z}_{25}$.

(21) Describe all group homomorphisms from $\mathbb{Z}_9$ to S_4.

(22) Define $C[0,1] = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$. Show that with respect to addition of maps $C[0,1]$ gives a group structure on $C[0,1]$. Using the above idea define a group structure on a the set of all group homomorphism from G_1 to G_2.

For $x \in [0,1]$, define

$$\phi_x : C[0,1] \to (\mathbb{R}, +)$$

by

$$f \mapsto f(x).$$

Show that ϕ_x is a group homomorphism. Find out the Kernal of ϕ_x.

(23) Let G_1, G_2 be two groups. Define a operation on $G_1 \times G_2$ by

$$(g_1, g_2)(h_1, h_2) = (g_1h_1, g_2h_2).$$

Show that with respect to above operation $G_1 \times G_2$ is a group.

Define a map $p_1 : G_1 \times G_2 \to G_1$ by $(g, h) \to g$. Show that p_1 is a group homorphism, $Ker(p_1) \cong G_2$

$$(G_1 \times G_2)/Ker(p_1) \cong G_1.$$

(24) S_n be the permutation group. For $1 \leq i \leq n$, define

$$H_i = \{\sigma \in S_n : \sigma(i) = i\}.$$

Show that H_i is a subgroup of S_n. Is it normal?

(25) Show that $\mathbb{Z}$ is a normal subgroup of $(\mathbb{R}, +)$. Also prove that

$$\frac{\mathbb{R}}{\mathbb{Z}} \cong S^1$$

where $S^1 = \{z = e^{i\theta} \in \mathbb{C} : |z| = 1\}$ is a group with respect to the usual multiplication of two complex numbers.

(26) Show that $\mathbb{Q}$ is a normal subgroup of $(\mathbb{R}, +)$.

(27) Does $\mathbb{Q}$ have any subgroups isomorphic to $\mathbb{Z} \times \mathbb{Z}$? Prove your answer.

(28) Prove that $(\mathbb{Q}, +) \ncong (\mathbb{Q}^*, \cdot)$.

(29) Is $(\mathbb{R}, +) \cong (\mathbb{R}^*, \cdot)$?

(30) Is $(\mathbb{R}, +) \cong (\mathbb{R}^+, \cdot)$?

Note: $\mathbb{R}^+$ is the set of all positive real numbers.

(31) Let G be a group. G is cyclic Group if and only if there exists a surjective group homomorphism from $\mathbb{Z} \to G$.

(32) Prove that every finitely generated subgroup of $(\mathbb{Q}, +)$ is cyclic.

(33) Prove that $(\mathbb{Q}, +)$ and $(\mathbb{Q} \times \mathbb{Q}, +)$ are not isomorphic as groups.

(34) Find all homorphisms from $\mathbb{Z}_3 \times \mathbb{Z}_{10}$ to $\mathbb{Z}_6 \times \mathbb{Z}_{25}$.

(35) Let G_1, G_2, G_3 be groups such that G_1 is a homomorphic image of both G_2 and G_3. If order of G_2 is 24 and order G_3 is 30, list the possibilities for G_1 (up to isomorphism).