# INTRODUCTION TO GROUPS AND SYMMETRY (MTH 203)

## Mid Semester Examination

Total Marks: 98 Time: 10 AM to 12 PM Date: 22/09/2019

**Instructions:** Solve all problems. Please write all steps properly. If you are using any theorem, fact, formula then write a complete statement.

**Problem-A.** For each of the following statements indicate whether it is **TRUE** or **FALSE**. You **DO NOT** need to provide justification for your answers in this problem. (30)

(1) Subtraction $-$ is an associative binary operation on $\mathbb{Z}$.

(2) There exists no non-zero group homomorphism from $\mathbb{Z}_{10}$ to $\mathbb{Z}_{21}$.

(3) There is no nontrivial homomorphism from $(\mathbb{Q}, +) \to (\mathbb{Z}, +)$.

(4) If $m, n, x, y \in \mathbb{Z}$ are such that $mx + ny = 3$ then $gcd(m, n) = 3$.

(5) Any normal subgroup of a group can be obtained as the kernal of some group homomorphism.

(6) Any permutation of order $\geq 3$ can be written as product of disjoint transpositions.

(7) There exist a subgroup of index two in $S_{10}$ which is not normal.

(8) Let $G_1$ and $G_2$ are two groups and $G_1$ has no proper nontrivial subgroup. There exists a non zero group homomorphism $\phi : G_1 \to G_2$ which is not one-one.

(9) $2^{36} - 1$ is divisible by 37.

(10) There is a group homomorphism $\phi : (\mathbb{Z}, +) \to \mathbb{Q}^*, \cdot)$ such that $\phi(2) = \frac{1}{3}$.

**Problem-B.** Solve all questions.

(1) Define the center of a group $G$ and show that it is a normal subgroup. (6)

(2) Let $\sigma = (2,4,5,6)(1,4,5)(3,1,6) \in S_6$. Find the order of $\sigma$. Write $\sigma$ as product of transposition. $(6+3)$

(3) Let $\sigma = (2,3)(2,5) \in S_5$. Find the cycle type of $\sigma$ and compute the number of all $\tau \in S_5$, which are congugate to $\sigma$. $(3+6)$

In the above both problems we have used cycle notation for $\sigma$.

(4) Let $G$ be a group and let $\sim$ be the relation defined as,

$$x \sim y \text{ if there exists } z \in G \text{ such that } x = zyz^{-1}.$$

Show that $\sim$ is an equivalence relation and if $x \sim y$ then the order of $x$ is equal to the order of $y$. $(6+3)$

(5) Let $H$ and $K$ be two normal subgroups of a group $G$, whose intersection is the trivial subgroup. Prove that every element of $H$ commutes with every element of $K$. (10)

(6) Let $\phi : G_1 \to G_2$ be a group homomorphism. Suppose $G_1$ is an abelian group. Show that $Im(\phi)$ is an abelian subgroup of $G_2$.

Prove or disprove $S_3 \simeq \mathbb{Z}_6$. $(6+4)$

(7) State the First Isomorphism Theorem.

Let $\mathbb{C}^*$ be a group of all nonzero complex number with respect to complex multiplication and $S^1 = \{z = x+iy \in \mathbb{C} : |z| = \sqrt{(x^2+y^2)} = 1\}$. Show that $S^1$ is a normal subgroup of $\mathbb{C}^*$ and

$$\frac{\mathbb{C}^*}{S^1} \cong (\mathbb{R}^+, \cdot)$$

Note: $\mathbb{R}^+$ denote the group of all positive real numbers with respect to the usual multiplication of real numbers. $(3+3+9)$