INTRODUCTION TO GROUPS AND SYMMETRY (MTH 203)

End Semester Examination

Total Marks: 100 Time: 9.15 AM to 12.15 PM Date: 24/11/2019

Instructions: Solve all problems. Please write all steps properly. If you are using any theorem, fact, formula then write a complete statement.

Problem-A. For each of the following statements indicate whether it is **TRUE** or **FALSE**. You **DO NOT** need to provide justification for your answers in this problem. **You will get** 3 **marks for correct answer,** -1 **marks for wrong answer and** 0 **marks for no answer. If your total mark in this section is negative then we will consider it** 0**.**

(1) There exists no element of infinite order in $(\mathbb{Q}/\mathbb{Z}, +)$.

(2) There exists 5 group homomorphisms from $\mathbb{Z}_{10} \to U_{15}$.

(3) Any finite subgroup of $(\mathbb{R}/\mathbb{Z}, +)$ is cyclic.

(4) There exists a surjective group homomorphism from $(\mathbb{R}^*, .)$ onto $(\mathbb{Q}^*, .)$.

(5) There exists a surjective homomphism $\phi : \mathbb{Z} \to S_3$.

(6) The Symmetry group $Isom_{S^1}(\mathbb{R}^2)$ of the circle in $\mathbb{R}^2$ is a finite group.

(7) There are 4 element of order 10 in S_7.

(8) If all subgroups of a group G are normal, then group is abelian.

(9) A subgroup H of a group G is a normal subgroup if and only if the number of left cosets of H is equal to the number of right cosets of H.

(10) Sign $(\sigma) = -1$, where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 4 & 6 & 3 & 7 & 2 & 9 & 5 & 8 & 10 \end{pmatrix} \in S_{10}.$$

(11) $SL_n(\mathbb{R})$ is not a normal subgroup of $GL_n(\mathbb{R})$. $(3 \times 11 = 33)$

Problem-B. Solve all questions. Each question carries 7 Marks.

(1) If in the group G, $a^5 = e, aba^{-1} = b^2$ for some $a, b \in G$, find $o(b)$.

(2) Let G be a group and the order of G is pq, where p and q are some prime numbers. Show that G is abelian or $Z(G)$ is trivial subgroup.
Note that $Z(G)$ denote the centre of the group.

(3) Show that there is no non-trivial group homomorphism from S_3 to $\mathbb{Z}/3\mathbb{Z}$.

(4) Let G be a group of order n and let m be an integer relatively prime to n. Show that if $x^m = y^m$, then $x = y$. Hence show that for each $z \in G$ there is a unique $x \in G$ such that $x^m = z$.

$(8 \times 4 = 32\ )$

Problem-C. Solve all problems. Each question carries 7 marks.

(1) Let $Isom(\mathbb{R}^n)$ denote the group of all isometries of $\mathbb{R}^n$. For $X \subset \mathbb{R}^n$, show that $Isom_X(\mathbb{R}^n)$ is a subgroup of $Isom(\mathbb{R}^n)$, where

$$Isom_X(\mathbb{R}^n) = \{\phi : \mathbb{R}^n \to \mathbb{R}^n : \phi \text{ is an isometry and } \phi(X) = X\}.$$

(2) Prove or disprove $GL_1(\mathbb{R}) \cong GL_1(\mathbb{C})$?

(3) Show that $GL_1(\mathbb{C})$ is isomorphic to a subgroup of $GL_2(\mathbb{R})$.

(4) Show that the only isometry of $\mathbb{R}^2$ fixing $(0, 1)$, $(1, 1)$ and $(0, 0)$ is the identity.

(5) Show that $SO(2)$ is a normal subgroup of $O(2)$, where

$$O(2) = \{A \in GL_2(\mathbb{R}) : AA^t = I\}, SO(2) = \{A \in O(2) : det(A) = 1\}.$$

$(7 \times 5 = 35)$

(P.T.O)