INTRODUCTION TO GROUPS AND SYMMETRY (MTH 203)
ASSIGNMENT-7

SUBMISSION DATE: 12-11-2019

Submit only Problem-A.

Problems-A:.

(1) Show that
$$SO(2) \cong U(1) \cong S^1 \cong \mathbb{R}/\mathbb{Z},$$
where $S^1 = \{z \in \mathbb{C}^* : |z| = 1\}$ is the unit circle group with respect to complex multiplication.

(2) Show that a finite subgroup of $SO(2)$ is cyclic, and a finite subgroup of $O(2)$ not contained in $SO(2)$ is D_n (dihedral group).

(3) Let $X \subset \mathbb{R}^n$. Show that $Isom_X(\mathbb{R}^n)$ is a subgroup of $Isom(\mathbb{R}^n)$.

(4) Let X be a bounded subset of $\mathbb{R}^n$. Show that $Isom_X(\mathbb{R}^n)$ does not contain any translation.

(5) Compute $Isom(\mathbb{R})$, write it in term of Seitz symbols. Show that it is isomorphic to the group of matrices
$$\{\begin{bmatrix} b & a \\ 0 & 1 \end{bmatrix} \mid a, b \in \mathbb{R}, b = \pm 1\}.$$

Problems-B:.

(1) Verify whether the following are true or false.
- The group $O(1)$ is the Group $\{1, -1\}$ under multiplication. The group $SO(1)$ is trivial.
- The group $SO(2)$ is the set of all rotations about the origin in $\mathbb{R}^2$.
- The group $O(2)$ is the set of all rotations about $(0, 0)$ together with reflections in straight lines through $(0, 0)$.
- The group $SO(3)$ is the groups of all rotations about straight lines through $(0, 0, 0)$ in $\mathbb{R}^3$.
- The set of all translations forms a normal subgroup of $Isom(\mathbb{R}^n)$ which is isomorphic to the group $\mathbb{R}^n$ under addition.

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(2) Prove that the linear map (with respect to standard basis of $\mathbb{R}^3$) corresponding to an element of the orthogonal group $O(3)$ is either:
- a rotation about some axis,
- a reflection in some plane,
- a rotation about an axis (say a) followed by a reflection in a plane perpendicular to that axis a. We call it rotatory reflection.

(3) Let X be a regular n-gon in $\mathbb{R}^2$. Show that $|Isom_X(\mathbb{R}^2)| \leq 2n$.

(4) Let G be a finite subgroup of $SO(3)$. Show that G is isomorphic to precisely one of the following groups:

$$C_n, A_4, S_4, S_5, D_n$$

where C_n is a cyclic group of order n for $n \geq 1$ and D_n is a diahedral group of order $2n$ for $n \geq 2$.

(5) Can you describe all finite subgroup of $SO(4)$ and more general of $SO(n)$? It is not a part of the syllabus.

(6) Can you find any relation between $SU(2)$ and $SO(3)$. Try to prove,

$$SU(2) \cong SO(3)/\mathbb{Z}_2.$$