Quantum Physics (PHY 201)
Assignment 3 (21-08-2010)

1. Prove that,

   a)

$$\int_{-\infty}^{\infty} e^{i(k-k')x} dx = (2\pi)\delta(k-k') \,.$$

   b)

$$\int_{-\infty}^{\infty} e^{i(p-p')x/\hbar} dx = (2\pi\hbar)\delta(p-p') \,.$$

   [Hint for question 1a): Show that $\frac{1}{(2\pi)}\int_{-\infty}^{\infty} e^{i(k-k')x}dx$ satisfies the defining properties of Dirac delta function. You would find the result $\int_{-\infty}^{\infty}\frac{sin(y)}{y}dy = \pi$ useful in this problem.]

2. The wave function in the spatial domain, $\Psi(x)$, is related to wave function in the momentum domain, $A(p)$, following,

$$\Psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} A(p) e^{\frac{ipx}{\hbar}} dp \,.$$

   Show that, if $\Psi(x)$ is a normalized wave function $A(p)$ also is a normalized wave function. (This is known as Parseval's theorem.)

3. Assuming $\Psi(x)$ is given by the relation as given in the question 2 above, show that,

   a)

$$\left\langle p^2 \right\rangle = \int_{-\infty}^{\infty} \Psi^*(x)\left(-\hbar^2\frac{d^2}{dx^2}\right)\Psi(x)dx \,,$$

   b)

$$\langle x \rangle = \int_{-\infty}^{\infty} A^*(p)\left(i\hbar\frac{d}{dp}\right)A(p)dp \,,$$

   c)

$$\left\langle x^2 \right\rangle = \int_{-\infty}^{\infty} A^*(p)\left(-\hbar^2\frac{d^2}{dp^2}\right)A(p)dp \,,$$

   [Assume that $\Psi(x)$ and $A(p)$ are normalized.]

4. The wave function of a particle is given by $\Psi(x)$. Find the velocity expectation value $\langle v \rangle$.

5. A particle moving in one dimension has a De Broglie wavelength somewhere in the range $\lambda = \lambda_0 \pm \Delta\lambda$. Assuming $\Delta\lambda/\lambda_0 \gg (\Delta\lambda/\lambda_0)^n$ for all $n > 1$ and n is an integer, find the minimum spread, Δx, of the wave associated with it.

6. A wave function $\Psi(x)$, in the range $|x| \leq 2$ is given by,

$$\Psi(x) \;\; = \tfrac{x^2}{2} - \tfrac{x^4}{8} \,.$$

Given that $\Psi(x) = 0$ for $|x| > 2$,
a) Check if $\Psi(x)$ is a normalized wave function and draw the normalized wave function as a function of x.
b) Find the expectation values $\langle x \rangle$ and $\langle x^2 \rangle$. Using these results find uncertainty Δx.
c) Using $\hat{p}_x = \frac{\hbar}{i}\frac{d}{dx}$ and $\hat{p}_x^2 = -\hbar^2 \frac{d^2}{dx^2}$ find expectation values $\langle p_x \rangle$, $\langle p_x^2 \rangle$ and using these results estimate uncertainty in the x component of momentum Δp_x.
d) Using the results of b) and c) above, find the uncertainty product, $\Delta x \Delta p_x$.

7. The momentum wave function $A(p)$, in the range $0 \leq p \leq 1$ is given by,

$$A(p) \;\; = p(1-p)(p-0.5)^2 \,.$$

Assume that $A(p)$ is zero for $p > 1$ or $p < 0$.
a) Check if $A(p)$ is a normalized wave function and plot the normalized wave function as a function of p.
b) Using the results of question 3b) and 3c) find $\langle x \rangle$ and $\langle x^2 \rangle$ and hence the uncertainty Δx.
c) Find $\langle p_x \rangle$, $\langle p_x^2 \rangle$ and hence Δp_x. Finally estimate the uncertainty product, $\Delta x \Delta p_x$.