MTH 510/615: Operator Theory and Operator Algebras

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Contents

1 Banach Algebras

1.1 Definition and Examples

All vector spaces in this course will be over $\mathbb{C}$.

Definition 1.1.1. 1. An algebra is a vector space A over $\mathbb{C}$ together with a bilinear multiplication under which A is a ring. In other words, for all $\alpha, \beta \in \mathbb{C}, a, b, c \in A$, we have

$$(\alpha a + \beta b)c = \alpha(ac) + \beta(bc) \text{ and } a(\alpha b + \beta c) = \alpha(ab) + \beta(ac)$$

2. An algebra A is said to be a normed algebra if there is a norm on A such that
   a) $(A, \|\cdot\|)$ is a normed linear space
   b) For all $a, b \in A$, we have $\|ab\| \leq \|a\|\|b\|$
3. A Banach algebra is a complete normed algebra.

Remark 1.1.2. 1. If X is a normed linear space, then $\|x + y\| \leq \|x\| + \|y\|$. Hence, the map

$$(x, y) \mapsto x + y$$

is jointly continuous. ie. If $x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x + y$.

2. Similarly, if A is a normed algebra, then the map

$$(x, y) \mapsto xy$$

is jointly continuous [Check!]

Example 1.1.3. 1. $A = \mathbb{C}$

2. $A = C[0, 1]$. More generally, $C(X)$ for X A compact, Hausdorff space.

   $A = C_b(X)$, where X is a locally compact Hausdorff space.

3. $A = C_0(X)$, where X is a locally compact Hausdorff space. [Exercise]

4. $A = c_0$, the space of complex sequences converging to 0.

   Note: All the above examples are abelian.

5. $A = M_n(\mathbb{C})$ for any $n \in \mathbb{N}$ with the operator norm
$$\|A\| = \sup\{\|A(x)\| : x \in \mathbb{C}^n, \|x\| \leq 1\}$$
6. $A = \mathcal{B}(X)$ for any Banach space X.
7. $A = L^1(\mathbb{R})$ with multiplication given by convolution

   Proof. For $f, g \in A$, we write
$$f * g(x) := \int_{\mathbb{R}} f(t)g(x-t)dt$$
   Now
$$\|f * g\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f * g(x)|dx \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(t)||g(x-t)|dtdx = \|f\|\|g\|$$
   by Fubini's theorem. The other axioms are easy to check. □
8. $A = \ell^1(\mathbb{Z})$ with multiplication given by convolution (proof is identical to the previous one). A is a commutative Banach algebra.

Definition 1.1.4. Let A be a Banach algebra.

1. A subset $I \subset A$ is called a <u>left ideal</u> of A if it is a vector subspace of A and
$$a \in A, b \in I \Rightarrow ab \in I$$
2. A <u>right ideal</u> is defined similarly.
3. In this course, an <u>ideal</u> will refer to a two-sided ideal, for which we write $I \lhd A$.
4. An ideal $I \lhd A$ is said to be proper if $I \neq \{0\}$ and $I \neq A$.
5. A <u>maximal ideal</u> is an ideal that is not properly contained in any proper ideal.

Example 1.1.5. 1. $A = C[0,1]$, then $I = \{f \in C(X) : f(1) = 0\}$ is a maximal ideal.

2. If $A = M_n(\mathbb{C})$, then A has no non-trivial ideals

   Proof. Let $\{0\} \neq J \lhd A$, then choose $0 \neq T \in J$, then $\exists T_{i,j} = a \neq 0$. Let $E_{k,l}$ be the permutation matrix obtained by switching the k^{th} row of the identity matrix with the l^{th} row. Then
$$T' := E_{1,j}TE_{i,1} \in J$$
   and $T'_{1,1} = a \neq 0$. Now let $F_{1,1}$ be the matrix with 1 in the $(1,1)^{th}$ entry and zero elsewhere. Then
$$\frac{1}{a}F_{1,1}T'F_{1,1} = F_{1,1} \in J$$
   Similarly, $F_{2,2}, F_{3,3}, \ldots, F_{n,n} \in J$. Adding them up, we have $I_{\mathbb{C}^n} \in J$ and since J is an ideal, this means that $J = A$. □

(End of Day 1)

3. If X is a locally compact Hausdorff space, then $C_0(X)$ is an ideal in $C_b(X)$, the space of bounded continuous functions on X
4. Let X be a Banach space and $A = \mathcal{B}(X)$, then set
$$\mathcal{F}(X) = \{T \in \mathcal{B}(X) : T \text{ has finite rank}\}$$
Then $\mathcal{F}(X)$ is an ideal in A.
5. If $A = \mathcal{B}(X)$, then the set $\mathcal{K}(X)$ of compact operators on X is a closed ideal in A. In fact, if H is a Hilbert space, then $\mathcal{K}(H) = \overline{\mathcal{F}(H)}$

Theorem 1.1.6. *If A is a Banach algebra, and $I \lhd A$ is a proper closed ideal, then A/I is a Banach algebra with the quotient norm*
$$\|a + I\| = \inf\{\|a + b\| : b \in I\}$$

Proof. 1. Clearly, A/I is an algebra.

2. Now we check that the axioms of the norm hold :
   a) If $\|a + I\| = 0$, then $\exists b_n \in I$ such that $\|a + b_n\| \to 0$. Since I is closed, this means that $a \in I$ and hence $a + I = 0$ in A/I
   b) Clearly, $\|a + I\| \geq 0$.
   c) If $a, b \in A$, then for any $c, d \in I$
   $$\|a + b + I\| \leq \|a + b + c + d\| \leq \|a + c\| + \|b + d\|$$
   This is true for any $c, d \in I$, so taking infimum gives $\|a + b + I\| \leq \|a + I\| + \|b + I\|$
3. We now check that A/I is a Banach algebra: If $a, b \in A$, then for any $c, d \in I$ we have
$$(a + c)(b + d) = ab + cb + ad + dc$$
where $x := cb + ad + dc \in I$. Hence
$$\|ab + I\| \leq \|ab + x\| \leq \|a + c\|\|b + d\|$$
This is true for any $c, d \in I$, so taking infimum gives $\|ab + I\| \leq \|a + I\|\|b + I\|$
4. A is complete (See [Conway, Theorem III.4.2])

□

Definition 1.1.7. Let A and B be Banach algebras.

1. A map $\varphi : A \to B$ is called a <u>homomorphism</u> of Banach algebras if

a) $\varphi : A \to B$ is a continuous (ie. bounded) linear transformation of normed linear spaces

b) $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$.

2. Recall that if φ is continuous, then

$$\|\varphi\| = \sup\{\|\varphi(a)\| : a \in A, \|a\| \leq 1\}$$

3. A bijective homomorphism whose inverse is also continuous is called an <u>isomorphism</u> of Banach algebras.

Example 1.1.8. 1. If $I \lhd A$ is a closed ideal, then the natural quotient map $\pi : A \to A/I$ is a homomorphism. Note that

$$\|\pi(a)\| = \|a + I\| \leq \|a + 0\| = \|a\|$$

Hence $\|\pi\| \leq 1$. We will see later (Theorem 1.2.9) that $\|\pi\| = 1$ if A is unital.

2. If $A = C(X)$ and $x_0 \in X$, then $f \mapsto f(x_0)$ is a continuous homomorphism.

3. Let A be any Banach algebra, and $\mathcal{B}(A)$ be the space of bounded linear operators on A. Define $\varphi : A \to \mathcal{B}(A)$ by

$$x \mapsto L_x, \text{ where } L_x(y) := xy$$

Then φ is a continuous homomorphism, called the <u>left regular representation</u> of A.

4. Let $X = [0,1]$, $H = L^2(X)$ and set $A = C(X), B = \mathcal{B}(H)$. Define

$$\varphi : A \to B \text{ by } f \mapsto M_f$$

where $M_f(g) := fg$. Then φ is a continuous homomorphism.

(End of Day 2)

Theorem 1.1.9. *Let $\varphi : A \to B$ be a homomorphism of Banach algebras and let $I = \ker(\varphi)$. Then*

1. *$I = \ker(\varphi)$ is a closed ideal in A*

2. *There is a unique injective homomorphism $\overline{\varphi} : A/I \to B$ such that $\overline{\varphi} \circ \pi = \varphi$. Furthermore,*

$$\|\overline{\varphi}\| = \|\varphi\|$$

Proof. We know from algebra that $\exists$ a unique homomorphism of rings $\overline{\varphi} : A/I \to B$ such that $\overline{\varphi} \circ \pi = \varphi$ which is given by

$$\overline{\varphi}(a + I) = \varphi(a)$$

It is easy to see that $\overline{\varphi}$ is linear as well, and so is a homomorphism of algebras. Furthermore, for any $c \in I$

$$\|\overline{\varphi}(a + I)\| = \|\varphi(a)\| = \|\varphi(a + c)\| \leq \|\varphi\|\|a + c\|$$

Taking infimum, we see that $\overline{\varphi}$ is continuous and $\|\overline{\varphi}\| \leq \|\varphi\|$. However, since $\|\pi\| \leq 1$ by Example 1.1.8(1),

$$\|\varphi\| = \|\overline{\varphi} \circ \pi\| \leq \|\overline{\varphi}\|\|\pi\| \leq \|\overline{\varphi}\|$$

Hence, $\|\overline{\varphi}\| = \|\varphi\|$. □

1.2 Invertible Elements

Definition 1.2.1. Let A be a Banach algebra

1. A is said to be unital if $\exists e \in A$ such that $ae = ea = a$ for all $a \in A$.
2. If A is unital with unit e, then we will write $1_A = 1 = e$, and assume that $\|1_A\| = 1$.
3. If A is unital, then we may assume $\mathbb{C} \subset A$ via the map $\alpha \mapsto \alpha 1_A$

Remark 1.2.2. ([Arveson, Theorem 1.4.2]) Let $(A, \|\cdot\|)$ be a complex algebra with a unit e that is also a Banach space. Furthermore, assume that the multiplication map

$$(x, y) \mapsto xy$$

is jointly continuous. Then there is a norm $\|\cdot\|_1$ on A that is equivalent to $\|\cdot\|$ such that $(A, \|\cdot\|_1)$ is a Banach algebra and $\|e\|_1 = 1$.

Example 1.2.3. 1. If X is compact Hausdorff, then $C(X)$ is unital.

2. If X is non-compact, then $C_0(X)$ is non-unital. In particular, c_0 is non-unital.
3. $M_n(\mathbb{C})$ is unital. So is $\mathcal{B}(X)$ for any Banach space X
4. $L^1(\mathbb{R})$ is non-unital

   Proof. Suppose $e \in L^1(\mathbb{R})$ is a unit, then for all $\epsilon > 0, \exists \delta > 0$ such that for any measurable $V \subset \mathbb{R}$

$$m(V) < \delta \Rightarrow \int_V |e(x)|dx < \epsilon$$

Let $V = (-\delta/4, \delta/4)$, then $m(V) = \delta/2 < \delta$. Now if $f = \chi_V$ is the characteristic function of V, then for any $x \in \mathbb{R}$

$$f(x) = e * f(x) = \int_{\mathbb{R}} e(t)f(x-t)dt = \int_{x-V} e(t)dt < \epsilon$$

However, if $x \in V$, then

$$1 = f(x) < \epsilon$$

so with $\epsilon = 1/2$, this gives a contradiction. □

5. $\ell^1(\mathbb{Z})$ is unital with unit (e_n) given by

$$e_n = \begin{cases} 1 & : n = 0 \\ 0 & : n \neq 0 \end{cases}$$

6. $L^\infty(X, \mu)$ is unital

Definition 1.2.4. Let A be a unital Banach algebra.

1. An element $a \in A$ is said to be invertible if $\exists b \in A$ such that $ab = ba = 1_A$. The inverse, if it exists, is unique, and is denoted by a^{-1}.

2. The General Linear group of A, denoted by $GL(A)$, is the set of all invertible elements in A.

Theorem 1.2.5. *If $a \in A$ is such that $\|1 - a\| < 1$, then $a \in GL(A)$. Furthermore, a^{-1} is given by the Neumann series*

$$a^{-1} = 1 + (1-a) + (1-a)^2 + \ldots = \sum_{k=0}^{\infty} (1-a)^k$$

Proof. Since the series on the RHS converges absolutely, the series

$$\sum_{n=0}^{\infty} (1-a)^n$$

converges to an element $b \in A$ (since A is a Banach space). Furthermore, writing $x = (1-a)$, by continuity of multiplication,

$$ab = \lim_{n\to\infty} \sum_{k=0}^{n} a(1-a)^k = \lim_{n\to\infty} \sum_{k=0}^{n} (1-x)x^k = \lim_{n\to\infty} (1 - x^{n+1}) = 1$$

Similarly, $ba = 1$ as well. □

Corollary 1.2.6. *1. $GL(A)$ is an open subset of A*

2. The map $x \mapsto x^{-1}$ from $GL(A)$ to $GL(A)$ is a homeomorphism.
In particular, $GL(A)$ is a topological group.

Proof. 1. If $a \in GL(A)$ and $b \in A$ such that

$$\|a - b\| < \frac{1}{\|a^{-1}\|}$$

Then $\|1 - a^{-1}b\| < 1$ and so $a^{-1}b$ is invertible, whence b is invertible.

2. WTS: $a \mapsto a^{-1}$ is continuous, so suppose $a_n \to a$ in $GL(A)$. Replacing a_n by $a_n a^{-1}$, we may assume WLOG that $a = 1$. Given $\delta > 0, \exists N \in \mathbb{N}$ such that

$$\|a_n - 1\| < \delta \quad \forall n \geq N$$

By Theorem 1.2.5,

$$a_n^{-1} = 1 + \sum_{k=1}^{\infty} (1 - a_n)^k$$

Hence,

$$\|a_n^{-1} - 1\| \leq \sum_{k=1}^{\infty} \|1 - a_n\|^k < \frac{\delta}{(1 - \delta)}$$

So given $\epsilon > 0$, choose $\delta > 0$ such that $\delta/(1 - \delta) < \epsilon$.

□

(End of Day 3)

Theorem 1.2.7. *Let A be a unital Banach algebra, then every ideal $I \lhd A$ is contained in a maximal ideal.*

Proof. Same proof as in Ring theory (using Zorn's Lemma). □

Theorem 1.2.8. *Let A be a unital Banach algebra*

1. *If $I \lhd A$ is a proper ideal, then $\overline{I}$ is a proper closed ideal.*
2. *Every maximal ideal in A is closed.*

Proof. If I is an ideal, then it is easy to check that $\overline{I}$ is an ideal. If $I \lhd A$ is proper, then $I \cap GL(A) = \emptyset$. Hence, $I \subset (A \setminus GL(A))$ which is closed, whence $\overline{I} \subset (A \setminus GL(A))$.

Finally, part (2) follows from part (1). □

Theorem 1.2.9. *Let A be a unital Banach algebra and $I \lhd A$ be a proper closed ideal. Let $\pi : A \to A/I$ be the natural homomorphism, then π is continuous and*

$$\|\pi\| = \|\pi(1)\| = 1$$

Proof. We saw in Example 1.1.8 that π is continuous and $\|\pi\| \leq 1$. Since $\|1_A\| = 1$,

$$\|1_A + I\| = \|\pi(1_A)\| \leq \|\pi\|\|1_A\| = \|\pi\| \leq 1$$

However, for any $b \in I, \|1_A + b\| \geq 1$ since $I \cap GL(A) = \emptyset$. Hence,

$$\|1_A + I\| = \inf\{\|1_A + b\| : b \in GL(A)\} \geq 1$$

□

1.3 Spectrum of an Element

Throughout this section, let A denote a unital Banach algebra with unit $1 \in A$

Definition 1.3.1. Let $a \in A$

1. The spectrum of a, denoted by $\sigma(a)$, is defined as

$$\sigma(a) := \{\lambda \in \mathbb{C} : (a - \lambda 1_A) \notin GL(A)\}$$

2. The resolvent of a, denoted by $\rho(a)$, is defined as

$$\rho(a) := \mathbb{C} \setminus \sigma(a)$$

Example 1.3.2. 1. If $T \in \mathcal{B}(\mathbb{C}^n)$, then $\sigma(T)$ is the set of eigen-values of T

2. If X is a Banach space,

$$\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is not bijective}\}$$

3. If $T \in \mathcal{B}(C[0,1])$ be the operator

$$T(f)(x) := \int_0^x f(t)dt$$

Then T is not surjective, because if $g \in \text{Image}(T)$, then g is a C^1 function. Hence, $0 \in \sigma(T)$. However, 0 is not an eigen-value of T [Exercise]

4. If $f \in C(X)$, then $\sigma(f) = f(X)$ is the range of f

5. If $A = \ell^\infty(X)$ for some set X, then for any $f \in A$, $\sigma(f) = \overline{f(X)}$ is the closure of the range of f in $\mathbb{C}$

Theorem 1.3.3. *For any $a \in A$, $\sigma(a)$ is a compact subset of the disc*

$$\{z \in \mathbb{C} : |z| \leq \|a\|\} \subset \mathbb{C}$$

Proof. 1. If $|\lambda| > \|a\|$, then $\|a/\lambda\| < 1$, so $(1 - a/\lambda) \in GL(A)$. Hence, $\lambda \in \rho(a)$. Hence,

$$\sigma(a) \subset \{z \in \mathbb{C} : |z| \leq \|a\|\}$$

2. The function $f : \lambda \mapsto (\lambda - a)$ is a continuous function $\mathbb{C} \to A$. Since $GL(A)$ is open,

$$\rho(a) = f^{-1}(GL(A))$$

is open, and hence $\sigma(a)$ is closed.

$\square$

Remark 1.3.4. Let A be a Banach algebra, $\Omega \subset \mathbb{C}$ be an open set and $F : \Omega \to A$ be a function.

1. We say that F is analytic if $\exists G : \Omega \to A$ continuous such that
$$\lim_{h\to 0}\left\|\frac{F(z+h)-F(z)}{h} - G(z)\right\| = 0 \quad \forall z \in \Omega$$
and in that case, we say that $F'(z) = G(z)$

2. Suppose F is analytic, and $\tau \in A^*$ is a bounded linear functional, then
$$H : \Omega \to \mathbb{C} \text{ given by } H = \tau \circ F$$
is analytic (in the usual sense) and $H' = \tau \circ G$.

(End of Day 4)

Lemma 1.3.5. *Let $a \in A$ and $F : \rho(a) \to A$ be given by*
$$F(z) = (z-a)^{-1}$$
Then F is analytic and $F'(z) = -(z-a)^{-2}$

Proof. Let $G : \rho(a) \to A$ be given by $z \mapsto -(z-a)^{-2}$. Then G is continuous because it is the composition of continuous functions. Now, for any $x, y \in GL(A)$
$$x^{-1} - y^{-1} = x^{-1}(y-x)y^{-1}$$
Applying this to $x = (z+h-a)$ and $y = (z-a)$, we have
$$F(z+h) - F(z) = (z+h-a)^{-1}(-h)(z-a)^{-1}$$
so
$$\left\|\frac{F(z+h)-F(z)}{h} - G(z)\right\| = \left\|[(z+h-a)^{-1} - (z-a)^{-1}](z-a)^{-1}\right\|$$
Now use the fact that $z \mapsto (z-a)^{-1}$ is continuous. □

Theorem 1.3.6 (Gelfand-Mazur)**.** *If A is a Banach algebra, then $\sigma(a) \neq \emptyset$ for any $a \in A$.*

Proof. Let $a \in A$, then we want to show that $\sigma(a) \neq \emptyset$. Clearly, if $a = 0$, then $0 \in \sigma(a)$, so we assume that $a \neq 0$. Suppose $\sigma(a) = \emptyset$, then $\rho(a) = \mathbb{C}$, so consider $F : \mathbb{C} \to A$ by
$$F(z) = (z-a)^{-1}$$
By Lemma 1.3.5, F is analytic and
$$F'(z) = -(z-a)^{-2} \tag{1.1}$$

As $z \to \infty$, $(1 - a/z) \to 1$, so by Corollary 1.2.6(2), we have

$$F(z) = z^{-1}\left(1 - \frac{a}{z}\right)^{-1} \to 0 \tag{1.2}$$

Hence, if $\tau \in A^*$, then consider

$$H(z) = \tau \circ F(z) = \tau((z-a)^{-1})$$

Then, H is entire by Equation 1.1 and bounded by Equation 1.2. So H is constant by Liouville's theorem. In particular,

$$H'(0) = \tau(a^{-2}) = 0$$

This is true for all $\tau \in A^*$, which is impossible since $a \neq 0$ and so $a^{-2} \neq 0$. $\square$

Corollary 1.3.7. *If A is a unital Banach algebra in which every non-zero element is invertible, then $A = \mathbb{C}1_A$*

Proof. Let $a \in A$, then $\exists \lambda$ such that $a - \lambda 1_A$ is not invertible. Hence, $a - \lambda 1_A = 0$, so $a = \lambda 1_A$. $\square$

Definition 1.3.8. For $a \in A$, the spectral radius of a is $r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}$

Remark 1.3.9.
1. By Theorem 1.3.3, $r(a) \leq \|a\|$
2. Since $\sigma(a)$ is compact, $\exists \lambda_0 \in \sigma(a)$ such that $r(a) = |\lambda_0|$

Example 1.3.10.
1. If X is compact, Hausdorff and $A = C(X)$, then $r(f) = \|f\|_\infty$ for all $f \in A$
2. If $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})$, then $r(T) = 0$, while $\|T\| = 1$

   More generally, if $T \in M_n(\mathbb{C})$ is nilpotent, then $r(T) = 0$ because the minimal polynomial of T is x^k for some $k \in \mathbb{N}$
3. Let H be a Hilbert space and $A = \mathcal{B}(H)$. Let $T \in A$ be a unitary operator, then
   a) $0 \notin \sigma(T)$ because T is invertible.
   b) If $\lambda \in \sigma(T)$, then $\overline{\lambda} \in \sigma(T^*) = \sigma(T^{-1})$.
   c) Furthermore, for any $\alpha \in \rho(T) \setminus \{0\}$, we have

   $$T^{-1} - \alpha^{-1} = \alpha^{-1}(\alpha - T)T^{-1}$$

   and so $\alpha^{-1} \in \rho(T^{-1})$. Hence, if $\lambda \in \sigma(T)$, then $\overline{\lambda}^{-1} \in \sigma(T)$.
   d) Since $\|T\| = 1$, it follows that

   $$\max\{|\lambda|, |\overline{\lambda}|^{-1}\} \leq 1$$

   Hence, $|\lambda| = 1$. This is true for all $\lambda \in \sigma(T)$, so $\sigma(T) \subset \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$

e) Since $\sigma(T) \neq \emptyset$, it follows that $r(T) = 1 = \|T\|$

Theorem 1.3.11 (Spectral Mapping Theorem)**.** *Let A be a unital Banach algebra, $a \in A$ and $p \in \mathbb{C}[z]$, then*

$$\sigma(p(a)) = p(\sigma(a)) = \{p(\lambda) : \lambda \in \sigma(a)\}$$

Proof. Note that if $p(z) = a_0 + a_1 z + \ldots + a_n z^n$, then $p(a) = a_0 1_A + a_1 a + \ldots + a_n a^n$. Now, if $\alpha \in \mathbb{C}$, then by the Fundamental theorem of algebra, $\exists \gamma, \beta_1, \beta_2, \ldots, \beta_n \in \mathbb{C}$ such that

$$p(z) - \alpha = \gamma(z - \beta_1)(z - \beta_2) \ldots (z - \beta_n)$$

Hence,

$$p(a) - \alpha = \gamma(a - \beta_1)(a - \beta_2) \ldots (a - \beta_n)$$

Hence,

$$\begin{aligned} \alpha \in \sigma(p(a)) &\Leftrightarrow \beta_i \in \sigma(a) \quad \text{for some } 1 \leq i \leq n \\ &\Leftrightarrow p(\lambda) - \alpha = 0 \quad \text{for some } \lambda \in \sigma(a) \\ &\Leftrightarrow \alpha \in p(\sigma(a)) \end{aligned}$$

□

(End of Day 5)

Theorem 1.3.12 (Spectral Radius Formula)**.** *For any $a \in A$,*

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}$$

In particular, this limit exists.

Proof.

1. By Theorem 1.3.3, $r(a) \leq \|a\|$. In fact, if $\lambda \in \sigma(a)$, then $\lambda^n \in \sigma(a^n)$ by the Spectral Mapping theorem. Hence, $|\lambda^n| \leq \|a^n\| \Rightarrow |\lambda| \leq \|a^n\|^{1/n}$. Hence,

$$r(a) \leq \liminf \|a^n\|^{1/n}$$

2. Conversely, let D be the open disc in $\mathbb{C}$ centred at 0 of radius $1/r(a)$ [$= +\infty$ if $r(a) = 0$]. If $\lambda \in D$, then $1 - \lambda a \in GL(A)$ [check!]. So if $\tau \in A^*$, consider the map

$$g : D \to \mathbb{C} \text{ given by } g(\lambda) := \tau((1 - \lambda a)^{-1})$$

As in Theorem 1.3.6, g is analytic, and so $\exists$ unique $\alpha_n \in \mathbb{C}$ such that

$$g(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$$

Now if $|\lambda| < 1/\|a\|$, then $\lambda \in D$ and $\|\lambda a\| < 1$, so by Theorem 1.2.5,

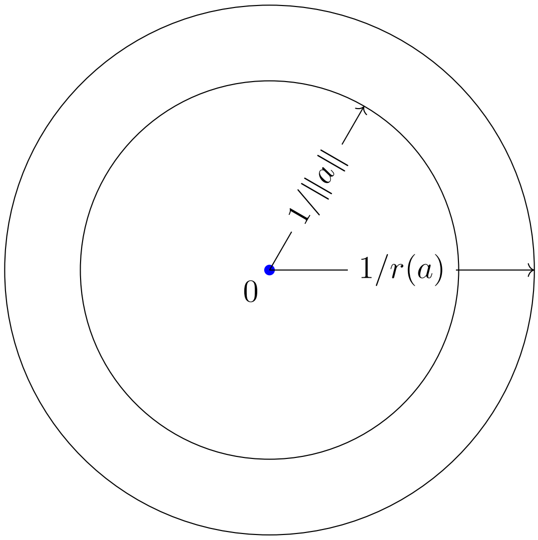
$$(1 - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$$

Hence,

$$g(\lambda) = \sum_{n=0}^{\infty} \tau(\lambda^n a^n) = \sum_{n=0}^{\infty} \lambda^n \tau(a^n)$$

So by uniqueness of the α_n, we have

$$g(\lambda) = \sum_{n=0}^{\infty} \tau(a^n)\lambda^n \quad \forall \lambda \in D$$



In particular, for fixed $\lambda \in D$, the series $\sum_{n=0}^{\infty} \tau(a^n)\lambda^n$ converges, and so the sequence $\{\tau(a^n)\lambda^n\}$ converges to 0, and is therefore bounded. This is true for all $\tau \in A^*$, so by the Uniform Boundedness principle, the sequence $\{\lambda^n a^n\}$ is a bounded sequence. Hence, $\exists M > 0$ such that for all $n \geq 0$,

$$\|\lambda^n a^n\| \leq M \Rightarrow \|a^n\|^{1/n} \leq M^{1/n}/|\lambda|$$

Taking lim sup on both sides, we get

$$\limsup \|a^n\|^{1/n} \leq \frac{1}{|\lambda|}$$

This is true for all $\lambda \in \mathbb{C}$ such that $|\lambda| < 1/r(a)$, and so

$$\limsup \|a^n\|^{1/n} \leq r(a)$$

□

Example 1.3.13. Let $A = \mathcal{B}(C[0,1])$ and $T \in A$ be

$$T(f)(x) = \int_0^x f(t)dt$$

Then

1. $\|T\| = 1$

2. $$\begin{aligned} T^2(f)(x) &= \int_0^x \int_0^t f(s)dsdt \\ \Rightarrow |T^2(f)(x)| &\leq \|f\|_\infty \int_0^x \int_0^t dsdt \\ &= \|f\|_\infty \int_0^x tdt \\ &= \|f\|_\infty \frac{x^2}{2} \\ \Rightarrow \|T^2\| &\leq \frac{1}{2} \end{aligned}$$

3. More generally,
$$\|T^n\| \leq \frac{1}{n!}$$

4. Hence,
$$r(T) \leq \lim_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n} = 0$$
Thus, $\sigma(T) = \{0\}$ even though T is not nilpotent.

1.4 Unital Commutative Banach Algebras

Throughout this section, let A denote a unital commutative Banach algebra

Definition 1.4.1. 1. A linear functional $\tau : A \to \mathbb{C}$ is said to be <u>multiplicative</u> if $\tau(ab) = \tau(a)\tau(b)$ for all $a, b \in A$.

Note: We do not require it to be continuous - it will be automatically.

2. The <u>Gelfand spectrum of A</u> is defined as
$$\Omega(A) = \{\tau : A \to \mathbb{C} : \tau \text{ is a non-zero multiplicative linear functional}\}$$

(End of Day 6)

Lemma 1.4.2. *Let A be a unital commutative Banach algebra.*

1. *If $\tau \in \Omega(A)$, then $\tau(1) = 1$*
2. *If $\tau \in \Omega(A)$, then $\ker(\tau)$ is a maximal ideal.*

3. *The map*

$$\mu : \tau \mapsto \ker(\tau)$$

defines a bijection between $\Omega(A)$ *and the set of all maximal ideals of* A

4. *Every* $\tau \in \Omega(A)$ *is continuous and* $\|\tau\| = \tau(1) = 1$

Proof. 1. For all $a \in A, \tau(a) = \tau(a \cdot 1) = \tau(a)\tau(1)$. Choose $a \in A$ such that $\tau(a) \neq 0$, then $\tau(1) = 1$.

2. If $\tau \in \Omega(A)$, then τ is surjective [Check!], and so τ induces an isomorphism $\overline{\tau} : A/\ker(\tau) \to \mathbb{C}$. Hence, $\ker(\tau)$ is a maximal ideal.

3. If $\tau \in \Omega(A)$, then $\ker(\tau)$ is maximal, so μ is well-defined.

   a) If $\tau_1(a) \neq \tau_2(a)$, then $a - \tau_2(a) \cdot 1 \in \ker(\tau_2) \setminus \ker(\tau_1)$, so the map μ is injective.

   b) If $I \lhd A$ is a maximal ideal, then I is closed, and so A/I is a Banach algebra. Furthermore, if $a + I \neq I$, then $W := \{x + ab : x \in I, b \in A\}$ is an ideal (Check!) of A that contains I. Since $W \neq I$, it must happen that $W = A$. Hence, $\exists x \in I$ and $b \in A$ such that $x+ab = 1_A$. Hence, $(a+I)(b+I) = 1_A+I$. Thus, every non-zero element in A/I is invertible.

   Thus, by Corollary 1.2.6, there is an isomorphism $\varphi : A/I \to \mathbb{C}$. Let $\pi : A \to A/I$ be the natural homomorphism then $\tau := \varphi \circ \pi$ is an element of $\Omega(A)$ and $\ker(\tau) = I$ so μ is surjective.

4. If $\tau \in \Omega(A)$ then $\tau : A \to \mathbb{C}$ is a linear functional.

   a) By part (2), $I := \ker(\tau)$ is a maximal ideal. By Theorem 1.2.8, I is closed. Hence, $\pi : A \to A/I$ is continuous by Theorem 1.2.9. Furthermore, since both A/I and $\mathbb{C}$ are finite dimensional, $\overline{\tau} : A/I \to \mathbb{C}$ as above is continuous. Since

   $$\tau = \overline{\tau} \circ \pi$$

   it follows that τ is continuous.

   b) Now for any $a \in A, \exists \alpha \in \mathbb{C}$ such that $a + I = \alpha + I$ and

   $$|\overline{\tau}(a + I)| = |\overline{\tau}(\alpha + I)| = |\alpha|$$

   By Theorem 1.1.6, $|\alpha| = \|\alpha + I\|$ and so $\|\overline{\tau}\| = 1$. Hence, $\|\tau\| \leq \|\overline{\tau}\|\|\pi\| \leq 1$. Since $\tau(1) = 1$, it follows that $\|\tau\| = 1$.

□

Theorem 1.4.3. *Let* A *be a unital commutative Banach algebra and* $a \in A$. *Then*

$$\sigma(a) = \{\tau(a) : \tau \in \Omega(A)\}$$

Proof. Suppose $\lambda \in \sigma(a)$, then $x := (a - \lambda \cdot 1)$ is not invertible. Let I be the principal ideal generated by x, then I is contained in a maximal ideal J by Theorem 1.2.7. Let τ be the corresponding element of $\Omega(A)$, then $J = \ker(\tau)$ so $\tau(x) = 0$ and so $\lambda = \tau(a)$. Conversely, if $\tau \in \Omega(A)$, then $x := a - \tau(a) \cdot 1$ is in $\ker(\tau)$, which is a proper ideal. Hence, x cannot be invertible and so $\tau(a) \in \sigma(a)$. □

Remark 1.4.4. Recall that A^* carries the weak-$*$ topology.

1. Banach-Alouglu theorem states that the set

$$B := \{\varphi \in A^* : \|\varphi\| \leq 1\}$$

is compact in the weak-$*$ topology.

2. $\Omega(A)$ inherits the weak-* topology and is pre-compact since $\Omega(A) \subset B$ by Lemma 1.4.2

Theorem 1.4.5. *$\Omega(A)$ is a compact Hausdorff space in the weak-$*$ topology.*

Proof. It suffices to show that $\Omega(A)$ is closed in B. So suppose $\tau_\alpha \to \tau$ with $\tau_\alpha \in \Omega(A)$ for all τ. In particular, $\tau_\alpha(1) = 1$ for all α. Hence, $\tau(1) = 1 \neq 0$. Hence, $\tau \neq 0$. Also, for any $a, b \in A$, we have

$$\tau_\alpha(ab) = \tau_\alpha(a)\tau_\alpha(b) \to \tau(a)\tau(b)$$

Hence, τ is multiplicative as well. □

Definition 1.4.6. Given $a \in A$, the <u>Gelfand Transform</u> of a is defined by

$$\hat{a} : \Omega(A) \to \mathbb{C}$$

by $\hat{a}(\tau) := \tau(a)$

Theorem 1.4.7. *Let A be a unital commutative Banach algebra.*

1. *For $a \in A, \hat{a} \in C(\Omega(A))$*

2. $\|\hat{a}\|_\infty = r(a)$

3. *The map*

$$\Gamma_A : A \to C(\Omega(A)) \text{ given by } a \mapsto \hat{a}$$

is a homomorphism of Banach algebras. This is called the <u>Gelfand representation</u> of A.

Proof. 1. Note that $\Omega(A)$ has the weak-$*$ topology, so for any net $\tau_\alpha \in \Omega(A)$, we say that $\tau_\alpha \to \tau$ iff $\tau_\alpha(x) \to \tau(x)$ for all $x \in A$. In particular, $\hat{a}(\tau_\alpha) = \tau_\alpha(a) \to \tau(a) = \hat{a}(\tau)$. Hence, $\hat{a} \in C(\Omega(A))$

2. Follows from Theorem 1.4.3.

3. Note that, for any $a, b \in A$ and $\tau \in \Omega(A)$,

$$\widehat{ab}(\tau) = \tau(ab) = \tau(a)\tau(b) = \hat{a}(\tau)\hat{b}(\tau)$$

This is true for all $\tau \in \Omega(A)$, and so $\widehat{ab} = \hat{a}\hat{b}$. Hence, Γ_A is multiplicative. Similarly, we see that Γ_A is also linear. Finally, Γ_A is continuous since

$$\|\Gamma_A(a)\| = \|\hat{a}\|_\infty = r(a) \leq \|a\|$$

by Theorem 1.3.3

□

(End of Day 7)

Definition 1.4.8. A is generated by $\{a, 1\}$ if $A = \overline{\{p(a) : p \in \mathbb{C}[z]\}}$

Theorem 1.4.9. *Let A be a unital Banach algebra generated by $\{1, a\}$. Then A is commutative, and the map*

$$\hat{a} : \Omega(A) \to \sigma(a), \text{ given by } \tau \mapsto \tau(a)$$

is a homeomorphism.

Proof. That A is commutative is clear. The map $\widehat{a}$ is surjective by Theorem 1.4.3. Also, if $\tau_1, \tau_2 \in \Omega(A)$ such that $\tau_1(a) = \tau_2(a)$, then since $\tau_1(1) = 1 = \tau_2(1)$, it follows that for any $p \in \mathbb{C}[z]$,

$$\tau_1(p(a)) = \tau_2(p(a))$$

Since both τ_1 and τ_2 are continuous, it follows that $\tau_1 = \tau_2$. Hence, $\widehat{a}$ is also injective. Thus,

$$\widehat{a} : \Omega(A) \to \sigma(a)$$

is bijective and continuous by Theorem 1.4.7. Since both sets are compact and Hausdorff, it is a homeomorphism. □

Definition 1.4.10. Let A be a unital commutative Banach algebra

1. The <u>radical</u> of A, denoted by $rad(A)$ is $\ker(\Gamma_A)$.

   Note:
   $$rad(A) = \{a \in A : r(a) = 0\} = \{a \in A : \sigma(a) = \{0\}\}$$
   and $rad(A)$ is the intersection of all maximal ideals of A.

2. A Banach algebra A is said to be <u>semi-simple</u> if $rad(A) = \{0\}$

   Note: A is semi-simple iff Γ_A is injective.

1.5 Examples of the Gelfand Spectrum

Remark 1.5.1. Let X be a compact Hausdorff space and $A = C(X)$. For any $x \in X$, the map

$$\tau_x : A \to \mathbb{C} \text{ given by } f \mapsto f(x)$$

is a multiplicative linear functional. So we get a function $X \to \Omega(A)$ which is clearly injective.

Theorem 1.5.2. *Let $I \lhd C(X)$ be a maximal ideal. Then $\exists x_0 \in X$ such that*

$$I = \ker(\tau_{x_0})$$

Proof. Let $I \lhd C(X)$ be a maximal ideal, then we claim that $\exists x_0 \in X$ such that

$$f(x_0) = 0 \quad \forall f \in I \qquad (*)$$

Suppose not, then for all $x \in X, \exists f_x \in I$ such that $f_x(x) \neq 0$. Then $\exists$ a neighbourhood V_x of X such that $f_x(y) \neq 0$ for all $y \in V_x$. Now the family $\{V_x : x \in X\}$ forms an open cover of X, and so must have a finite subcover, say $\{V_{x_1}, V_{x_2}, \ldots, V_{x_n}\}$. Define

$$h = \sum_{i=1}^{n} f_{x_i}\overline{f_{x_i}}$$

Then $h \in I$ since I is an ideal, and if $x \in X$ then $\exists 1 \leq i \leq n$ such that $x \in V_{x_i}$. Hence, $f_{x_i}(x) \neq 0$ and so $h(x) > 0$. Thus, $h > 0$ on X. Hence, $h \in GL(C(X))$ and so $I = C(X)$. This is a contradiction, and so the claim $(*)$ is true. Thus, $I \subset \ker(\tau_{x_0})$ and since I is maximal, it follows that $I = \ker(\tau_{x_0})$. □

Theorem 1.5.3. *Let $A = C(X)$, then the map*

$$\mu : X \to \Omega(A) \textit{ given by } x \mapsto \tau_x$$

is a homeomorphism.

Proof. By Remark 1.5.1, Lemma 1.4.2 and Theorem 1.5.2, the map μ is bijective. Now suppose $x_\alpha \to x$ in X. Then, for any $f \in C(X)$

$$f(x_\alpha) \to f(x) \Leftrightarrow \tau_{x_\alpha}(f) \to \tau_x(f)$$

Hence, $\tau_{x_\alpha} \to \tau_x$ in the weak-$*$ topology. Hence, μ is a continuous bijection between two compact sets. Hence, μ is a homeomorphism. □

Remark 1.5.4. Let $A = \ell^1(\mathbb{Z})$, then for any $\lambda \in \mathbb{T}$, define

$$\tau_\lambda : \ell^1(\mathbb{Z}) \to \mathbb{C} \text{ given by } (a_n) \mapsto \sum_{n=0}^{\infty} a_n \lambda^n$$

Note that τ_λ is well-defined since the series on the right-hand side converges absolutely. Furthermore, $\tau_\lambda \in \Omega(A)$ [Check!]

(End of Day 8)

Theorem 1.5.5. *The map*

$$\mu : \mathbb{T} \to \Omega(A) \text{ given by } \lambda \mapsto \tau_\lambda$$

is a homeomorphism.

Proof. As before, μ is injective and continuous. Since $\mathbb{T}$ and $\Omega(A)$ are both compact, it suffices to prove that μ is surjective. So suppose $\tau \in \Omega(A)$ and let $a \in A$ be given by

$$a_n = \begin{cases} 1 & : n = 1 \\ 0 & : n \neq 1 \end{cases}$$

and let $\lambda := \tau(a)$. Then

1. $|\lambda| = |\tau(a)| \leq \|a\| = 1$
2. $a \in A$ is invertible with inverse b given by

   $$b_n = \begin{cases} 1 & : n = -1 \\ 0 & : n \neq -1 \end{cases}$$

   Hence,

   $$\left|\frac{1}{\lambda}\right| = |\tau(b)| \leq \|b\| \leq 1$$

   Hence, $|\lambda| = 1$
3. Consider $\tau \in \Omega(A)$, then for $\lambda := \tau(a)$, we have $|\lambda| = 1$ as above, and

   $$\tau_\lambda(a) = \lambda = \tau(a)$$

4. Now note that a^k is the sequence

   $$(a^k)_n = \begin{cases} 1 & : n = k \\ 0 & : n \neq k \end{cases}$$

   and so A is generated by a as a Banach algebra. Since $\tau(a) = \tau_\lambda(a)$, it follows that $\tau = \tau_\lambda$

□

Definition 1.5.6. Let $f : \mathbb{T} \to \mathbb{C}$ be continuous, then we say that f has an absolutely convergent Fourier series if

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n \text{ and } \sum_{n\in\mathbb{Z}} |a_n| < \infty$$

The Wiener algebra $\mathcal{W}$ is the set of all such functions.

Theorem 1.5.7 (Gelfand-Weiner). *If $f \in \mathcal{W}$ has no zeroes in $\mathbb{T}$, then $1/f \in \mathcal{W}$*

Proof. Let $A = \ell^1(\mathbb{Z})$ and let

$$\Gamma_A : A \to C(\mathbb{T})$$

be the Gelfand transform. Note that $\mathcal{W} = R(\Gamma_A)$. Hence, if $f \in \mathcal{W}$ such that f has no zeroes in $\mathbb{T}$, then write $f = \hat{a}$ and note that

$$\tau_\lambda(a) \neq 0 \quad \forall \lambda \in \mathbb{T}$$

Hence, $\tau(a) \neq 0$ for all $\tau \in \Omega(A)$. By Theorem 1.4.3, $0 \notin \sigma(a)$, and so $a \in GL(A)$. Let $b = a^{-1}$ then $g := \hat{b}$ is the inverse of f in $\mathcal{W}$. □

1.6 Spectral Permanence Theorem

Throughout this section, let A be a unital Banach algebra and $B \subset A$ a subalgebra of A such that $1_A \in B$.

Remark 1.6.1. We say that $b \in B$ is <u>invertible in B</u> if $\exists b' \in B$ such that $bb' = 1$.

1. $GL(B) \subset GL(A)$
2. For $b \in B$, we write

   $$\sigma_B(b) = \{\lambda \in \mathbb{C} : (b - \lambda 1_A) \text{ is invertible in } B\}$$

   and distinguish it from $\sigma_A(b)$
3. By part (i), it follows that $\sigma_A(b) \subset \sigma_B(b)$

Example 1.6.2. Let $A = C(\mathbb{T})$ and $B \subset A$ be the subalgebra generated by $\zeta \in A$, where $\zeta(z) = z$. Hence,

$$B = \overline{\{p(z) : p \in \mathbb{C}[z]\}}$$

Then

1. By Example 1.3.2(4), $\sigma_A(\zeta) = \zeta(\mathbb{T}) = \mathbb{T}$
2. Claim: $\sigma_B(\zeta) = D := \{z \in \mathbb{C} : |z| \leq 1\}$. By Theorem 1.4.3,

   $$\sigma_B(\zeta) = \{\tau(\zeta) : \tau \in \Omega(B)\}$$

   So we claim: $\Omega(B) = D$.

   a) For each $\lambda \in D$, define $\tau_\lambda(p(z)) = p(\lambda)$. By the Maximum modulus principle,

      $$|p(\lambda)| \leq \sup_{|z|=1} |p(z)| = \|p\|_B$$

      Hence, τ_λ extends to a bounded linear functional on B, and is clearly multiplicative [Check!]

b) Now given $\tau \in \Omega(B)$, let $\lambda = \tau(\zeta)$. Then, $|\lambda| \leq \|\zeta\|_B = 1$. Also, for any $p(z) \in \mathbb{C}[z]$,

$$\tau(p(z)) = p(\tau(\zeta)) = p(\lambda) = \tau_\lambda(p(z))$$

Since $\tau = \tau_\lambda$ on a dense set, it follows that $\tau = \tau_\lambda$ on B.

Hence, $\Omega(B) \cong D$ and so $\sigma_B(\zeta) = \zeta(D) = D$.

(End of Day 9)

Theorem 1.6.3. *Let B be a closed subalgebra of a unital Banach algebra A containing the unit of A. If $b \in B$, then $\partial\sigma_B(b) \subset \sigma_A(b)$*

Proof. Suppose not, then $\exists \lambda \in \partial\sigma_B(b) \setminus \sigma_A(b)$. Hence, $(b - \lambda) \in GL(A)$ and $\exists (\lambda_n) \subset \rho_B(b)$ such that $\lambda_n \to \lambda$. Hence, $(b - \lambda_n) \in GL(B) \subset GL(A)$. But the continuity of the inverse map in $GL(A)$, we have

$$(b - \lambda_n)^{-1} \to (b - \lambda)^{-1} \text{ in } GL(A)$$

But, $(b - \lambda_n)^{-1} \in B$ for all n and so $(b - \lambda)^{-1} \in B$, whence $\lambda \notin \sigma_B(b)$. This is a contradiction. $\square$

Definition 1.6.4. Let $K \subset \mathbb{C}$ be a compact set, then $\mathbb{C} \setminus K$ has exactly one unbounded component, which we denote by X_∞. List the other bounded components $X_1, X_2, \ldots, X_n$, so that

$$\mathbb{C} \setminus K = X_\infty \sqcup X_1 \sqcup X_2 \sqcup \ldots \sqcup X_n$$

Each such $X_i, 1 \leq i \leq n$ is called a <u>hole</u> in K.

Lemma 1.6.5. *Let X be a connected topological space and $K \subset X$ be a closed set such that $\partial K = \emptyset$. Then either $K = X$ or $K = \emptyset$.*

Proof. If $\partial K = \emptyset$, then $X = \text{int}(K) \sqcup X \setminus K$ can be expressed as a union of disjoint open sets. Since X is connected, either $\text{int}(K) = \emptyset$ or $X \setminus K = \emptyset$. If $K \neq X$, it follows that $\text{int}(K) = \emptyset$. But then $K = \text{int}(K) \sqcup \partial K = \emptyset$. $\square$

Corollary 1.6.6. *Let $1_A \in B \subset A$ as above and $b \in B$. If X is a component of $\mathbb{C} \setminus \sigma_A(b)$, then either $X \cap \sigma_B(b) = \emptyset$ or $X \subset \sigma_B(b)$*

Proof. Since $\partial\sigma_B(b) \subset \sigma_A(b)$, it follows that the unbounded component of $\mathbb{C} \setminus \sigma_A(b)$ must intersect $\sigma_B(b)$ trivially. So suppose X is a hole in $\sigma_A(b)$, then let $K = X \cap \sigma_B(b)$ as a closed subspace of X. The boundary $\partial_X(K)$ of K relative to X is

$$\partial_X(K) = \overline{K} \cap \overline{X \setminus K} = K \cap \overline{X \setminus K}$$

Now note that $K \subset \sigma_B(b)$ and

$$X \setminus K = \{x \in X : x \notin \sigma_B(b)\} = X \cap \rho_B(b) \subset \rho_B(b)$$

But Theorem 1.6.3,

$$\partial_X(K) \subset \partial\sigma_B(b) \subset \sigma_A(b) \subset \mathbb{C} \setminus X$$

But $\partial_X(K) \subset X$, so $\partial_X(K) = \emptyset$. The previous lemma now implies that either $K = \emptyset$ or $K = X$ as required. $\square$

Theorem 1.6.7 (Spectral Permanence Theorem)**.** *Let $1_A \in B \subset A$ as above and $b \in B$. Then $\sigma_B(b)$ is obtained from $\sigma_A(b)$ by adjoining to it some (and perhaps none) of its holes.*

For instance, if $\sigma_A(b) = \mathbb{T}$, then $\sigma_B(b)$ must be either $\mathbb{T}$ or $\mathbb{D}$. Compare this with Example 1.6.2.

Corollary 1.6.8. *Let $1_A \in B \subset A$ as above and $b \in B$. If $\sigma_A(b)$ has no holes, then $\sigma_B(b) = \sigma_A(b)$. In particular, if $\sigma_A(b) \subset \mathbb{R}$, then $\sigma_B(b) = \sigma_A(b)$.*

1.7 Exercises

1. Let $X = C[0,1]$ with the supremum norm, and let $T : X \to X$ be given by

$$Tf(x) = \int_0^x f(t)dt$$

   a) Prove that $T \in \mathcal{B}(X)$

   b) Prove that T does not have any eigen-values. (See Example 1.3.2)

2. Let X be a Banach space

   a) If $A, B \subset X$ are two compact sets, then prove that

$$A + B = \{x + y : x \in A, y \in B\}$$

   is compact.

   [*Hint:* The operation $+ : X \times X \to X$ is continuous]

   b) Prove that $\mathcal{K}(X)$ is a subspace of $\mathcal{B}(X)$

   Also read [Conway, Theorem II.4.2]. This proves that $\mathcal{K}(X)$ is a closed ideal in $\mathcal{B}(X)$

3. Let X be a locally compact Hausdorff space. Prove that $C_0(X)$ is a Banach algebra. (See Example 1.1.3)

4. Let $\{A_n\}$ be a sequence of Banach algebras. Define

$$B = \{(a_n) : a_n \in A_n \quad \forall n \text{ and } \sup \|a_n\| < \infty\}$$

   a) Prove that B is an algebra under the operations of component-wise addition, scalar multiplication and multiplication.

b) For any $(a_n) \in B$, define

$$\|(a_n)\| := \sup \|a_n\|$$

and prove that B is a Banach algebra with respect to this norm.

Note: B is called the *direct sum* of the A_n's and is denoted by

$$\bigoplus_{n=1}^{\infty} A_n$$

5. Let $H = \ell^2(\mathbb{N})$ and $T \in \mathcal{B}(H)$ be given by

$$T((x_n)) = (0, x_1, x_2, \ldots)$$

and let $\lambda \in \mathbb{C}$.

a) If $|\lambda| > 1$, then prove that $\lambda \notin \sigma(T)$

b) If $|\lambda| \leq 1$, then prove that $e_1 = (1, 0, 0, \ldots)$ is not in the range of $(T - \lambda)$

[*Hint:* Consider the case where $\lambda = 0$ separately]

Conclude that $\sigma(T) = \{z \in \mathbb{C} : |z| \leq 1\}$

6. Let $A = C^1[0,1]$ be the space of all continuously differentiable functions on $[0,1]$ with the norm

$$\|f\| := \|f\|_\infty + \|f'\|_\infty$$

a) Prove that A is a Banach algebra under this norm.

b) Let $f(x) = x$, then prove that $r(f) = 1$ and $\|f\| = 2$.

7. Let $A = C^1[0,1]$ as above. Let $\zeta : [0,1] \to \mathbb{C}$ be the inclusion.

a) Show that ζ generates A as a Banach algebra (See Definition 1.4.8)

b) For $t \in [0,1]$, define $\tau_t : A \to \mathbb{C}$ by

$$\tau_t(f) := f(t)$$

Show that the map $[0,1] \to \Omega(A)$ given by $t \mapsto \tau_t$ is a homeomorphism.

c) Conclude that the Gelfand representation of Theorem 1.4.7 is not surjective.

8. Let A be the set of all 2×2 complex matrices of the form

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

for some $a, b \in \mathbb{C}$. Think of A as a subset of $\mathcal{B}(\mathbb{C}^2)$, and equip A with the operator norm.

a) Show that A is a unital commutative Banach algebra

b) Determine $\Omega(A)$

c) Show that the Gelfand transform $\Gamma_A : A \to C(\Omega(A))$ is not injective.

The next 3 problems indicate that the theory developed for unital commutative Banach algebras translates to the non-unital case almost verbatim.

9. Let A be a non-unital Banach algebra, and set $\tilde{A} = A \times \mathbb{C}$. Define algebraic operations on $\tilde{A}$ by

   a) $(a, \alpha) + (b, \beta) = (a + b, \alpha + \beta)$

   b) $\beta(a, \alpha) = (\beta a, \beta\alpha)$

   c) $(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)$

   and define

$$\|(a, \alpha)\| := \|a\| + |\alpha|$$

   Then, prove that

   a) $\tilde{A}$ is a unital Banach algebra

   b) The map $a \mapsto (a, 0)$ from A to $\tilde{A}$ is an injective homomorphism.

   $\tilde{A}$ is called the unitization of A.

10. Let A be a commutative non-unital Banach algebra, and let $\Omega(A)$ be defined as in Definition 1.4.1.

    a) Prove that $\Omega(A) \cup \{0\}$ is a compact set in the weak-$*$ topology. Conclude that $\Omega(A)$ is a locally compact, Hausdorff space.

    b) For any $a \in A$, define $\hat{a}$ as in Definition 1.4.6. Prove that $\hat{a} \in C_0(\Omega(A))$ by treating $0 \in A^*$ as the "point at infinity".

11. Let A be a commutative non-unital Banach algebra and $\tilde{A}$ its unitization.

    a) For each $\tau \in \Omega(A) \cup \{0\}$, define $\tilde{\tau} \in \Omega(\tilde{A})$ by $\tilde{\tau}((a, \alpha)) = \tau(a) + \alpha$. Prove that the map

$$\tau \mapsto \tilde{\tau}$$

    defines a bijection from $\Omega(A) \cup \{0\}$ to $\Omega(\tilde{A})$

    b) For each $a \in A$, define $\sigma(a) = \sigma_{\tilde{A}}((a, 0))$. Prove that

$$\sigma(a) = \{\tau(a) : \tau \in \Omega(A)\} \cup \{0\}$$

    Note: For each $a \in A, 0 \in \sigma(a)$. This is one crucial difference between the non-unital and unital cases.

12. Let A be a unital Banach algebra and $a, b \in A$.

a) Prove that the series

$$\sum_{n=0}^{\infty} \frac{a^n}{n!}$$

converges in A. We denote its sum by e^a

b) Prove that $\|e^a\| \leq e^{\|a\|}$

c) If $ab = ba$, then prove that $e^{a+b} = e^a e^b$ [*Hint:* Prove the Binomial theorem in this setting]

13. Let A be a Banach algebra.

   a) Let $\{A_\alpha\}$ be a family of Banach subalgebras of A. Prove that $\bigcap_\alpha A_\alpha$ is a Banach algebra.

   b) Let $S \subset A$ be any set. Prove that $\exists B \subset A$ such that

      i. $S \subset B$

      ii. B is a Banach algebra

      iii. If $C \subset A$ is any Banach algebra such that $S \subset C$, then $B \subset C$.

   B is called the Banach algebra <u>generated by S</u>

14. Let A be a unital Banach algebra and let $B \subset A$ be a maximal commutative subalgebra (ie. B is commutative, and if C is any commutative subalgebra of A such that $B \subset C$, then $B = C$).

   a) Prove that $1_A \in B$

   b) For any $b \in B$, prove that $\sigma_B(a) = \sigma_A(b)$

2 C*-Algebras

2.1 Operators on Hilbert Spaces

Throughout this section, let H and K be complex Hilbert spaces and $\mathcal{B}(H,K)$ be the collection of bounded operators from H to K. We write $\mathcal{B}(H)$ for $\mathcal{B}(H,H)$.

Definition 2.1.1.

1. A function $u : H \times K \to \mathbb{C}$ is called a <u>sesqui-linear form</u> if, for all $x, y, z \in H$ or K and for all $\alpha, \beta \in \mathbb{C}$
   a) $u(\alpha x + \beta y, z) = \alpha u(x,z) + \beta u(y,z)$
   b) $u(x, \alpha y + \beta z) = \overline{\alpha} u(x,y) + \overline{\beta} u(x,z)$
2. A sesqui-linear form $u : H \times K \to \mathbb{C}$ is called <u>bounded</u> if $\exists M \geq 0$ such that $|u(x,y)| \leq M\|x\|\|y\|$ for all $(x,y) \in H \times K$

If $T \in \mathcal{B}(H,K)$, then $u(x,y) := \langle Tx, y\rangle$ is a bounded sesqui-linear form.

Theorem 2.1.2. *If $u : H \times K \to \mathbb{C}$ is a bounded sesqui-linear form with bound M, then $\exists$ unique operators $T \in \mathcal{B}(H,K)$ and $S \in \mathcal{B}(K,H)$ such that*

$$u(x,y) = \langle Tx, y\rangle = \langle x, Sy\rangle$$

Proof. For each $y \in K$, define $L_y : H \to \mathbb{C}$ by $L_y(x) = u(x,y)$. Then L_x is a bounded linear functional on H. By the Riesz representation theorem, $\exists b_y \in H$ such that

$$L_y(x) = \langle x, s_y\rangle$$

Define $S : K \to H$ by $S(y) = s_y$. Then S is linear [Check!]. For any $y \in K$ such that $\|y\| \leq 1$, $\|s_y\| = \|L_y\| \leq M$, then $\|S\| \leq M$. □

(End of Day 10)

Definition 2.1.3. If $T \in \mathcal{B}(H,K)$ the unique operator $S \in \mathcal{B}(K,H)$ such that

$$\langle Tx, y\rangle = \langle x, Sy\rangle$$

is called the <u>adjoint</u> of T and is denoted by T^*

Example 2.1.4.

1. If $H = \mathbb{C}^n$ and $T = (a_{i,j}) \in \mathcal{B}(H)$, then $T^* = (\overline{a_{j,i}})$

2. If $H = L^2[0,1]$ and $k \in L^2([0,1] \times [0,1])$, we define

$$T(f)(x) = \int_0^1 k(x,y)f(y)dy$$

Then $T \in \mathcal{B}(H)$ is called the Volterra integral operator with kernel k and

$$\|T\| \leq \|k\|_2$$

In this case

$$T^*(f)(x) = \int_0^1 \overline{k(y,x)}f(y)dy$$

Proof. For any $f, g \in H$ let $h := T^*(g)$, then we have

$$\int_0^1 \int_0^1 k(x,y)f(y)\overline{g(x)}dydx = \int_0^1 f(x)\overline{h(x)}dx$$

By taking conjugates and using Fubini, we have

$$\int_0^1 \int_0^1 \overline{k(x,y)}g(x)dx\overline{f(y)}dy = \int_0^1 h(y)\overline{f(y)}dy$$

This must be true for any $f \in H$, so

$$T^*(g)(y) = h(y) = \int_0^1 \overline{k(x,y)}g(x)dx$$

$\square$

3. If $H = \ell^2$ and $S \in \mathcal{B}(H)$ is given by

$$S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$$

S is called the right shift operator and

$$S^*(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$$

4. Let $H = L^2[0,1]$ and $f \in C[0,1]$. Define $T_f \in \mathcal{B}(H)$ by

$$T_f(g) := fg$$

Note that $\|T_f\| \leq \|f\|_\infty$ (See Example 1.1.8(4)) and

$$(T_f)^* = T_{\overline{f}}$$

Theorem 2.1.5. *For $T, S \in \mathcal{B}(H)$ and $\alpha, \beta \in \mathbb{C}$*

1. $(\alpha T + S)^* = \overline{\alpha}T^* + S^*$

2. $(TS)^* = S^*T^*$

3. $(T^*)^* = T$

4. *If* $T \in GL(\mathcal{B}(H))$*, then* $T^* \in GL(\mathcal{B}(H))$ *and* $(T^{-1})^* = (T^*)^{-1}$

Proof. Obvious by definition. $\square$

Theorem 2.1.6. *If* $T \in \mathcal{B}(H)$*, then*

$$\|T\| = \|T^*\| = \|T^*T\|^{1/2}$$

Proof. For $x \in H$ with $\|x\| \leq 1$, we have

$$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx\rangle = \langle T^*Th, h\rangle \\ &\leq \|T^*Th\|\|h\| \leq \|T^*T\| \\ &\leq \|T^*\|\|T\| \end{aligned}$$

Taking sup gives $\|T\| \leq \|T^*\|$. The reverse inequality is true since $T^{**} = T$. Hence, $\|T\| = \|T^*\|$. But then the inequalities above show that

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\|$$

which proves the theorem. $\square$

Definition 2.1.7. Let $T \in \mathcal{B}(H)$. We say that T is

1. normal if $TT^* = T^*T$

2. unitary if $TT^* = T^*T = I$

3. self-adjoint if $T = T^*$

4. a projection if $T = T^* = T^2$

   Note: Every projection $T \in \mathcal{B}(H)$ is associated to a unique closed subspace $M = T(H) \subset H$. Conversely, if M is a closed subspace of H, then $H = M \oplus M^{\perp}$, so there is a natural projection $T \in \mathcal{B}(H)$ such that $T(H) = M$.

5. an isometry if $\|Tx\| = \|x\|$ for all $x \in H$

Theorem 2.1.8. $T \in \mathcal{B}(H)$ *is self-adjoint iff* $\langle Tx, x\rangle \in \mathbb{R}$ *for all* $x \in H$

Proof. If T is self-adjoint, then for any $x \in H$, we have

$$\langle Tx, x\rangle = \langle x, T^*x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$$

Conversely, if $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in H$, then

$$\langle Tx, x\rangle = \langle T^*x, x\rangle$$

as above. Consider $S = (T - T^*)$, then $S = S^*$ and

$$\begin{aligned} 0 = \langle S(x+\alpha y), x+\alpha y\rangle &= \langle Sx, x\rangle + \overline{\alpha}\langle Sx, y\rangle + \alpha\langle Sy, x\rangle + |\alpha|^2\langle Sy, y\rangle \\ &= \overline{\alpha}\langle Tx, y\rangle - \overline{\alpha}\langle x, Ty\rangle + \alpha\langle Ty, x\rangle + \alpha\langle y, Tx\rangle \\ \Rightarrow \overline{\alpha}\langle Tx, y\rangle + \alpha\langle Ty, x\rangle &= \overline{\alpha}\langle T^*x, y\rangle - \alpha\langle T^*y, x\rangle \end{aligned}$$

First put $\alpha = 1$ and then $\alpha = i$, to get

$$\begin{aligned} \langle Tx, y\rangle + \langle Ty, x\rangle &= \langle T^*x, y\rangle - \langle T^*y, x\rangle \\ -i\langle Tx, y\rangle + i\langle Ty, x\rangle &= -i\langle T^*x, y\rangle - i\langle T^*y, x\rangle \end{aligned}$$

Multiplying the first equation by i and adding gives that

$$\langle Tx, y\rangle = \langle T^*x, y\rangle$$

which proves that $T = T^*$. ☐

(End of Day 11)

Theorem 2.1.9. *If $T \in \mathcal{B}(H)$ is self-adjoint, then*

$$\|T\| = \sup\{|\langle Tx, x\rangle| : x \in H, \|x\| = 1\}$$

Proof. Let $\beta := \sup\{|\langle Tx, x\rangle| : x \in H, \|x\| = 1\}$, then by Cauchy-Schwartz, $\beta \leq \|T\|$. Conversely, since $T = T^*$, we have that for any $x, y \in H$ with $\|x\| = \|y\| = 1$,

$$\langle T(x \pm y), x \pm y\rangle = \langle Tx, x\rangle \pm 2\text{Re}\langle Tx, y\rangle + \langle Ty, y\rangle$$

Hence,

$$\begin{aligned} 4\text{Re}\langle Tx, y\rangle &= \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle \\ &\leq \beta(\|x+y\|^2 + \|x-y\|^2) \\ &= 2\beta(\|x\|^2 + \|y\|^2) \\ &= 4\beta \end{aligned}$$

Now if $\lambda\langle Tx, y\rangle = |\langle Tx, y\rangle|$ with $|\lambda| = 1$, we may replace x by λx to get the required inequality. ☐

Corollary 2.1.10. *If $T \in \mathcal{B}(H)$ and $\langle Tx, x\rangle = 0$ for all $x \in H$, then $T = 0$*

Proof. Since $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in H$, T is self-adjoint by Theorem 2.1.8. Hence, $T = 0$ by Theorem 2.1.9. ☐

Corollary 2.1.11. *$T \in \mathcal{B}(H)$ is an isometry iff $T^*T = I$*

Proof. T is an isometry iff $\langle Tx, Tx\rangle = \langle x, x\rangle$ for all $x \in H$. This is equivalent to $\langle (T^*T - I)x, x\rangle = 0$, so the theorem now follows from the previous corollary. ☐

Theorem 2.1.12. *$T \in \mathcal{B}(H)$ is normal iff $\|Tx\| = \|T^*x\|$ for all $x \in H$.*

Proof. For all $x \in H$,

$$\begin{aligned} \|Tx\|^2 &= \|T^*x\|^2 \\ \Leftrightarrow \langle T^*Tx, x\rangle &= \langle TT^*x, x\rangle \\ \Leftrightarrow \langle (T^*T - TT^*)x, x\rangle &= 0 \end{aligned}$$

The theorem now follows from Corollary 2.1.10. □

Definition 2.1.13. Let A be a Banach algebra.

1. An involution on A is a map $\delta : A \to A$ such that for all $a, b \in A$ and $\alpha \in \mathbb{C}$,
   a) $\delta(\delta(a)) = a$
   b) $\delta(ab) = \delta(b)\delta(a)$
   c) $\delta(\alpha a + b) = \overline{\alpha}\delta(a) + \delta(b)$
2. We write $a^* := \delta(a)$
3. A is called a C^*-algebra if there is an involution $a \mapsto a^*$ on A such that

$$\|a^*a\| = \|a\|^2 \quad \forall a \in A$$

Remark 2.1.14. 1. By property (a), $a \mapsto a^*$ is bijective

2. If A is unital, then for any $a \in A$,

$$a^* = a^* \cdot 1 = (1^* \cdot a)^* \Rightarrow a = 1^* \cdot a$$

   and similarly, $a = a \cdot 1^*$. By the uniqueness of the identity, $1 = 1^*$

3. If A is unital, then for any $\alpha \in \mathbb{C}$, $\alpha^* := (\alpha \cdot 1)^* = \overline{\alpha}$
4. If A is a Banach algebra and $a \mapsto a^*$ is an involution such that $\|a\|^2 \leq \|a^*a\|$ for all $a \in A$, then A is a C^*-algebra.

   Proof. We need to show that $\|a^*a\| \leq \|a\|^2$. Since A is a Banach algebra, we know that $\|a^*a\| \leq \|a^*\|\|a\|$, so it suffices to prove that $\|a^*\| \leq \|a\|$.
   But since $\|a^*\|^2 \leq \|(a^*)^*a^*\| = \|aa^*\| \leq \|a\|\|a^*\|$, it follows that $\|a^*\| \leq \|a\|$. □

Example 2.1.15. 1. If $A = \mathbb{C}$ with the usual norm. Then $z \mapsto \overline{z}$ is an involution on $\mathbb{C}$ that makes it a C^*-algebra.

2. If H is a Hilbert space, then $\mathcal{B}(H)$ is a C*-algebra by Theorem 2.1.6. In particular, $M_n(\mathbb{C})$ is a C* algebra in which

$$(a_{i,j})^* := (\overline{a_{j,i}})$$

3. Similarly, $\mathcal{K}(H)$ is a C*-algebra [If $T \in \mathcal{K}(H)$, then $T^* \in \mathcal{K}(H)$]. Note that if H is infinite dimensional, then $\mathcal{K}(H)$ is non-unital.

(End of Day 12)

4. If X is a locally compact Hausdorff space, then $C_0(X)$ is a C*-algebra with involution $f^*(x) = \overline{f(x)}$. This is unital iff X is compact.

5. If (X, μ) is a measure space, then $L^\infty(X, \mu)$ is a C* algebra with the same involution as above.

6. Let $A = C^1[0,1]$ be the Banach algebra with norm
$$\|f\| := \|f\|_\infty + \|f'\|_\infty$$
(See section 1.7, §6) The map $f \mapsto f^* := \overline{f}$ is an involution in A. However, if $f(x) = x$, then
$$\|f\|^2 = (1+1)^2 = 4, \text{ while } \|f^*f\| = \|f^2\| = 1+2 = 3$$
and so A is not a C^*-algebra with respect to this involution and norm.

Lemma 2.1.16. *If A is a C^*-algebra, then for any $a \in A$,*

1. $\|a\| = \|a^*\|$

2. $\|aa^*\| = \|a\|^2$

Proof. 1. Note that $\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|$. Hence, $\|a\| \leq \|a^*\|$. The other inequality follows from the fact that $(a^*)^* = a$.

2. Note that $\|aa^*\| = \|(a^*)^*a\| = \|a^*\|^2 = \|a\|^2$ (by part (i)).

□

Definition 2.1.17. Let $T \in \mathcal{B}(H)$, then consider
$$A := \overline{\{p(T,T^*) : p \text{ is a polynomial in two non-commuting variables }\}}$$

1. A is a subalgebra of $\mathcal{B}(H)$. Since A is closed, A is a Banach algebra.

2. If p is a polynomial as above, then $p(T,T^*)^* \in A$ since the latter is also a polynomial in T and T^*. So if $a \in A$, then $\exists p_n$ as above such that $p_n(T,T^*) \to a$. By the previous lemma, $p_n(T,T^*)^* \to a^*$. Hence, $a^* \in A$, and so A is a C* algebra.

3. If $B \subset \mathcal{B}(H)$ is any C*-algebra containing $\{1, T\}$, then $T^* \in B$. Hence, for any polynomial p as above, $p(T,T^*) \in B$, whence $A \subset B$. Hence, A is the smallest C* algebra containing $\{1, T\}$.

Thus, A is called the C*-algebra generated by T and is denoted by $C^*(T)$.

Note: $C^*(T)$ is commutative iff T is normal, and in that case

$$C^*(T) = \overline{\{p(T, T^*) : p \in \mathbb{C}[x, y]\}}$$

Theorem 2.1.18. *If A is a C^* algebra, then for any $a \in A$, we have*

$$\begin{aligned} \|a\| &= \sup\{\|ax\| : x \in A, \|x\| \leq 1\} \\ &= \sup\{\|xa\| : x \in A, \|x\| \leq 1\} \\ &= \sup\{\|x^* a y\| : x, y \in A, \|x\|, \|y\| \leq 1\} \end{aligned}$$

Proof. Assume $a \neq 0$. Since A is a Banach algebra, $\|ax\| \leq \|a\|$ for all $x \in A$ such that $\|x\| \leq 1$. Furthermore, if $x = a^*/\|a\|$, then $\|x\| = 1$ by Lemma 2.1.16, and $\|ax\| = \|a\|$. This proves the first equality. The second is similar and the third follows from the first two. □

Definition 2.1.19.
1. A function $\varphi : A \to B$ between two C^* algebras is called a <u>$*$-homomorphism</u> if φ is a homomorphism of Banach algebras, and $\varphi(a^*) = \varphi(a)^*$
2. A bijective $*$-homomorphism is called an isomorphism of C^*-algebras.

Example 2.1.20.
1. If $A = C(X)$ and $x_0 \in X$, then $\varphi : A \to \mathbb{C}$ given by $f \mapsto f(x_0)$ is a $*$-homomorphism.
2. If $A = C(X)$ and $\{x_1, x_2, \ldots, x_n\} \subset X$ (with possible repeats). Define $\varphi : A \to M_n(\mathbb{C})$ by
$$f \mapsto \begin{pmatrix} f(x_1) & 0 & \ldots & 0 \\ 0 & f(x_2) & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & f(x_n) \end{pmatrix}$$
This is a $*$-homomorphism from A to $M_n(\mathbb{C})$
3. Conversely, if $T = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \in M_n(\mathbb{C})$ be a diagonal matrix. Let $X = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} = \sigma(T)$, and define $\varphi : C(X) \to M_n(\mathbb{C})$ by
$$f \mapsto f(T) := \begin{pmatrix} f(\lambda_1) & 0 & \ldots & 0 \\ 0 & f(\lambda_2) & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & f(\lambda_n) \end{pmatrix}$$
4. If $A = C[0, 1]$ and $B = \mathcal{B}(L^2[0, 1])$, then define $\varphi : A \to B$ by
$$f \mapsto M_f$$
(See Example 2.1.4). Then φ is a $*$-homomorphism.

(End of Day 13)

Definition 2.1.21. Let A be a C^*-algebra, then an ideal of A is an ideal $I \lhd A$ of A such that $a \in I \Rightarrow a^* \in I$.

Remark 2.1.22. If A is a C^*-algebra and $I \lhd A$ is closed. Then

1. There is a well-defined involution on A/I given by
$$(a+I)^* := a^* + I$$
Since A/I is a Banach algebra by Theorem 1.1.6, by Remark 2.1.14, we need to prove that
$$\|a+I\|^2 \leq \|a^*a + I\|$$
We will prove this later.

2. Furthermore, that will show that the natural map $\pi : A \to A/I$ is a $*$-homomorphism (it is already continuous since $\|a+I\| \leq \|a\|$)

3. If $\varphi : A \to B$ is a $*$-homomorphism, then $I = \ker(\varphi)$ is an ideal in A. Hence, by part (i), A/I is a C^*-algebra, and there is an injective homomorphism
$$\overline{\varphi} : A/I \to B \text{ given by } a + I \mapsto \varphi(a)$$
such that $\overline{\varphi} \circ \pi = \varphi$. Note that $\overline{\varphi}$ is a $*$-homomorphism.

Theorem 2.1.23. *Let A be a non-unital C^*-algebra, then $\exists$ a C^* algebra $\tilde{A}$ such that*

1. *$\tilde{A}$ is unital*

2. *There is an isometric $*$-homomorphism $\mu : A \to \tilde{A}$ such that $\mu(A) \lhd \tilde{A}$ and $\tilde{A}/\mu(A)$ is a one-dimensional vector space.*

3. *If B is any unital C^* algebra and $\varphi : A \to B$ a $*$-homomorphism, then $\exists$ a unique $*$-homomorphism $\tilde{\varphi} : \tilde{A} \to B$ such that $\tilde{\varphi}(1_{\tilde{A}}) = 1_B$ and $\tilde{\varphi} \circ \mu = \varphi$.*

4. *If $(\tilde{A}', \mu')$ is a pair satisfying properties (i)-(iii), then there is an isomorphism $\psi : \tilde{A} \to \tilde{A}'$ such that $\psi \circ \mu = \mu'$*

The algebra $\tilde{A}$ is called the <u>unitization</u> of A

Proof. Let $\mathcal{B}(A)$ denote the space of bounded operators on A (treated as Banach space) and let $\mu : A \to \mathcal{B}(A)$ be the left-regular representation (See Example 1.1.8)
$$a \mapsto L_a \text{ where } L_a(b) := ab$$
Let $\tilde{A} := \{L_a + \lambda \cdot 1_{\mathcal{B}(A)} : a \in A, \lambda \in \mathbb{C}\}$, and define an involution on $\tilde{A}$ by
$$(L_a + \lambda \cdot 1)^* := L_{a^*} + \overline{\lambda} \cdot 1$$
Now

1. Note that the map from $A \to \mathcal{B}(A)$ given by

$$a \mapsto L_a$$

is isometric by Theorem 2.1.18. Hence, its image is closed in $\mathcal{B}(A)$. Now it follows from [Conway, § III.4.3] that $\tilde{A}$ is closed. Since it is clearly a linear subspace, and an algebra [Check!], it follows that $\tilde{A}$ is a Banach algebra. It now remains to check that

$$\|X\|^2 \leq \|X^*X\| \quad \forall X \in \tilde{A}$$

If $X = L_a + \lambda 1$, then

$$\begin{aligned}
\|X\|^2 &= \sup_{\|b\|\leq 1} \|(L_a + \lambda 1_A)(b)\|^2 = \sup_{\|b\|\leq 1} \|ab + \lambda b\|^2 \\
&= \sup_{\|b\|\leq 1} \|(ab + \lambda b)^*(ab + \lambda b)\| \\
&= \sup_{\|b\|\leq 1} \|b^*(X^*X(b)\| \\
&\leq \sup_{\|b\|\leq 1} \|X^*X(b)\| \leq \|X^*X\|
\end{aligned}$$

2. By Theorem 2.1.18, $\|L_a\| = \|a\|$ and so μ is an isometry. By Definition, μ is a $*$-homomorphism, and $\mu(A) = \{L_a : a \in A\} \lhd \tilde{A}$ [Check!]

   We now need to prove that $\tilde{A}/\mu(A)$ is one-dimensional : Now, $\tilde{A}/\mu(A)$ has dimension atmost 1. If it had dimension zero, then $\mu(A) = \tilde{A}$, and so $1_{\mathcal{B}(A)} = L_a$ for some $a \in A$. But then, $ab = b$ for all $b \in A$. Taking $*$'s, we see that $b^*a^* = b^*$, and so $ca = c$ for all $c \in A$. Hence, $a = 1_A$ which contradicts the assumption that A is non-unital. Hence, $\tilde{A}/\mu(A)$ is one-dimensional.

3. If $\varphi : A \to B$ is a $*$-homomorphism with B unital, then define $\tilde{\varphi} : \tilde{A} \to B$ by

$$L_a + \lambda 1 \mapsto \varphi(a) + \lambda 1_B$$

   Then $\tilde{\varphi}$ is well-defined (since the map μ is injective) and satisfies all the required conditions.

4. Exercise.

□

If A is already unital, then the map μ constructed above is an isomorphism, so we just write $\tilde{A} = A$ in that case.

2.2 Spectrum of an Element

Remark 2.2.1. Let A be a C^*-algebra, then for $a \in A$, we define $\sigma(a) = \sigma_{\tilde{A}}(a)$ if A is non-unital.

Definition 2.2.2. Let A be a C^* algebra, then $a \in A$ is called

1. <u>normal</u> if $aa^* = a^*a$
2. <u>self-adjoint</u> if $a = a^*$
3. <u>positive</u> if $\exists b \in A$ such that $a = b^*b$

   Note:

   a) Every positive element is self-adjoint.

   b) If $T \in \mathcal{B}(H)$ is a positive operator, then $\langle Tx, x \rangle \geq 0$ for all $x \in H$.
4. If A is unital, then a is <u>unitary</u> if $aa^* = a^*a = 1$
5. a <u>projection</u> if $a = a^* = a^2$

(End of Day 14)

Remark 2.2.3. Let A be a C^*-algebra and $a \in A$, then $\exists$ unique b, c self-adjoint such that $a = b + ic$

Proof. Let $b = (a + a^*)/2, c = i(a^* - a)/2$, then $a = b + ic$. Suppose $a = b' + ic'$, then $b' - b = i(c' - c)$. Take $*$'s to note that $b' - b = -i(c' - c)$, and so $b' - b = c' - c' = 0$. $\square$

Theorem 2.2.4. *Let $\tau : A \to \mathbb{C}$ be a non-zero homomorphism, then*

1. *If $a = a^*$, then $\tau(a) \in \mathbb{R}$*
2. *$\tau(a^*) = \overline{\tau(a)}$ for all $a \in A$*
3. *If $a \in A$ is positive, then $\tau(a) \geq 0$*
4. *If A is unital and $u \in A$ is unitary, then $|\tau(u)| = 1$*
5. *If $p \in A$ is a projection, then $\tau(p) \in \{0, 1\}$.*

Proof. If A is unital, then $\tau(1) = 1$ by Lemma 1.4.2. If A is non-unital, then we may extend τ to a map $\tilde{\tau} : \tilde{A} \to \mathbb{C}$ such that $\tilde{\tau}(1) = 1$. Therefore, we assume WLOG that A is unital and that $\tau(1) = 1$.

1. By Lemma 1.4.2, $\|\tau\| = 1$. Hence, if $t \in \mathbb{R}$, then

   $$|\tau(a+it)|^2 \leq \|a+it\|^2 = \|(a+it)^*(a+it)\| = \|(a-it)(a+it)\| = \|a^2+t^2\| \leq \|a^2\|+t^2$$

   So if $\tau(a) = \alpha + i\beta$, then

   $$|\alpha|^2 + (\beta + t)^2 \leq \|a^2\| + t^2 \Rightarrow |\alpha|^2 + 2t\beta \leq \|a^2\|$$

   If $\beta \neq 0$, then let $t \to \pm\infty$ to obtain a contradiction. Hence, $\beta = 0$ and so $\tau(a) \in \mathbb{R}$

2. If $a \in A$, then write $a = b + ic$, where b, c are self-adjoint as in Remark 2.2.3. Then $\tau(b), \tau(c) \in \mathbb{R}$ by part (i) and $a^* = b - ic$. Hence, $\tau(a) = \tau(b) + i\tau(c)$ and $\tau(a^*) = \tau(b) - i\tau(c) = \overline{\tau(a)}$

3. If $b \in A$, then $\tau(b^*b) = \tau(b^*)\tau(b) = \overline{\tau(b)}\tau(b) = |\tau(b)|^2 \geq 0$.

4. $1 = \tau(1) = \tau(u^*u) = \tau(u^*)\tau(u) = \overline{\tau(u)}\tau(u)$

5. $\tau(p) = \overline{\tau(p)} = \tau(p)^2$. The only two numbers in $\mathbb{C}$ that satisfy these properties are $\{0, 1\}$

□

Remark 2.2.5. Let A be a C^*-algebra and $a \in A$, then (as in Definition 2.1.17), we consider

$$B := \overline{\{p(a, a^*) : p \text{ is a polynomial in two non-commuting variables}\}}$$

Then, as in Definition 2.1.17, B is a C^*-algebra, which we call the <u>C^*-algebra generated by a</u>, and is denoted by $C^*(a)$.

Note: If C is the Banach algebra generated by a, then $C \subset B$. However, $C \neq B$ in general.

Theorem 2.2.6 (Spectral Permanence Theorem)**.** *Let $B \subset A$ be a subalgebra such that $1_A \in B$. For any $b \in B$, $\sigma_B(b) = \sigma_A(b)$*

Proof. 1. Suppose $b \in B$ is self-adjoint, consider $C = C^*(b)$ to be the C^*-algebra generated by $\{1, b\}$. Then C is commutative since $b = b^*$. Hence,

   $$\sigma_C(b) = \{\tau(b) : \tau \in \Omega(C)\}$$

   by Theorem 1.4.3. By Theorem 2.2.4, $\sigma_C(b) \subset \mathbb{R}$. In particular, $\sigma_C(b) = \partial\sigma_C(b)$. Hence by Remark 1.6.1 and Theorem 1.6.3, we have

   $$\sigma_A(b) \subset \sigma_C(b) = \partial\sigma_C(b) \subset \sigma_A(b) \Rightarrow \sigma_A(b) = \sigma_C(b)$$

   Similarly, $\sigma_B(b) = \sigma_C(b)$.

2. Now suppose b is not self-adjoint. By Remark 1.6.1, we need to show that $\sigma_B(b) \subset \sigma_A(b)$. Let $\lambda \in \sigma_B(b)$ and let $c := b - \lambda 1$. If c is invertible in A, then $\exists d \in A$ such that $cd = 1 = dc$. Hence, $c^*d^* = d^*c^* = 1$. Hence,

$$(d^*d)(cc^*) = (cc^*)(d^*d) = 1$$

So (cc^*) is invertible in A. Since cc^* is self-adjoint, it follows from the first part that cc^* is invertible in B. Hence, $\exists c' \in B$ such that $cc^*c' = 1$. Hence, c is right-invertible in B. Similarly, c is left-invertible in B. Hence, c is invertible [Why?], and $\lambda \notin \sigma_B(b)$. This is a contradiction.

□

Corollary 2.2.7. *Let A be a C^* algebra and $a \in A$*

1. *If $a = a^*$, then $\sigma(a) \subset \mathbb{R}$*
2. *If a is unitary, then $\sigma(a) \subset \mathbb{T}$*
3. *If a is a projection, then $\sigma(a) \subset \{0, 1\}$*

Proof. In all cases, let $B := C^*(a)$ (which is commutative). By Theorem 1.4.3,

$$\sigma_B(a) = \{\tau(a) : \tau \in \Omega(B)\}$$

But by Spectral Permanence, $\sigma_A(a) = \sigma_B(a)$. Now apply Theorem 2.2.4. □

Remark 2.2.8. It is also true that if a is positive (as in Definition 2.2.2), then $\sigma(a) \subset [0, \infty)$. However, the proof is much harder as we do not know that the element $b \in A$ (which satisfies $b^*b = a$) is an element of $C^*(a)$, and so we cannot apply Theorem 1.4.3 directly.

Lemma 2.2.9. *If $a \in A$ is self-adjoint, then $r(a) = \|a\|$*

Proof. Since $a = a^*$, $\|a\|^2 = \|aa^*\| = \|a^2\|$. Now note that $a^2 = (a^2)^*$, so

$$\|a^4\| = \|(a^2)^*(a^2)\| = \|a^2\|^2 = \|a\|^4$$

So by induction, $\|a^{2^n}\| = \|a\|^{2^n}$ for all $n \in \mathbb{N}$. So by Theorem 1.3.12,

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \|a\|$$

□

(End of Day 15)

Theorem 2.2.10. *There is atmost one norm on an involutive algebra making it a C^* algebra.*

Proof. If $\|\cdot\|$ and $\|\cdot\|'$ are two norms under which A is a C^*-algebra, then for any $a \in A$, we have
$$\|a\|^2 = \|a^*a\| = r(a^*a) = \|a^*a\|_2' = \|a\|'^2$$
□

Theorem 2.2.11. *Let $\varphi : A \to B$ be a $*$-homomorphism. Then*

1. *If $\varphi(1_A) = 1_B$, then $\sigma_B(\varphi(a)) \subset \sigma_A(a)$ for all $a \in A$*
2. *$\|\varphi(a)\| \leq \|a\|$ for all $a \in A$*
3. *If φ is injective, then $\|\varphi(a)\| = \|a\|$ for all $a \in A$.*

Proof.
1. If $\lambda \notin \sigma_A(a)$, then $\exists b \in A$ such that $(a - \lambda 1_A)b = b(a - \lambda 1_A) = 1_A$. Apply φ to this expression to see that $\lambda \notin \sigma_B(\varphi(a))$.
2. If A is non-unital, extend φ to a map $\tilde{\varphi} : \tilde{A} \to \tilde{B}$ such that $\tilde{\varphi}(1_A) = 1_B$. If A is unital, then set $C := \overline{\text{Image}(\varphi)}$. Note that $\varphi(1_A)$ is the unit in C. Hence, if $a \in A$, then set $b := a^*a$ (so that b is self-adjoint), and note that
$$\begin{aligned} &\sigma_C(\varphi(b)) \subset \sigma_A(b) \Rightarrow r_C(\varphi(b)) \leq r_A(b) \\ &\Rightarrow \|\varphi(b)\| \leq \|b\| \text{ (by Lemma 2.2.9)} \\ &\Rightarrow \|\varphi(a)\|^2 = \|\varphi(a^*a)\| \leq \|a^*a\| = \|a\|^2 \end{aligned}$$
3. Suppose φ is injective, define a new norm on A by
$$\|a\|' := \|\varphi(a)\|$$
Then $\|\cdot\|'$ satisfies all the requirements to make $(A, \|\cdot\|')$ a C^*-algebra [Check!]. By uniqueness of the norm, we have
$$\|\varphi(a)\| = \|a\|' = \|a\| \quad \forall a \in A$$

□

2.3 Unital Commutative C^* algebras

Lemma 2.3.1. *Let $\mathcal{A} \subset C(X, \mathbb{R})$ be an closed subalgebra of real continuous functions such that*

1. *$\mathcal{A}$ contains the constant functions*
2. *For all $x, y \in X, x \neq y, \exists f \in \mathcal{A}$ such that $f(x) \neq f(y)$.*

   Note: If this happens, we say that $\mathcal{A}$ separates points of X

Then, for any $f, g \in \mathcal{A}$

1. $|f| \in \mathcal{A}$

2. $\max\{f, g\}, \min\{f, g\} \in \mathcal{A}$

Proof. Since, for any f and g in $C(X)$, we have

$$\max\{f,g\} = \frac{1}{2}[f + g + |f - g|]$$
$$\min\{f,g\} = \frac{1}{2}[f + g - |f - g|]$$

it suffices to prove part (i).
Let $f \in \mathcal{A}$, then there is $m > 0$ such that $|f(x)| \leq m$ for each $x \in X$. Then defining $g(x) := \frac{|f(x)|}{m}$ for each $x \in X$ we see that $g(x) \in [0, 1]$ for each $x \in X$. Since $\mathcal{A}$ is a subspace of $C(X)$ it is enough to prove that $g \in \mathcal{A}$. By the Weierstrass approximation theorem, there is a sequence p_n of polynomials such that $p_n \to \sqrt{\cdot}$ uniformly on $[0, 1]$. Hence,

$$p_n\left(\frac{f^2}{m^2}\right) \longrightarrow \sqrt{\frac{f^2}{m^2}} = g$$

uniformly on $[0, 1]$. Since $\mathcal{A}$ is an algebra containing the constants,

$$p_n\left(\frac{f^2}{m^2}\right) \in \mathcal{A} \text{ for each } n \in \mathbb{N}$$

Since $\mathcal{A}$ is closed, g is in $\mathcal{A}$ as required. □

Lemma 2.3.2. *Let $\mathcal{A}$ and X satisfy the hypotheses of Lemma 2.3.1, then for any pair of real numbers α, β and any pair of distinct points $x, y \in X$, there is a function $g \in \mathcal{A}$ such that $g(x) = \alpha$ and $g(y) = \beta$*

Proof. Since $x \neq y$ we can choose and $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. Then the function g defined by

$$g(u) = \frac{\alpha(f(u) - f(y)) - \beta(f(x) - f(u))}{f(x) - f(y)}$$

is an element of $\mathcal{A}$ since $\mathcal{A}$ is an algebra, and it satisfies the required properties. □

Theorem 2.3.3 (Stone-Weierstrass)**.** *Let $\mathcal{A} \subset C(X, \mathbb{R})$ be an closed subalgebra of real continuous functions such that*

1. $\mathcal{A}$ contains the constant functions

2. $\mathcal{A}$ separates points of X

Then $\mathcal{A} = C(X, \mathbb{R})$

Proof. Let $f \in C(X)$, and $\epsilon > 0$ be given. For any $\tau, \sigma \in X$, by Lemma 2.3.2, there is a function $f_{\tau\sigma} \in \mathcal{A}$ such that $f_{\tau\sigma}(\tau) = f(\tau)$ and $f_{\tau\sigma}(\sigma) = f(\sigma)$. Define

$$U_{\tau\sigma} := \{t \in X : f_{\tau\sigma}(t) < f(t) + \epsilon\}$$
$$V_{\tau\sigma} := \{t \in X : f_{\tau\sigma}(t) > f(t) - \epsilon\}$$

Then $U_{\tau\sigma}$ and $V_{\tau\sigma}$ are open sets containing τ and σ respectively. By the compactness of X, there is a finite set $\{t_1, t_2, \ldots, t_n\}$ such that $\{U_{t_i\sigma}\}_{i=1}^n$ covers X. Let $f_\sigma := \min\{f_{t_i\sigma} : 1 \leq i \leq n\}$ then f_σ is an element of $\mathcal{A}$ (by Lemma 2.3.1) and satisfies

$$f_\sigma(t) < f(t) + \epsilon \quad \forall \quad t \in X$$
$$f_\sigma(t) > f(t) - \epsilon \quad \forall \quad t \in V_\sigma := \bigcap_{i=1}^n V_{t_i\sigma}$$

We now select a finite subcover $\{V_{\sigma_j}\}_{j=1}^m$ from $\{V_\sigma\}$ for X and define $g := \max\{f_{\sigma_j} : 1 \leq j \leq m\}$. Then g is in $\mathcal{A}$ by Lemma 2.3.1, and it satisfies

$$f(t) - \epsilon < g(t) < f(t) + \epsilon \qquad\qquad (t \in X)$$

Hence, to every $\epsilon > 0$ there is an element $g \in \mathcal{A}$ such that $\|f - g\|_\infty < \epsilon$. Since $\mathcal{A}$ is closed, we see that f is in $\mathcal{A}$. This is true for every f in $C(X)$ and hence the theorem is proved. $\square$

(End of Day 16)

Theorem 2.3.4 (Stone Weierstrass)**.** *Let $\mathcal{A} \subset C(X)$ be a closed subalgebra of the space of complex-valued continuous functions on a compact Hausdorff space X. Suppose that*

1. *$\mathcal{A}$ contains the constant functions*
2. *$\mathcal{A}$ separates points of X*
3. *If $f \in \mathcal{A}$, then $f^* \in \mathcal{A}$*

Then $\mathcal{A} = C(X)$

Proof. Let $\mathcal{B} := \{\mathrm{Re}(f) : f \in \mathcal{A}\} \subset C(X, \mathbb{R})$. Then $\mathcal{B}$ satisfies all the hypotheses of Theorem 2.3.3. If $f \in C(X)$, then write $f = g + ih$, where g, h are real-valued. By Theorem 2.3.3, $g, h \in \mathcal{B}$, and so $f \in \mathcal{A}$. $\square$

Theorem 2.3.5 (Gelfand-Naimark)**.** *Let A be a unital commutative C^* algebra, and let $\Omega(A)$ denote its Gelfand spectrum. Then the Gelfand transform*

$$\Gamma_A : A \to C(\Omega(A))$$

is an isometric isomorphism of C^* algebras.

Proof. Let $\mathcal{A} := R(\Gamma_A)$, then

1. Γ_A is isometric: Suppose $a \in A$, we want to show that $\|a\| = \|\hat{a}\|_\infty = r(a)$. As in Lemma 2.2.9, it suffices to prove that $\|a^{2^n}\| = \|a\|^{2^n}$ for all $n \in \mathbb{N}$. Since A is commutative,

$$\|a^2\| = \|(a^2)^* a^2\|^{1/2} = \|(a^*a)(a^*a)\|^{1/2} = (\|a^*a\|^2)^{1/2} = (\|a\|^4)^{1/2} = \|a\|^2$$

   By induction, we may show that $\|a^{2^n}\| = \|a\|^{2^n}$ for all $n \in \mathbb{N}$, so Γ_A is injective.

2. Γ_A is surjective:
   a) $\mathcal{A}$ is closed since A is complete and Γ_A is isometric.
   b) Since A is unital, $\mathcal{A}$ contains $1_{C(X)}$. Hence, $\mathcal{A}$ contains the constant functions.
   c) If $\tau, \mu \in \Omega(A)$ are two different element, then $\exists a \in A$ such that $\tau(a) \neq \mu(a)$. This is equivalent to the fact that $\hat{a}(\tau) \neq \hat{a}(\mu)$. Hence, $\mathcal{A}$ separates points of X
   d) Suppose $\hat{a} \in \mathcal{A}$, then $\hat{a}^* = \hat{a^*} \in \mathcal{A}$.

   So $\mathcal{A}$ satisfies all the hypotheses of the Stone-Weierstrass theorem. Hence, Γ_A is surjective.

□

Theorem 2.3.6. *Let A be a unital C^* algebra and $a \in A$ be such that $A = C^*(a)$. Then the map*

$$\hat{a} : \Omega(A) \to \sigma(a) \text{ given by } \tau \mapsto \tau(a)$$

is a homeomorphism.

Proof. Note that $\hat{a}$ is clearly continuous.

1. $\hat{a}$ is injective: If $\hat{a}(\tau) = \hat{a}(\mu)$, then $\tau(a) = \mu(a)$. By Theorem 2.2.4(2), this implies that $\tau(a^*) = \mu(a^*)$. Since $\tau(1) = \mu(1) = 1$, it follows that $\tau(p(a, a^*)) = \mu(p(a, a^*))$ for any polynomial p in two non-commuting variables. Hence, $\tau = \mu$ on A.

2. $\hat{a}$ is surjective: Follows from Theorem 1.4.9.

Since $\Omega(A)$ and $\sigma(a)$ are compact, $\hat{a}$ is a homeomorphism. □

Note: This is different from Theorem 1.4.9 since the Banach algebra generated by a may be strictly smaller than $C^*(a)$.

Remark 2.3.7. Let $a \in A$ be as in Theorem 2.3.6, then there is an isomorphism

$$\mu : C(\sigma(a)) \to C(\Omega(A))$$

given by $f \mapsto f \circ \hat{a}$

(End of Day 17)

Theorem 2.3.8. *Let A be a C^*-algebra and $a \in A$ be normal. Then there is an isometric $*$-isomorphism*

$$\Theta : C(\sigma(a)) \to C^*(a)$$

such that

$$\Theta(p(z, \overline{z})) = p(a, a^*)$$

for any polynomial $p \in \mathbb{C}[x, y]$. This map Θ is called the continuous functional calculus and we write

$$f(a) := \Theta(f)$$

for any $f \in C(\sigma(a))$.

Proof. By Remark 2.3.7, there is a $*$-isomorphism $\mu : C(\sigma(a)) \to C(\Omega(A))$. Furthermore, if $p(z) = z$, then

$$\mu(p)(\tau) = p \circ \hat{a}(\tau) = p(\tau(a)) = \tau(a) = \hat{a}(\tau)$$

Hence, $\mu(p) = \hat{a}$. Now, by the Gelfand-Naimark theorem, we have a $*$-isomorphism

$$\Gamma_A : C^*(a) \to C(\Omega(A)) \text{ given by } a \mapsto \hat{a}$$

Note that $\Gamma_A^{-1}(\hat{a}) = a$, so the map

$$\Theta : C(\sigma(a)) \to C^*(a) \text{ given by } \Theta = \Gamma_A^{-1} \circ \mu$$

is a $*$-isomorphism such that

$$\Theta(p) = a$$

Similarly, if $q(z) = \overline{z}$, then $\Theta(q) = a^*$. Hence, for any polynomial $p \in \mathbb{C}[x, y]$, we have

$$\Theta(p(z, \overline{z})) = p(a, a^*)$$

$\square$

Theorem 2.3.9 (Spectral Mapping Theorem)**.** *Let A be a C^*-algebra and $a \in A$ be a normal element. Then for any $f \in C(\sigma(a))$,*

$$\sigma(f(a)) = f(\sigma(a))$$

Proof. Note that $f \mapsto f(a)$ is an isometric $*$-isomorphism from $C := C(\sigma(a))$ to $B := C^*(a)$. Hence,

$$\sigma_B(f(a)) = \sigma_C(f)$$

By the Spectral Permanence theorem,

$$\sigma_B(f(a)) = \sigma_A(f(a))$$

By Example 1.3.2, $\sigma_C(f) = f(\sigma(a))$. $\square$

Corollary 2.3.10. *Let $a \in A$ be a normal element, then $\|a\| = r(a)$*

Compare this with Lemma 2.2.9

Proof. Let $f \in C(\sigma(a))$ be the function $f(z) = z$, then $\|f\|_\infty = r(a)$. But $f(a) = a$, so $\|a\| = \|f\|_\infty$ since the continuous functional calculus is isometric. $\square$

Theorem 2.3.11. *Let A be a unital C^* algebra and $a \in A$ be a normal element.*

1. *If $\sigma(a) \subset \mathbb{R}$, then $a = a^*$*
2. *If $\sigma(a) \subset [0, \infty)$, then a is positive*
3. *If $\sigma(a) \subset \mathbb{T}$, then a is unitary*
4. *If $\sigma(a) \subset \{0, 1\}$, then a is a projection*

Compare this with Corollary 2.2.7

Proof. Let $f \mapsto f(a)$ denote the functional calculus from $C(\sigma(a)) \to C^*(a) \subset A$. In particular, if $p(z) = z$, then

$$a = p(a) \text{ and } a^* = p^*(a) = \overline{p}(a)$$

1. If $\sigma(a) \subset \mathbb{R}$, then $p = \overline{p}$ in $C(\sigma(a))$, so $a = a^*$
2. Let $f(t) = t^{1/2}$, then $b := f(a)$ is normal and $\sigma(b) = f(\sigma(a)) \subset \mathbb{R}$, so b is self-adjoint. Now, $b^*b = b^2 = a$, so a is positive
3. Note that $p\overline{p} = \overline{p}p = 1$ on $C(\sigma(a))$, so $aa^* = a^*a = 1_A$
4. Again, $p = p^2 = p^*$, so $a = a^2 = a^*$ is a projection.

$\square$

2.4 Spectrum of a Normal Operator

The goal of this section is to understand the spectrum of a normal operator, and understand what it can say about the operator in light of the continuous functional calculus. We begin by analyzing the spectrum of any bounded operator in $\mathcal{B}(H)$. For $T \in \mathcal{B}(H)$, we write $\ker(T)$ and $R(T)$ to denote the kernel and range of T respectively.

Definition 2.4.1. We say that an operator $T \in \mathcal{B}(H)$ is <u>bounded below</u> if $\exists c > 0$ such that $\|T(x)\| \geq c\|x\|$ for all $x \in H$

Lemma 2.4.2. *Let $T \in \mathcal{B}(H)$ be bounded below, then*

1. *T is injective*
2. *$R(T)$ is closed in H*

Proof. 1. This is trivial from the definition.

2. If $(y_n) \subset R(T)$ such that $y_n \to y$, then write $y_n = T(x_n)$. Since (y_n) is Cauchy and

$$\|y_n - y_m\| \geq c\|x_n - x_m\|$$

implies that (x_n) is Cauchy. Since H is complete, $\exists x \in H$ such that $x_n \to x$. Since $T \in \mathcal{B}(H)$, $T(x_n) \to T(x)$, and so $y = T(x) \in R(T)$ as required.

$\square$

Theorem 2.4.3. *Let $T \in \mathcal{B}(H)$, then TFAE:*

1. *T is bounded below*
2. *T is left-invertible in $\mathcal{B}(H)$ (ie. $\exists S \in \mathcal{B}(H)$ such that $ST = I$)*

Proof.

1. If T is left-invertible with left-inverse $S \in \mathcal{B}(H)$, then for all $x \in H$

$$\|x\| = \|ST(x)\| \leq \|S\|\|T(x)\|$$

so $c := \|S\|^{-1}$ works.

2. Conversely, if T is bounded below by a constant $c > 0$, then T is injective, and $R(T)$ is closed. So let $M < H$ such that $H = R(T) \oplus M$. Then define $S : H \to H$ by

$$S(T(x), m) := x$$

One can check that this map is well-defined and it is bounded since

$$\|x\|^2 \leq c^{-2}\|T(x)\|^2 \leq c^{-2}\|T(x)\|^2 + c^{-2}\|m\|^2 = c^{-2}\|(T(x), m)\|^2$$

Hence, $S \in \mathcal{B}(H)$ and clearly, $ST = I$ holds.

$\square$

Theorem 2.4.4. *Let $T \in \mathcal{B}(H)$, then T is invertible if and only if T is bounded below and $R(T)$ is dense in H.*

Proof. If T is invertible, then $c = \|T^{-1}\|^{-1}$ works, so T is bounded below. Furthermore, the range $R(T)$ is H, so it is, in particular, dense in H.

Conversely, if T is bounded below and $R(T)$ is dense, then T is injective, and $R(T) = H$ because it is closed. Hence, T is surjective. By the bounded inverse theorem, T is invertible. $\square$

Definition 2.4.5. Let $T \in \mathcal{B}(H)$.

1. The point spectrum of T, denoted by $\sigma_p(T)$, is the set of all eigen-values of T.
2. The approximate spectrum of T is the set

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : (T - \lambda) \text{ is not bounded below}\}$$

Note that
$$\sigma_p(T) \subset \sigma_{ap}(T) \subset \sigma(T)$$
The following example shows that these inclusions may be strict. Before we do that, we show that $\sigma_{ap}(T)$ is always non-empty.

Theorem 2.4.6. *For any $T \in \mathcal{B}(H)$, $\partial\sigma(T) \subset \sigma_{ap}(T)$. In particular, $\sigma_{ap}(T) \neq \emptyset$*

Proof. Suppose $\lambda \in \partial\sigma(T) \setminus \sigma_{ap}(T)$, then $\exists \lambda_n \in \rho(T)$ such that $\lambda_n \to \lambda$, and $(T-\lambda)$ is bounded below, say by $c > 0$. Since $\lambda \in \sigma(T)$, $(T-\lambda)$ is not invertible. Hence, it must happen that $R(T-\lambda)$ is not dense in H. Equivalently, $\exists x \in R(T-\lambda)^{\perp}$ which is non-zero. Now define
$$x_n = \frac{(T-\lambda_n)^{-1}(x)}{\|(T-\lambda_n)^{-1}(x)\|}$$
Then $(T-\lambda_n)x_n$ is a scalar multiple of x, and so
$$(T-\lambda_n)(x_n) \perp (T-\lambda)(x_n)$$
Hence, by Pythagoras' theorem,
$$\begin{aligned}\|(T-\lambda)(x_n)\|^2 &\leq \|(T-\lambda)(x_n)\|^2 + \|(T-\lambda_n)(x_n)\|^2 \\ &= \|(\lambda-\lambda_n)(x_n)\|^2 \\ &= |\lambda-\lambda_n|^2 \to 0\end{aligned}$$
This contradicts the fact that $(T-\lambda)$ is bounded below. $\square$

(End of Day 18)

Example 2.4.7. Let $S : \ell^2 \to \ell^2$ be the right-shift operator
$$S((x_n)) = (0, x_1, x_2, \ldots)$$
We wish to determine $\sigma(S), \sigma_{ap}(S)$ and $\sigma_p(S)$. Note that S^* is the left-shift operator
$$S^*((x_n)) = (x_2, x_3, \ldots)$$

1. If $|\lambda| < 1$, then we claim that $\lambda \in \sigma(S)$. To see this, note that S is not surjective, so $0 \in \sigma(S)$. So it suffices to consider the case where $\lambda \neq 0$. Then,
$$\lambda \in \sigma(S) \Leftrightarrow \overline{\lambda} \in \sigma(S^*)$$
But if $z = \overline{\lambda}$ then $|z| < 1$, so if $x = (z, z^2, z^3, \ldots)$, then $z \in \ell^2$, and
$$S^*(x) = (z^2, z^3, \ldots) = zx$$
and so z is an eigen-value of S^*, whence $z \in \sigma(S^*)$. Hence,
$$\lambda \in \sigma(S)$$

2. In fact, this shows that if $D = \{z \in \mathbb{C} : |z| < 1\}$, then

$$D \subset \sigma(S^*) \Rightarrow D \subset \sigma(S)$$

However, $\sigma(S)$ is closed, and $\|S\| = 1$, so by Theorem 1.3.3

$$\sigma(S) = \overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$$

3. Now if $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, then

$$\|(S - \lambda)x\| \geq |\|Sx\| - \|\lambda x\| = (1 - \lambda)\|x\|$$

so $(S - \lambda)$ is bounded below, whence $\lambda \notin \sigma_{ap}(S)$. By the previous theorem, it follows that

$$\sigma_{ap}(S) = \{z \in \mathbb{C} : |z| = 1\}$$

4. Finally, $\sigma_p(S) = \emptyset$ (HW). Hence,

$$\begin{aligned} \sigma(S) &= \{z \in \mathbb{C} : |z| \leq 1\} \\ \sigma_{ap}(S) &= \{z \in \mathbb{C} : |z| = 1\} \\ \sigma_p(S) &= \emptyset \end{aligned}$$

We now examine the case of a normal operator. But before that, we need a rather useful lemma.

Lemma 2.4.8. *For any $A \in \mathcal{B}(H)$,*

1. $\ker(A) = R(A^*)^\perp$
2. $\overline{R(A^*)} = \ker(A)^\perp$
3. $\overline{R(A^*A)} = \ker(A)^\perp$

Proof. 1. For any $x, y \in H$, we have

$$\begin{aligned} & x \in \ker(A) \\ \Leftrightarrow\ & \langle Ax, y \rangle = 0 \quad \forall y \in H \\ \Leftrightarrow\ & \langle x, A^*y \rangle = 0 \quad \forall y \in H \\ \Leftrightarrow\ & x \in R(A^*)^\perp \end{aligned}$$

2. Follows from part (i) and the fact that for any subspace $W \subset H$

$$\overline{W} = (W^\perp)^\perp$$

3. By part (ii), $\overline{R(A^*A)} \subset \overline{R(A^*)} = \ker(A)^\perp$, so it suffices to prove that

$$R(A^*) \subset \overline{R(A^*A)}$$

Let $y \in \operatorname{ran}(A^*)$ and write $y = A^*(x)$ for some $x \in H$. Express

$$x = u + v \text{ where } u \in \ker(A^*), v \in \ker(A^*)^\perp$$

Then $y = A^*(v)$. Now by part (ii) applied to A^*, $\exists w \in \operatorname{ran}(A)$ such that

$$\|v - w\| < \epsilon$$

Write $w = Au$ for some $u \in H$. Then

$$\|y - A^*Au\| = \|A^*v - A^*w\| \le \epsilon\|A\|$$

□

Theorem 2.4.9. *If $T \in \mathcal{B}(H)$ is a normal operator, then $\sigma(T) = \sigma_{ap}(T)$*

Proof. Since one inclusion is trivial, we show that $\sigma(T) \subset \sigma_{ap}(T)$. So fix $\lambda \notin \sigma_{ap}(T)$, then we wish to show that $\lambda \notin \sigma(T)$. Since $\lambda \notin \sigma_{ap}(T)$, $(T - \lambda)$ is bounded below. By Theorem 2.4.4, it now suffices to show that $R(T - \lambda)$ is dense in H. Equivalently by Lemma 2.4.8, we wish to show that

$$R(T - \lambda)^\perp = \ker((T - \lambda)^*) = \{0\}$$

But since $(T - \lambda)$ is normal, by Theorem 2.1.12,

$$\|(T - \lambda)(x)\| = \|(T - \lambda)^*(x)\| \quad \forall x \in H$$

Since $(T - \lambda)$ is bounded below, it follows that $(T - \lambda)^*$ is also bounded below, and hence injective. This completes the proof. □

Theorem 2.4.10. *Let $T \in \mathcal{B}(H)$ be a normal operator. If $\lambda \in \sigma(T)$ is an isolated point of $\sigma(T)$, then λ is an eigen-value of T.*

Proof. Since λ is an isolated point, let $f = \chi_{\{\lambda\}} \in C(\sigma(T))$ and $P = f(T)$. Since $f = \overline{f} = f^2$, it follows that P is an orthogonal projection and $P \neq 0$ since $f \neq 0$. Furthermore,

$$(z - \lambda)f(z) = 0 \quad \forall z \in \sigma(T)$$

and so $(T - \lambda)P = 0$. Hence, any non-zero vector in $P(H)$ is an eigen-vector associated to λ. □

Definition 2.4.11. Let $T \in \mathcal{B}(H)$ and $M \subset H$ a closed subspace of H

1. M is said to be <u>invariant</u> under T if $T(M) \subset M$
2. M is said to be <u>reducing</u> for T if M is invariant under T and T^*

For a general $T \in \mathcal{B}(H)$, the existence of a non-trivial invariant subspace is an open problem. However, for normal operators, the problem is more tractable because of the functional calculus. We give one such example.

Theorem 2.4.12. *If $T \in \mathcal{B}(H)$ is a normal operator such that $\sigma(T)$ is disconnected, then T has a non-trivial invariant subspace.*

Proof. HW. $\square$

(End of Day 19)

2.5 Positive Operators and Polar Decomposition

Recall that a complex number $z \in \mathbb{C}$ can be expressed in the form $z = r\omega$ where $r \in \mathbb{R}_+$ is a positive real number and $\omega \in S^1$. We now prove the existence of a polar decomposition of an operator in $\mathcal{B}(H)$, where the role of r is played by a positive operator, and $e^{i\theta}$ by a partial isometry (both of which are defined below).

Throughout this section, for an operator $T \in \mathcal{B}(H)$, we write $\ker(T)$ and $R(T)$ for its kernel and range respectively.

Lemma 2.5.1. *An operator $T \in \mathcal{B}(H)$, then TFAE:*

1. *$\exists S \in \mathcal{B}(H)$ such that $T = S^*S$*
2. *$\langle Tx, x\rangle \geq 0$ for all $x \in H$*

If either of these conditions hold, then we say that T is a positive operator (See Definition 2.2.2)

Proof. If T is positive, then $\exists S \in \mathcal{B}(H)$ such that $T = S^*S$, and so

$$\langle Tx, x\rangle = \|Sx\|^2 \geq 0 \quad \forall x \in H$$

Conversely, if $\langle Tx, x\rangle \geq 0$ for all $x \in H$, then T is self-adjoint (and hence normal) by Theorem 2.1.8. By Theorem 2.3.11, it suffices to show that $\sigma(T) \subset [0, \infty)$. By Corollary 2.2.7, $\sigma(T) \subset \mathbb{R}$, so we show that if $\lambda \in \mathbb{R}, \lambda < 0$, then $\lambda \notin \sigma(T)$. To see this, fix $x \in H$, and note that

$$\begin{aligned}\|(T-\lambda)x\|^2 &= \|Tx\|^2 - 2\lambda\langle Tx, x\rangle + \lambda^2\|x\|^2 \\ &\geq -2\lambda\langle Tx, x\rangle + \lambda^2\|x\|^2 \\ &\geq \lambda^2\|x\|^2\end{aligned}$$

since $\lambda < 0$ and $\langle Tx, x\rangle \geq 0$. Hence, $(T - \lambda)$ is bounded below. Since $(T - \lambda)$ is self-adjoint and hence normal, it follows from Theorem 2.4.9 that $\lambda \notin \sigma(T)$. $\square$

Note that every positive operator is self-adjoint by Theorem 2.1.8. Furthermore, if $A \in \mathcal{B}(H)$, then $T := A^*A$ is a positive operator, and hence we may apply the continuous functional calculus to T. Since $\sigma(T) \subset \mathbb{R}_+$, we may apply the square root function $t \mapsto \sqrt{t}$ to T, which leads to the following definition.

Definition 2.5.2.

1. Let $A \in \mathcal{B}(H)$, then we define
$$|A| = (A^*A)^{1/2}$$
Note that if A is normal, then this coincides with applying the modulus function to A.

2. An operator $W \in \mathcal{B}(H)$ is called a partial isometry if
$$x \in \ker(W)^\perp \Rightarrow \|W(x)\| = \|x\|$$
The space $\ker(W)^\perp$ is called the initial space of W and $R(W)$ is called the final space of W. Note that both are closed subspaces of H.

   Note: A partial isometry is an isometry iff its initial space is H

Lemma 2.5.3. *Let W be a partial isometry, then W^*W and WW^* are projections onto the initial and final space of W respectively.*

Proof. Let $p := W^*W$, then

1. For $x \in \ker(W)^\perp$ and $y \in \ker(W)$, we have
$$\langle p(x), y\rangle = \langle W(x), W(y)\rangle = 0$$
Hence, $p(x) \in \ker(W)^\perp$.

2. Furthermore, for $x \in \ker(W)^\perp$, then
$$\langle W(x), W(x)\rangle = \langle x, x\rangle$$
So by the polarization identity,
$$\langle W(x), W(y)\rangle = \langle x, y\rangle \quad \forall x, y \in \ker(W)^\perp$$
Thus, if $x \in \ker(W)^\perp$, then for any $y \in H$, we write $y = y' + y''$ where $y' \in \ker(W), y'' \in \ker(W)^\perp$, then
$$\begin{aligned}\langle p(x), y\rangle &= \langle W(x), W(y)\rangle = \langle W(x), W(y'')\rangle \\ &= \langle x, y''\rangle = \langle x, y\rangle\end{aligned}$$
Hence, $p(x) = x$, so p is a projection.

3. If $p(x) = x$, then for any $y \in \ker(W)$,
$$\langle x, y\rangle = \langle W(x), W(y)\rangle = 0$$
so $x \in \ker(W)^\perp$, so p is a projection onto $\ker(W)^\perp$.

The argument for $q := WW^*$ is similar. $\square$

Theorem 2.5.4 (Polar Decomposition)**.** *Let $A \in \mathcal{B}(H)$, then $\exists$ a partial isometry $W \in \mathcal{B}(H)$ such that*
$$A = W|A|$$
Furthermore, if $A = UP$ with P positive and U a partial isometry such that $\ker(U) = \ker(P)$, then $P = |A|$ and $U = W$ must hold.

This unique expression $A = W|A|$ is called the polar decomposition of A.

Proof. For $x \in H$, we have
$$\|Ax\|^2 = \langle Ax, Ax\rangle = \langle A^*Ax, x\rangle = \langle |A|^2 x, x\rangle = \langle |A|x, |A|x\rangle = \||A|x\|^2$$
Hence,
$$W : R(|A|) \to R(A) \text{ given by } W(|A|x) = Ax$$
is an isometry. By Lemma 2.4.8(3),
$$R(A^*A) = \ker(A)^\perp$$
But since $A^*Ax = |A|(|A|x)$, it follows that
$$\overline{R(|A|)} = \ker(A)^\perp$$
Hence W extends to an isometry
$$W : \ker(A)^\perp \to \overline{R(A)}$$
Now extend W to $\ker(A)$ to be zero, so we get a partial isometry. And clearly, $W|A| = A$ holds.

As for uniqueness, note that $A^*A = PU^*UP$ and U^*U is the projection E onto the initial space of U, $\ker(U)^\perp = \ker(P)^\perp = \overline{R(P)}$. Thus, $A^*A = PEP = P^2$. By the uniqueness of the positive square root, it follows that $P = |A|$. Since
$$Ax = U|A|x = W|A|x$$
it follows that U and W agree on $R(|A|)$, which is a dense subset of both their initial spaces. Hence, $U = W$ must hold. $\square$

One simple example of how the polar decomposition may be used is the following rather useful result.

Corollary 2.5.5. *For any $T \in \mathcal{B}(H), T \in \mathcal{K}(H)$ if and only if $T^*T \in \mathcal{K}(H)$*

Proof. If $T \in \mathcal{K}(H)$ then $T^*T \in \mathcal{K}(H)$ since $\mathcal{K}(H)$ is an ideal. Conversely, if $S := T^*T \in \mathcal{K}(H)$, then $S^n \in \mathcal{K}(H)$ for all $n \geq 1$. Hence, $p(S) \in \mathcal{K}(H)$ for any polynomial $p(z) \in \mathbb{C}[z]$ such that $p(0) = 0$. Now, since S is self-adjoint, $\sigma(S) \subset \mathbb{R}$, so by the Weierstrass approximation theorem, $f(S) \in \mathcal{K}(H)$ for any $f \in C(\sigma(S))$ such that $f(0) = 0$. In particular,

$$|T| = \sqrt{T^*T} \in \mathcal{K}(H)$$

Now it follows that $T \in \mathcal{K}(H)$ because of the polar decomposition and the fact that $\mathcal{K}(H)$ is an ideal. □

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2.6 Exercises

1. Let A be a unital C^*-algebra, then prove that $\|1_A\| = 1$

2. Let H be a Hilbert space. Prove that $T \in \mathcal{B}(H)$ is left-invertible iff $\ker(T) = \{0\}$ and $T(H)$ is a closed subspace of H. [*Hint:* Every closed subspace has an orthogonal complement]

3. Let $\varphi : A \to B$ be a $*$-homomorphism between two commutative C^*-algebras. Prove that the transpose

$$\varphi^t : \Omega(B) \to \Omega(A) \text{ given by } \tau \mapsto \tau \circ \varphi$$

   is continuous. Furthermore, if φ is an isomorphism, then prove that φ^t is a homeomorphism.

4. Let X and Y be two compact Hausdorff spaces. Prove that X is homeomorphic to Y iff there is a $*$-isomorphism $C(X) \cong C(Y)$.

5. Let H be a Hilbert space, $T \in \mathcal{B}(H)$, $W < H$ a closed subspace of H. W is said to be <u>invariant under T</u> if $T(W) \subset W$, and W is said to be <u>reducing with respect to T</u> if W is invariant under T and T^*

   If $P \in \mathcal{B}(H)$ be the orthogonal projection onto W, then prove that

   a) W is invariant under T iff $PTP = TP$

   b) W is reducing with respect to T iff $TP = PT$

6. Let $T \in \mathcal{B}(H)$ be a normal operator such that $\sigma(T)$ is disconnected. Prove that T has a non-trivial invariant subspace.

3 The Spectral Theorem

3.1 The Finite Dimensional Case

Let H be a finite dimensional complex Hilbert space

Definition 3.1.1. An operator $T \in \mathcal{B}(H)$ is said to be diagonalizable if H has an orthonormal basis consisting of eigen-vectors of T.

Remark 3.1.2. Note that the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

is *diagonalizable* in the sense that it is similar to a diagonal matrix. However, it is not diagonalizable in the sense of the above definition because the basis of eigen-vectors are not orthogonal.

Lemma 3.1.3. *If $T \in \mathcal{B}(H)$ is diagonalizable, then T is normal.*

Proof. Let $\beta \subset H$ be an orthonormal basis consisting of eigen-vectors of T. For any $v, w \in \beta$ suppose $Tv = \lambda v, Tw = \mu w$. If $v \neq w$, then

$$\langle T^* v, w\rangle = \overline{\mu}\langle v, w\rangle = 0 = \overline{\lambda}\langle v, w\rangle$$

and if $v = w$, then

$$\langle T^* v, w\rangle = \overline{\lambda}\langle v, w\rangle$$

In either case, we see that $T^*(v) = \overline{\lambda} v$. Hence,

$$TT^* v = |\lambda|^2 v = T^* T v$$

This is true for all $v \in \beta$, so $TT^* = T^*T$. $\square$

Lemma 3.1.4. *If $T \in \mathcal{B}(H)$ is normal and $v \in H$ is an eigen-vector of T corresponding to the eigen value λ, then v is an eigen-vector of T^* corresponding to the eigen value $\overline{\lambda}$*

Proof. Suppose $Tv = \lambda v$, then $\|(T - \lambda)v\| = 0$. But $(T - \lambda)$ is normal, so by Theorem 2.1.12,

$$\|(T^* - \overline{\lambda})v\| = 0$$

and so $T^* v = \overline{\lambda} v$ $\square$

Lemma 3.1.5. *Let $T \in \mathcal{B}(H)$. If $W \subset H$ is a subspace such that $T(W) \subset W$, then $T^*(W^\perp) \subset W^\perp$*

Proof. If $x \in W^\perp$, then for any $y \in W$, we have $Ty \in W$, so

$$\langle T^*x, y\rangle = \langle x, Ty\rangle = 0$$

Hence, $T^*x \in W^\perp$ as required. □

Theorem 3.1.6 (Spectral Theorem)**.** *Let $T \in \mathcal{B}(H)$ be normal, then T is diagonalizable.*

Proof. We induct on $\dim(H)$. Since H is a complex Hilbert space, T has an eigen-value and a corresponding eigen-vector v. Then the subspace $\langle v\rangle$ spanned by v is invariant under T^* (by Lemma 3.1.4). Hence, $W := \langle v\rangle^\perp$ is invariant under T (by Lemma 3.1.5). Similarly, $T^*(W) \subset W$. Hence,

$$T|_W \in \mathcal{B}(W)$$

is a normal operator. By induction, W has an ONB β' consisting of eigen vectors of T. Then, $\beta' \cup \{v\}$ forms an ONB for H consisting of eigen-vectors of T. □

Lemma 3.1.7. *If T is normal, and $\lambda \neq \mu \in \sigma(T)$, then the corresponding eigen-spaces are orthogonal.*

Proof. Suppose T is normal, and $x \in E_\lambda, y \in E_\mu$, then $T^*y = \overline{\mu}y$ (by Lemma 1.3), so

$$\lambda\langle x, y\rangle = \langle Tx, y\rangle = \langle x, T^*y\rangle = \langle x, \overline{\mu}y\rangle = \mu\langle x, y\rangle$$

Since $\lambda \neq \mu$, it follows that $\langle x, y\rangle = 0$ □

Theorem 3.1.8. *$T \in \mathcal{B}(H)$ is diagonalizable iff there exist mutually orthogonal projections $\{P_1, \ldots, P_n\}$ and complex numbers $\{\lambda_1, \ldots, \lambda_n\}$ such that*

$$I = \sum_{i=1}^n P_i \text{ and } T = \sum_{i=1}^n \lambda_i P_i$$

Proof.

1. Suppose T is diagonalizable, then T is normal by Lemma 1.2. Let $\{\lambda_1, \ldots, \lambda_n\}$ be the distinct eigen-values of T and let E_{λ_i} be the corresponding eigen-spaces. Then the E_{λ_i} are mutually orthogonal spaces by Lemma 1.6. Since T is diagonalizable, they span H, so

$$H = \oplus_{i=1}^n E_{\lambda_i}$$

Let P_i denote the projection onto E_{λ_i}. Then

$$I = \sum_{i=1}^n P_i$$

and the $\{P_i\}$ are mutually orthogonal ($P_iP_j = P_jP_i = 0$ if $i \neq j$). Furthermore,

$$T = \sum \lambda_i P_i$$

clearly holds.

2. Conversely, if $T = \sum \lambda_i P_i$ for some mutually orthogonal projections, then for $E_i := P_i(H)$, we have

$$H = I(H) = \sum_{i=1}^{n} E_i$$

and $E_i \cap E_j = \{0\}$, so the above sum must be a direct sum. Also,

$$Tx = TP_i x = \sum \lambda_j P_j P_i x = \lambda_i x \quad \forall x \in E_i$$

Let β_i be a basis for E_i, then

$$\beta := \cup_{i=0}^{n} \beta_i$$

forms a basis for H (since $H = \sum E_i$) and β consists of eigen-vectors of T.

$\square$

Theorem 3.1.9. *Let H be a complex Hilbert space of dimension n, let $H_0 = \mathbb{C}^n$ and $\{e_1, \ldots, e_n\}$ be the standard ONB for H_0. Then $T \in \mathcal{B}(H)$ is diagonalizable iff $\exists$ a unitary operator*

$$U : H \to H_0$$

such that $S := UTU^{-1} \in \mathcal{B}(H_0)$ satisfies

$$S(e_i) = \lambda_i e_i$$

for some sequence $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n$. Furthermore, in that case

$$\sup_i \{|\lambda_i|\} \leq \|T\|$$

Proof. 1. Suppose T is diagonalizable, then there is an ONB $\{x_1, x_2, \ldots, x_n\}$ of H such that

$$T(x_i) = \lambda_i x_i \quad \forall 1 \leq i \leq n$$

Define $U(x_i) = e_i$, and extend U to a linear operator $H \to H_0$. Now note that

$$\langle U(x_i), e_j \rangle = \delta_{i,j} = \langle x_i, U^*(e_j) \rangle$$

Hence, $U^*(e_j) = x_j$ for all $1 \leq j \leq n$. Hence,

$$UU^* = U^*U = I$$

Furthermore, if $S = UTU^{-1} \in \mathcal{B}(H_0)$, we have

$$S(e_i) = UTU^{-1}(e_i) = \lambda_i e_i \quad \forall 1 \leq i \leq n$$

And finally, for each $1 \leq i \leq n$,

$$|\lambda_i| = \|\lambda_i e_i\| = \|S(e_i)\| \leq \|S\| = \|UTU^{-1}\| \leq \|T\|$$

2. Conversely, suppose $S = UTU^{-1}$ as in the statement of the theorem, then let $x_i := U^{-1}(e_i)$. Since U is a unitary, $\{x_1, x_2, \ldots, x_n\}$ forms an ONB for H. A simple calculation shows that $T(x_i) = \lambda_i x_i$ as required. Furthermore, each λ_i is an eigen value of T, so $\sup\{|\lambda_i|\} = r(T) \leq \|T$.

$\square$

Definition 3.1.10. Let H and H_0 be two Hilbert spaces. Two operators $T \in \mathcal{B}(H)$ and $S \in \mathcal{B}(H_0)$ are said to be unitarily equivalent if $\exists$ a unitary operator $U : H \to H_0$ such that $S = UTU^{-1}$

Note:

1. Unitary equivalence is an equivalence relation. We write $S \sim_U T$
2. If $S \sim_U T$, then $\sigma(S) = \sigma(T)$

   Proof. $S - \lambda I = U(T - \lambda I)U^{-1}$ $\square$

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3.2 Multiplication Operators

Definition 3.2.1. Let (X, μ) be a σ-finite measure space.

1. For two measurable function $f, g : X \to \mathbb{C}$, we say that $f = g$ a.e. if

$$\mu(\{x \in X : f(x) \neq g(x)\}) = 0$$

   This defines an equivalence relation on the set of measurable functions on X.

2. For any $1 \leq p < \infty$, we say f is p-summable if

$$\int_X |f(x)|^p < \infty$$

   The equivalence classes of measurable p-summable functions forms a vector space, denoted by $L^p(X, \mu)$. Furthermore, the function

$$\|f\|_p := \left(\int_X |f(x)|^p\right)^{1/p}$$

   defines a norm on $L^p(X, \mu)$ under which it is a Banach space.

3. Note that $L^2(X, \mu)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle := \int_X f\overline{g}d\mu$$

4. For any $\varphi : X \to \mathbb{C}$ be measurable and $M > 0$, we define

$$A_M := \{x \in X : |\varphi(x)| > M\}$$

We say that φ is <u>essentially bounded</u> if $\exists M > 0$ such that

$$\mu(A_M) = 0$$

5. We define $L^\infty(X, \mu)$ to be the vector space of (equivalence classes of) essentially bounded functions. The function

$$\|\varphi\|_\infty := \inf\{M > 0 : \mu(A_M) = 0\}$$

defines a norm on $L^\infty(X, \mu)$.

6. Note that $L^\infty(X, \mu)$ is a C^*-algebra with respect to this norm and point-wise multiplication.

Theorem 3.2.2. *Let $\varphi : X \to \mathbb{C}$ be essentially bounded, then we define*

$$M_\varphi : L^2(X, \mu) \to L^2(X, \mu) \text{ given by } f \mapsto \varphi f$$

Then

1. $M_\varphi \in \mathcal{B}(L^2(X, \mu))$
2. $\|M_\varphi\| \leq \|\varphi\|_\infty$
3. *If $\varphi = \psi$ a.e., then $M_\varphi = M_\psi$*

Proof. For any $f \in L^2(X, \mu)$, let $M > 0$ such that $\mu(A_M) = 0$, then we have

$$\|M_\varphi(f)\|^2 = \int_X |\varphi(x) f(x)|^2 d\mu = \int_{X \setminus A_M} |\varphi(x) f(x)|^2 \leq M^2 \int_{X \setminus A_M} |f(x)|^2 d\mu \leq M^2 \|f\|^2$$

Hence, $M_\varphi \in \mathcal{B}(L^2(X, \mu))$ and

$$\|M_\varphi\| \leq M$$

This is true for all $M > 0$ such that $\mu(A_M) = 0$, and so

$$\|M_\varphi\| \leq \|\varphi\|_\infty$$

This proves (i) and (ii). Part (iii) follows from the definition. □

Example 3.2.3. A multiplication operator should be thought of as a generalization of a diagonal matrix.

1. If $X = \{1, 2, 3, \ldots, n\}$ and μ is the counting measure, then
   a) $L^2(X, \mu) \cong \mathbb{C}^n$ with the usual inner product.

b) A multiplication operator $M_\varphi : \mathbb{C}^n \to \mathbb{C}^n$ corresponds to a *diagonal matrix*

$$M_\varphi(e_n) = \lambda_n e_n$$

where $\lambda_n = \varphi(n)$

2. If $X = \mathbb{N}$ and μ is the counting measure, then
   a) $\varphi : X \to \mathbb{C}$ is essentially bounded iff the sequence $\lambda_n := \varphi(n)$ is bounded.
   b) The multiplication operator $M_\varphi : \ell^2 \to \ell^2$ corresponds to an *infinite diagonal matrix*

$$M_\varphi(e_n) = \lambda_n e_n$$

Definition 3.2.4. Let H be a Hilbert space. $T \in \mathcal{B}(H)$ is said to be diagonalizable if $\exists$ a σ-finite measure space (X, μ) such that T is unitarily equivalent to the multiplication operator on $L^2(X, \mu)$.

In other words, $\exists \varphi \in L^\infty(X, \mu)$ and a unitary operator $U : H \to L^2(X, \mu)$ such that

$$M_\varphi = UTU^{-1}$$

Theorem 3.2.5. *Let (X, μ) be a σ-finite measure space. The map*

$$\Delta : \varphi \to M_\varphi$$

from $L^\infty(X, \mu)$ to $\mathcal{B}(L^2(X, \mu))$ is an isometric $*$-homomorphism.

Proof. Δ is clearly a $*$-homomorphism. We need to show that $\|M_\varphi\| = \|\varphi\|_\infty$. We know that $\|M_\varphi\| \leq \|\varphi\|_\infty$. To prove the reverse inequality, consider $0 < c < \|\varphi\|_\infty$, then

$$A_c := \{x \in X : |\varphi(x)| > c\}$$

has positive measure. Choose $E \subset A_c$ such that $0 < \mu(E) < \infty$ (this is possible since (X, μ) is σ-finite). Now $\chi_E \in L^2(X, \mu)$ and

$$|\varphi(x)\chi_E(x)| \geq c\chi_E(x) \quad \forall x \in X$$

Hence by squaring and integrating

$$\|M_\varphi \chi_E\|_2 \geq c\|\chi_E\|_2$$

and so $\|M_\varphi\| \geq c$ since $\|\chi_E\| \neq 0$. This is true for all $0 < c < \|\varphi\|_\infty$, and so

$$\|M_\varphi\| \geq \|\varphi\|_\infty$$

as required. $\square$

Corollary 3.2.6. *If $T \in \mathcal{B}(H)$ is diagonalizable, then T is normal.*

Proof. Choose a unitary $U : H \to L^2(X,\mu)$ and a $\varphi \in L^\infty(X,\mu)$ such that $M_\varphi = UTU^{-1}$, then

$$T = U^{-1}M_\varphi U$$

and so $T^* = U^{-1}M_\varphi^* U$ since $U = U^*$. By Theorem 3.2.5, $M_\varphi^* = M_{\overline{\varphi}}$, so

$$TT^* = U^{-1}M_\varphi M_{\overline{\varphi}} U \text{ and } T^*T = U^{-1}M_{\overline{\varphi}} M_\varphi U$$

Since φ and $\overline{\varphi}$ commute, T is normal. $\square$

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Definition 3.2.7. If $\varphi \in L^\infty(X,\mu)$, $\lambda \in \mathbb{C}$ and $r > 0$, define

$$B(\lambda, r) := \{z \in \mathbb{C} : |z - \lambda| < r\}$$

The essential range of φ is defined as

$$\text{ess-range}(\varphi) := \{\lambda \in \mathbb{C} : \mu(\varphi^{-1}(B(\lambda, r)) > 0 \quad \forall r > 0\}$$

In other words, $\lambda \in \mathbb{C}$ is not in the essential range of φ iff $\exists r > 0$ such that

$$\mu(\{x \in X : |f(x) - \lambda| < r\}) = 0$$

Equivalently, $\lambda \notin \text{ess-range}(\varphi)$ iff $\exists r > 0$ such that

$$|\varphi(x) - \lambda| \geq r \text{ a.e.}$$

Note that the essential range does not depend on the choice of representative in the equivalence class of φ.

Theorem 3.2.8. *Let (X,μ) be a σ-finite measure space, then*

$$\sigma(M_\varphi) = \textit{ess-range}(\varphi)$$

Proof. Suppose $\lambda \notin \text{ess-range}(\varphi)$, then $\exists r > 0$ such that

$$|\varphi(x) - \lambda| \geq r \text{ a.e.}$$

Let $E := \{x \in X : |\varphi(x) - \lambda| < r\}$, then $\mu(E) = 0$, so define

$$\psi(x) := \begin{cases} \frac{1}{\varphi(x)-\lambda} & : x \notin E \\ 0 & : x \in E \end{cases}$$

Then $|\psi(x)| \leq 1/r$ for all $x \in X$, so $\psi \in L^\infty(X,\mu)$ and for any $f \in L^2(X,\mu)$, we have

$$(M_\varphi - \lambda I)M_\psi f(x) = f(x) \quad \forall x \notin E$$

Since $\mu(E) = 0$, this means that $(M_\varphi - \lambda I)M_\psi = I$. Similarly, $M_\psi(M_\varphi - \lambda I) = I$ and so $\lambda \notin \sigma(M_\varphi)$. Hence,

$$\sigma(M_\varphi) \subset \text{ess-range}(\varphi)$$

Now if $\lambda \in \text{ess-range}(\varphi)$, then we construct a sequence $(f_n) \subset L^2(X, \mu)$ of unit vectors such that

$$\|(M_\varphi - \lambda I)f_n\| \to 0$$

For each $n \in \mathbb{N}$, the set

$$E_n := \{x \in X : |\varphi(x) - \lambda| \leq 1/n\}$$

has positive measure. Since μ is σ-finite, choose $F_n \subset E_n$ such that $0 < \mu(F_n) < \infty$ and define $f_n := \mu(F_n)^{-1/2}\chi_{F_n}$, so that

$$|(\varphi(x) - \lambda)f_n(x)| \leq \frac{1}{n}|f_n(x)| \quad \forall x \in X$$

Squaring and integrating gives

$$\|(M_\varphi - \lambda I)f_n\|_2 \leq \frac{1}{n} \to 0$$

□

Remark 3.2.9. We have proved that if $\varphi \in L^\infty(X, \mu)$ is such that M_φ is invertible, then $M_\varphi^{-1} = M_\psi$ for some $\psi \in L^\infty(X, \mu)$. This is a reflection of the fact that

$$A := \{M_\varphi : \varphi \in L^\infty(X, \mu)\}$$

is a maximal Abelian subalgebra of $\mathcal{B}(L^2(X, \mu))$ [See Problem 14 of Section 1.7]

Definition 3.2.10. 1. Let $S \subset \mathcal{B}(H)$ be any set. The commutant of S is defined as

$$S' := \{T \in \mathcal{B}(H) : Ta = aT \quad \forall a \in S\}$$

Note that S' is a linear subspace of $\mathcal{B}(H)$ that is closed under composition. Furthermore, if S is closed under taking adjoints, then so is S'. Hence, if S is a C*-subalgebra of $\mathcal{B}(H)$, then so is S'.

2. If $S \subset \mathcal{B}(H)$, then $S'' := (S')'$. Note that $S \subset S''$.

Lemma 3.2.11. *$A \subset \mathcal{B}(H)$ is a maximal Abelian subalgebra if and only if $A = A'$.*

Proof. Omitted. □

Definition 3.2.12. A C*-algebra $A \subset \mathcal{B}(H)$ is called a von Neumann algebra if $A = A''$

Remark 3.2.13. We have just shown that if $A \subset \mathcal{B}(H)$ is a maximal Abelian subalgebra, then A is a von Neumann algebra. In particular,

$$L^\infty(X, \mu) \hookrightarrow \mathcal{B}(L^2(X, \mu))$$

is a von Neumann algebra.

3.3 The Spectral Theorem

Definition 3.3.1. Let X be a compact metric space and μ be a positive measure on X defined on a σ-algebra $\mathcal{M}$ on X.

1. μ is called a Borel measure if the domain of μ includes all Borel sets (equivalently, all the open sets)
2. μ is called inner regular if for any $A \in \mathcal{M}$, we have
$$\mu(A) = \sup\{\mu(K) : K \subset A \text{ compact }\}$$
3. μ is called outer regular if for any $A \in \mathcal{M}$, we have
$$\mu(A) = \inf\{\mu(U) : A \subset A \text{ open }\}$$
4. μ is called Radon if μ is a Borel measure that is both inner and outer regular and $\mu(K) < \infty$ for any compact set $K \subset X$. [Equivalently, $\mu(X) < \infty$]

Remark 3.3.2. Let X be a compact metric space and μ a Radon measure on X.

1. Then every continuous function $f : X \to \mathbb{C}$ is measurable and
$$|\int_X f d\mu| \leq \|f\|_\infty \mu(X) < \infty$$
Hence, the map
$$\Lambda_\mu : f \mapsto \int_X f d\mu$$
defines a bounded linear functional on $C(X)$.
2. Furthermore, if $f \geq 0$ in $C(X)$, then $\Lambda(f) \in [0, \infty)$. Such a linear functional on $C(X)$ is called positive.

Theorem 3.3.3 (Riesz Representation Theorem)**.** *Let X be a compact Hausdorff space and*
$$\Lambda : C(X) \to \mathbb{C}$$
be a positive linear functional. Then $\exists$ a unique Radon measure μ on X such that
$$\Lambda(f) = \int_X f d\mu \quad \forall f \in C(X)$$

Proof. Omitted. □

(End of Day 23)

Definition 3.3.4. Let H be a Hilbert space and $T \in \mathcal{B}(H)$ a normal operator. Recall that

$$C^*(T) = \overline{\{p(T, T^*) : p \in \mathbb{C}[x, y]\}}$$

is the smallest C^*-algebra containing T. A vector $e \in H$ is called cyclic with respect to T if the set

$$C^*(T)e := \{Ae : A \in C^*(T)\}$$

is dense in H

Example 3.3.5. 1. Let $H = L^2[0, 1]$ and $T \in \mathcal{B}(H)$ be given by

$$Tf(x) = xf(x)$$

Then take $e(x) \equiv 1$, then $e \in H$ is a cyclic vector for T.

Proof. Note that $Te(x) = x$, so $C^*(T)e$ contains all polynomials. By Weierstrass' theorem and Lusin's theorem, the polynomials are dense in $L^2[0, 1]$. $\square$

2. Let $H = \mathbb{C}^2$ and $T(x, y) = (x, 0)$, then
   a) For any $e \in H$, $C^*(T)(e) \subset \mathbb{C} \oplus \{0\}$, and so T does not have a cylic vector.
   b) However, if $H_1 := \mathbb{C} \oplus \{0\}$, and $H_2 = \{0\} \oplus \mathbb{C}$, then $T(H_i) \subset H_i$ and $T|_{H_i} \in \mathcal{B}(H_i)$ has a cyclic vector each.

Theorem 3.3.6 (Spectral Theorem - Special Case)**.** *Suppose $T \in \mathcal{B}(H)$ is a normal operator which has a cyclic vector, then T is diagonalizable.*

Proof. Let $e \in H$ be a cyclic vector for T. Let $X := \sigma(T)$. Define

$$\Lambda : C(X) \to \mathbb{C} \text{ by } f \mapsto \langle f(T)e, e\rangle$$

Since the map $f \mapsto f(T)$ is linear, Λ is a linear map. Furthermore, if $f \geq 0$ in $C(X)$, $\exists g \in C(X)$ such that $g = \overline{g}$ and $g^2 = f$. Hence,

$$g(T) = g(T)^* \text{ and } g(T)^2 = f(T)$$

Thus,

$$\langle f(T)e, e\rangle = \langle g(T)e, g(T)e\rangle \geq 0$$

Thus, Λ is a positive linear functional on $C(X)$. Hence by the Riesz Representation theorem, $\exists$ a unique Radon measure μ on X such that

$$\langle f(T)e, e\rangle = \int_X f d\mu$$

Now consider $C(X)$ as a subspace of $L^2(X, \mu)$. For any $f, g \in C(X)$, we have

$$\begin{aligned}
\langle f, g\rangle &= \int_X f\overline{g} d\mu \\
&= \int_X \overline{g} f d\mu \\
&= \langle g(T)^* f(T)e, e\rangle \\
&= \langle f(T)e, g(T)e\rangle
\end{aligned}$$

So we define $U : C(X) \to H$ by

$$U(f) := f(T)e$$

Then

1. U preserves inner product. Since $C(X)$ is dense in $L^2(X, \mu)$ by Lusin's theorem, U extends to a unitary from $L^2(X, \mu)$ to its range.

2. The range of U contains $C^*(T)e$. Since this is dense in H, the range of U is all of H.

Furthermore, let $\varphi \in L^\infty(X, \mu)$ be the function $\varphi(z) = z$, then for any $g \in C(X)$, we have

$$UM_\varphi(g) = U(\varphi g) = \varphi(T)g(T)e = Tg(T)e = TU(g)$$

Hence,

$$T = UM_\varphi U^{-1}$$

since the two operators agree on $C(X)$ which is dense in $L^2(X, \mu)$. □

Remark 3.3.7. Let $\{H_n\}$ be a sequence of separable Hilbert spaces and $A_n \in \mathcal{B}(H_n)$ such that

$$\sup \|A_n\| < \infty$$

Then let

$$H := \bigoplus_{n=1}^{\infty} H_n := \{(x_n) : x_n \in H_n \text{ and } \sum_{n=1}^{\infty} \|x_n\|_{H_n}^2 < \infty\}$$

1. H is a Hilbert space with inner product given by

$$\langle (x_n), (y_n) \rangle := \sum_{n=1}^{\infty} \langle x_n, y_n \rangle$$

2. The operator $A : H \to H$ defined by

$$A((x_n)) := (A_n(x_n))$$

is a bounded linear operator. We denote this operator by

$$A = \bigoplus_{n=1}^{\infty} A_n$$

Lemma 3.3.8. *Suppose $\{H_n\}$ is a finite or infinite sequence of Hilbert spaces and $A_n \in \mathcal{B}(H_n)$ such that $\sup \|A_n\| < \infty$. If each A_n is diagonalizable, then $A := \oplus_{n=1}^{\infty} A_n$ is diagonalizable.*

Proof. For each $n \in \mathbb{N}$, there is a σ-finite measure space $(X_n, \mathcal{M}_n, \mu_n)$, unitaries $U_n : H_n \to L^2(X_n, \mu_n)$ and $\varphi_n \in L^\infty(X_n, \mu_n)$ such that

$$A_n = U_n^{-1} M_{\varphi_n} U_n$$

Now set $X := \sqcup X_n$ be the disjoint union with the σ-algebra

$$\mathcal{M} := \{E \subset X : E \cap X_n \in \mathcal{M}_n \quad \forall n \in \mathbb{N}\}$$

and measure μ given by

$$\mu(E) := \sum_{n=1}^{\infty} \mu_n(E \cap X_n)$$

Then μ is clearly a measure on X, and it is σ-finite since each μ_n is. [Check!].

$$L^2(X, \mu) = \bigoplus_{n=1}^{\infty} L^2(X_n, \mu_n)$$

Then, $U := \bigoplus U_n$ defines a unitary operator from $H := \oplus H_n$ to $L^2(X, \mu)$ and define $\varphi : X \to \mathbb{C}$ by

$$\varphi|_{X_n} = \varphi_n$$

Then φ_n is μ-essentially bounded. And

$$A = U^{-1} M_\varphi U$$

$\square$

Lemma 3.3.9. *Let H be a separable Hilbert space and $T \in \mathcal{B}(H)$ be a normal operator, then $\exists$ closed subspaces $\{H_n\}$ of H such that*

1. *Each H_n is reducing for T*
2. *$T|_{H_n}$ has a cyclic vector $x_n \in H_n$*
3. *If $n \neq m$, then $H_n \perp H_m$*
4. *$H = \bigoplus H_n$*

Proof. For any $x \in H$, define

$$H_x := \overline{C^*(T)x}$$

Define

$$\mathcal{F} := \{S \subset H : \forall x, y \in S, H_x \perp H_y\}$$

Then $\mathcal{F}$ can be partially ordered by inclusion. If $\mathcal{C}$ is a chain in $\mathcal{F}$, then the union

$$T := \bigcup_{S \in \mathcal{C}} S$$

is also a member of $\mathcal{F}$ and is an upper bound for $\mathcal{C}$. Hence, $\mathcal{F}$ satisfies the conditions of Zorn's lemma, and so must have a maximal element E.

We claim that $H = \sum_{e\in E} H_e$. For if not, then $\exists x \in H$ such that $x \perp \sum_{e\in E} H_e$. Then, $E \cup \{x\}$ would be a member of $\mathcal{F}$ contradicting the maximality of E. Hence,

$$H = \bigoplus_{e\in E} H_e$$

Since H is separable, E must be countable, thus proving the theorem. $\square$

Theorem 3.3.10 (Spectral Theorem - General Case)**.** *If H is a separable Hilbert space and $T \in \mathcal{B}(H)$ a normal operator, then T is diagonalizable.*

Proof. Theorem 3.3.6 + Lemma 3.3.8 + Lemma 3.3.9. $\square$

3.4 Exercises

1. Let $T \in \mathcal{K}(H)$ be a compact normal operator. Show that every non-zero spectral value is an eigen-value of T.

   [*Hint:* If $\lambda \in \sigma(T) \setminus \{0\}$, by Theorem 2.4.9, there is a sequence of unit vectors $(x_n) \subset H$ such that $\|T(x_n) - \lambda x_n\| \to 0$. Choose a subsequence (x_{n_j}) such that $T(x_{n_j})$ converges. Show that this limit vector is, in fact, an eigen-vector of T]

2. Let $T \in \mathcal{K}(H)$ be a normal operator. Show that H has an orthonormal basis consisting of eigen-vectors of T.

   [*Hint:* Use the ideas of Theorem 3.1.6. Use the previous problem, and replace the induction argument by Zorn's lemma.]

3. Let X be a compact metric space and $\Lambda : C(X) \to \mathbb{C}$ be a positive linear functional (as in Remark 3.3.2). Without using the Riesz Representation theorem, prove that Λ is bounded and that

   $$\|\Lambda\| = \Lambda(\mathbf{1})$$

   where $\mathbf{1}$ denotes the contant function 1.

4. Let X be a compact Hausdorff space and μ a positive Borel measure on X. For any $\varphi \in C(X)$, prove that

   $$\text{ess-range}(\varphi) = \varphi(X)$$

   For the remaining problems, let H be a Hilbert space, $T \in \mathcal{B}(H)$ a normal operator with a cyclic vector $e \in H$. Furthermore, set $X := \sigma(T)$ and let μ be the (positive) Radon measure obtained in the Spectral Theorem (Theorem 3.3.6).

5. If $f \in C(X)$ is such that $f(T)e = 0$, then show that $f = 0$ in $C(X)$.

6. For any $\lambda \in X$ and any open neighbourhood $U \subset X$ of λ, show that $\mu(U) > 0$

   Note: For any positive Borel measure μ on a set X, the support of μ is defined to be
   $$\text{supp}(\mu) := \{x \in X : \mu(U) > 0 \quad \forall \text{ open } U \text{ such that } x \in U\}$$
   The above problem shows that if μ is the measure obtained in Theorem 3.3.6, then
   $$\text{supp}(\mu) = \sigma(T)$$

7. Show that $\lambda \in \mathbb{C}$ is an eigen-value of T iff $\mu(\{\lambda\}) \neq 0$.

   Note: Let (X, μ) be a σ-finite measure space.

   a) A point $x \in X$ is called an atom of the measure μ if $\mu(\{x\}) > 0$

   b) Note that if $\lambda \in \sigma(T)$ is an isolated point, then $\mu(\{\lambda\}) > 0$ by the previous theorem. Hence we have obtained Theorem 2.4.10.

8. Prove that for any $\epsilon > 0$ there exist a normal operator $S \in \mathcal{B}(H)$ with finite spectrum such that $\|S - T\| < \epsilon$.

   [*Hint:* If ψ is a simple function, then prove that the induced multiplication operator $M_\psi \in \mathcal{B}(L^2(X, \mu))$ has finite spectrum]

3.5 Complex Measures

This section sketches the theory of complex measures. For details, see [Rudin, Chapter 6]

Definition 3.5.1. Let $\mathcal{M}$ be a σ-algebra on a set X.

1. If $E \in \mathcal{M}$, a partition of E is a countable family $\{E_i\} \subset \mathcal{M}$ of mutually disjoint sets such that $E = \sqcup E_i$.

2. A complex measure on X is a function
   $$\mu : \mathcal{M} \to \mathbb{C}$$
   such that for any $E \in \mathcal{M}$ and any partition $\{E_i\}$ of E, one has
   $$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$$
   where the RHS is a convergent series in $\mathbb{C}$.

Remark 3.5.2. If μ is a complex measure on X

1. If $E \in \mathcal{M}$, then for any partition $\{E_i\}$ of E, the series

$$\sum_{i=1}^{\infty} \mu(E_i)$$

converges in $\mathbb{C}$. In particular, any rearrangement of that series converges, and so the series must converge absolutely (by Riemann's theorem). Hence,

$$\sum_{i=1}^{\infty} |\mu(E_i)| < \infty$$

2. We want to find a positive measure λ on X such that

$$|\mu(E)| \leq \lambda(E) \quad \forall E \in \mathcal{M}$$

In particular, for any partition $\{E_i\}$ of E, one must have

$$\lambda(E) \geq \sum_{i=1}^{\infty} |\mu(E_i)|$$

Therefore, we define a set function $\lambda : \mathcal{M} \to [0, \infty)$ by

$$\lambda(E) := \sup \sum_{i=1}^{\infty} |\mu(E_i)|$$

where the supremum is taken over all partitions of E.

(End of Day 24)

Theorem 3.5.3. *Let μ be a complex measure on X. Then the function λ defined above is a positive measure on X.*

Proof. Clearly,

$$\lambda(E) \geq 0 \quad \forall E \in \mathcal{M}$$

and $\lambda(\emptyset) = 0$. Hence it suffices to prove countable additivity. So let $\{E_i\}$ be a partition of $E \in \mathcal{M}$. We WTS:

$$\lambda(E) = \sum_{i=1}^{\infty} \lambda(E_i)$$

1. Suppose $t_i \in \mathbb{R}$ such that $t_i < \lambda(E_i)$ for all i. Then each E_i has a partition $\{A_{i,j}\}$ such that

$$\sum_{j=1}^{\infty} |\mu(A_{i,j})| > t_i$$

Since $\{A_{i,j}\}$ forms a partition for E, it follows that

$$\sum_{i=1}^{\infty} t_i \leq \sum_{i,j} |\mu(A_{i,j})| \leq \lambda(E)$$

Taking supremum over all possible $\{t_i\}$ proves that

$$\sum_{i=1}^{\infty} \lambda(E_i) \leq \lambda(E)$$

2. Conversely, let $\{A_j\}$ be a partition of E, then for each $j \in \mathbb{N}$, $\{A_j \cap E_i\}$ is a partition of A_j, and for each $i \in \mathbb{N}$, $\{A_j \cap E_i\}$ is a partition of E_i. Hence,

$$\begin{aligned}\sum_j |\mu(A_j)| &= \sum_j \left| \sum_i \mu(A_j \cap E_i) \right| \\ &\leq \sum_j \sum_i |\mu(A_j \cap E_i)| \\ &= \sum_i \sum_j |\mu(A_j \cap E_i)| \\ &\leq \sum_i \lambda(E_i)\end{aligned}$$

This is true for any partition $\{A_j\}$ of E, and so taking supremum gives

$$\lambda(E) \leq \sum_i \lambda(E_i)$$

□

Remark 3.5.4. 1. The measure λ defined above is unique in the following sense: If ν is any other positive measure on X such that

$$|\mu(E)| \leq \nu(E) \quad \forall E \in \mathcal{M}$$

Then $\lambda(E) \leq \nu(E)$ for all $E \in \mathcal{M}$

Proof. If ν is any other measure as above, then for any $E \in \mathcal{M}$, and any partition $\{E_i\}$ of E, we have

$$\sum_i |\mu(E_i)| \leq \sum_i \nu(E_i) = \nu(E)$$

This is true for any partition $\{E_i\}$, so taking supremum gives $\lambda(E) \leq \nu(E)$ for all $E \in \mathcal{M}$. □

This measure λ is called the <u>total variation</u> of μ and is denoted by $|\mu|$

2. If μ is a complex measure and $\alpha \in \mathbb{C}$, then $\alpha\mu$ is a complex measure defined by

$$(\alpha\mu)(E) := \alpha\mu(E)$$

Now we claim that $|\alpha\mu\| = |\alpha||\mu|$

Proof. Let $\gamma := \alpha\mu$, and let $E \in \mathcal{M}$ and $\{E_i\}$ be a partition of E, then

$$\sum_i |\gamma(E_i)| = |\alpha| \sum_i |\mu(E_i)| \leq |\alpha||\mu|(E)$$

Taking supremum gives that

$$|\gamma|(E) \leq |\alpha||\mu|$$

Replacing α by $1/\alpha$ gives the reverse inequality □

3. Similarly, if μ and γ are two complex measures, then

$$(\mu + \gamma)(E) := \mu(E) + \gamma(E)$$

defines a complex measure such that

$$|\mu + \gamma|(E) \leq |\mu|(E) + |\gamma|(E) \quad \forall E \in \mathcal{M}$$

Theorem 3.5.5. *Let μ be a complex measure on X, then $|\mu|(X) < \infty$.*

Proof. Omitted. □

Theorem 3.5.6. *Let $M(X)$ be the set of all complex measures on X. Define*

$$(\mu + \lambda)(E) := \mu(E) + \lambda(E) \text{ and } (\alpha\mu)(E) := \alpha\mu(E)$$

Then $M(X)$ is a vector space under these operations. Furthermore, The function

$$\|\mu\| := |\mu|(X)$$

defines a norm on $M(X)$.

Proof. By Theorem 3.5.5, $\|\cdot\|$ is a well-defined real-valued function on $M(X)$ such that $\|\mu\| \geq 0$ for all $\mu \in M(X)$. Furthermore,

1. If $\|\mu\| = 0$, then, for any $E \in \mathcal{M}$, we have

$$|\mu(E)| \leq |\mu|(E) \leq |\mu|(X) = \|\mu\| = 0 \Rightarrow \mu(E) = 0 \quad \forall E \in \mathcal{M}$$

2. The other two conditions of the norm follow from Remark 3.5.4.

□

Definition 3.5.7. Let μ be a positive measure on X and $\lambda \in M(X)$. We say that λ is absolutely continuous with respect to μ if

$$\forall E \in \mathcal{M}, \mu(E) = 0 \Rightarrow \lambda(E) = 0$$

If this happens, we write $\lambda \ll \mu$

Example 3.5.8.

1. Let μ be any measure on X and $\varphi \in L^1(X, \mu)$. Define

$$\lambda(E) := \int_E \varphi d\mu$$

Then $\lambda \in M(X)$ and $\lambda \ll \mu$

2. If μ any complex measure, then $\mu \ll |\mu|$

Theorem 3.5.9 (Radon-Nikodym Theorem). *Let μ be a positive measure on X and $\lambda \in M(X)$ such that $\lambda \ll \mu$. Then $\exists$ unique $\varphi \in L^1(\mu)$ such that*

$$\lambda(E) = \int_E \varphi d\mu$$

Proof. Omitted. □

Proposition 3.5.10. *Let μ be a complex measure on X, then $\exists h \in L^1(X, |\mu|)$ such that $|h(x)| = 1$ for all $x \in X$ and*

$$\mu(E) = \int_E h d|\mu| \quad \forall E \subset X \text{ measurable}$$

Furthermore, this h is unique a.e. $[|\mu|]$

Proof. Omitted. □

Definition 3.5.11. Let μ be a complex Borel measure on X, then

1. We say μ is regular if $|\mu|$ is regular (as in Definition 3.3.1). Write $M_B(X)$ for the set of all regular complex Borel measures on X, and we think of $M_B(X)$ as a subspace of $M(X)$.

2. For any $f : X \to \mathbb{C}$ measurable, we define

$$\int_X f d\mu := \int_X f h d|\mu|$$

where h is as above. This is well-defined by the uniqueness of h.

3. The map $\Lambda_\mu : C(X) \to \mathbb{C}$ given by

$$f \mapsto \int_X f d\mu$$

defines a bounded linear functional on X with

$$\|\Lambda_\mu\| = \|\mu\|$$

Proof. Clearly,

$$\left|\int_X f d\mu\right| \leq \int_X |fh| d|\mu| \leq \|f\|_\infty |\mu|(X)$$

since $|h| = 1$ on X. Hence, Λ_μ is bounded and $\|\Lambda_\mu\| \leq \|\mu\|$. Now since $h \in L^1(X, |\mu|), \exists (f_n) \in C(X)$ such that

$$f_n \to \overline{h} \text{ in } L^1(X, |\mu|)$$

Replacing f_n by $f_n/\|f_n\|$ if need be, we may assume that $\|f_n\| = 1$. Since $h \in L^\infty(X, |\mu|)$, it follows that

$$\Lambda_\mu(f_n) = \int_X f_n h d|\mu| \to \int_X |h|^2 d|\mu| = |\mu|(X) = \|\mu\|$$

Hence, $\|\Lambda_\mu\| = \|\mu\|$ □

Theorem 3.5.12 (Riesz Representation Theorem)**.** *Let X be a compact Hausdorff space and $\Lambda : C(X) \to \mathbb{C}$ be a bounded linear functional. Then $\exists$ a unique complex Borel measure μ on X such that*

$$\Lambda = \Lambda_\mu$$

In other words, the map

$$M_B(X) \to C(X)' \textit{ given by } \mu \mapsto \Lambda_\mu$$

is an isometric isomorphism of normed linear spaces.

Proof. Omitted. □

(End of Day 25)

3.6 Borel Functional Calculus

Given a normal operator $T \in \mathcal{B}(H)$, we would like to define $f(T)$ when $f : \sigma(T) \to \mathbb{C}$ is not necessarily continuous.

Definition 3.6.1. Let $X \subset \mathbb{C}$ be compact. Set

$$B(X) = \{f : X \to \mathbb{C} : f \text{ Borel-measurable and bounded}\}$$

Note that

1. $B(X)$ is a normed linear space under the supremum norm. It is a Banach space (because the pointwise limit of a sequence of measurable functions is again measurable and the uniform limit of a sequence of bounded functions is bounded)
2. $B(X)$ is a C^*-algebra under the point-wise operations
3. $C(X) \subset B(X)$.

4. $C(X) \neq B(X)$ in general, since $B(X)$ contains characteristic functions χ_E for any Borel set $E \subset X$, and these may not be continuous (unless X is discrete).

Given $f \in B(X)$, we wish to make sense of

$$f(T) \in \mathcal{B}(H)$$

We do this by constructing a $*$-homomorphism

$$\widehat{\Theta} : B(X) \to \mathcal{B}(H)$$

which extends the continuous functional calculus.

Definition 3.6.2. 1. Let A be a C^*-algebra and H a Hilbert space. A $*$-representation of A on H is a $*$-homomorphism $\pi : A \to \mathcal{B}(H)$. We write (H, π) for the representation.

2. A $*$-representation $\pi : A \to \mathcal{B}(H)$ is called cyclic if $\exists e \in H$ such that the set

$$\{\pi(a)(e) : a \in A\}$$

is dense in H

3. A $*$-representation $\pi : A \to \mathcal{B}(H)$ is called non-degenerate if the set

$$\{\pi(a)x : a \in A, x \in H\}$$

is dense in H.

Example 3.6.3. 1. Any cyclic representation is a non-degenerate representation.

2. Let $T \in \mathcal{B}(H)$ be a normal operator, then if $A = C(\sigma(T))$, then the continuous functional calculus defines a representation of A. If T has a cyclic vector, then this is a cyclic representation.

3. If $T \in \mathcal{B}(\mathbb{C}^2)$ is given by $T(x, y) = (x, 0)$, then this is not a non-degenerate representation of $C(\sigma(T))$

4. Let $X \subset \mathbb{C}$ and μ be a Borel measure on X. Let $H := L^2(X, \mu)$ and define

$$\pi : C(X) \to \mathcal{B}(H) \text{ given by } f \mapsto M_f$$

Then π is a cyclic representation because the set

$$\{\pi(f)(1) : f \in C(X)\}$$

is dense in H.

Definition 3.6.4. 1. A sequence of operators $(T_n) \in \mathcal{B}(H)$ is said to converge <u>strongly</u> to $T \in \mathcal{B}(H)$ if, for each $x \in H$

$$T_n(x) \to T(x)$$

If this happens, we write $T_n \xrightarrow{s} T$

2. A sequence of operators $(T_n) \in \mathcal{B}(H)$ is said to converge <u>weakly</u> to $T \in \mathcal{B}(H)$ if, for each $x, y \in H$

$$\langle T_n(x), y\rangle \to \langle Tx, y\rangle$$

If this happens, we write $T_n \xrightarrow{w} T$

Example 3.6.5. 1. If $T_n \to T$ in the norm of $\mathcal{B}(H)$, then $T_n \xrightarrow{s} T$

2. If $T_n \xrightarrow{s} T$, then $T_n \xrightarrow{w} T$ by Cauchy-Schwartz.

3. Let $S \in \mathcal{B}(\ell^2)$ be given by the left-shift operator

$$S((x_1, x_2, \dots)) = (x_2, x_3, \dots)$$

Then let $T_n := S^n$. Note that

$$T_n((x_1, x_2, \dots)) = (x_{n+1}, x_{n+2}, \dots)$$

a) $\|T_n\| = 1$ for all n (Exercise)

b) However, for any $x \in H$, we have

$$\sum |x_n|^2 < \infty$$

So

$$\|T_n(x)\|^2 = \sum_{j=n+1}^{\infty} |x_j|^2 \to 0$$

Thus, $T_n \xrightarrow{s} 0$

4. Let $H = \ell^2(\mathbb{N})$ and let S be the right-shift operator

$$S((x_n)) = (0, x_1, x_2, \dots)$$

Let $T_n := S^n$, so that

$$T((x_n)) = (\underbrace{0, 0, \dots, 0}_{n \text{ times}}, x_1, x_2, \dots)$$

a) T_n does not converge strongly to 0 because each T_n is an isometry.

b) Claim: $T_n \xrightarrow{w} 0$

Proof. If $x, y \in H$, and $\epsilon > 0$, choose $N \in \mathbb{N}$ such that

$$\sum_{n=N}^{\infty} |y_n|^2 < \epsilon^2$$

Then, for any $n \geq N$, we have

$$|\langle T_n(x), y\rangle| = |\sum_{n=N}^{\infty} x_{n-N}\overline{y_n}| \leq \|x\| \left(\sum_{n=N}^{\infty} |y_n|^2\right)^{1/2} < \epsilon\|x\|$$

by the Cauchy-Schwartz inequality Hence, $\langle T_n(x), y\rangle \to 0$ □

Definition 3.6.6. A representation $\widehat{\pi} : B(X) \to \mathcal{B}(H)$ is called a <u>σ-representation</u> if for every uniformly bounded $(f_n) \in B(X)$

$$f_n \to 0 \text{ pointwise } \Rightarrow \widehat{\pi}(f_n) \xrightarrow{s} 0$$

Lemma 3.6.7. *Let $\widehat{\pi} : B(X) \to \mathcal{B}(H)$ be a representation such that for every uniformly bounded $(f_n) \in B(X)$*

$$f_n \to 0 \textit{ pointwise} \Rightarrow \widehat{\pi}(f_n) \xrightarrow{w} 0$$

Then $\widehat{\pi}$ is a σ-representation.

Proof. Suppose this condition holds, and (f_n) a uniformly bounded sequence such that $f_n \to 0$ pointwise. Then for any $x \in H$, consider

$$\|\widehat{\pi}(f_n)(x)\|^2 = \langle \widehat{\pi}(f_n^* f_n)(x), x\rangle$$

But, $(f_n^* f_n)$ is a uniformly bounded sequence converging pointwise to 0. Hence by hypothesis, the RHS converges to 0, and hence

$$\widehat{\pi}(f_n)(x) \to 0 \quad \forall x \in H$$

as required □

Theorem 3.6.8. *Let X be a compact metric space and H a Hilbert space. Every non-degenerate $*$-representation $\pi : C(X) \to \mathcal{B}(H)$ extends uniquely to a σ-representation $\widehat{\pi} : B(X) \to \mathcal{B}(H)$*

Proof. We prove this using the following lemmas. □

(End of Day 26)

Solved Exercises 1,2,5,6, and 7 from section 3.4

(End of Day 27)

Lemma 3.6.9. *Let $S(X)$ denote the set of all simple Borel-measurable functions on X. Then $S(X)$ is dense in $B(X)$*

Proof. Clearly, $S(X)$ is a subalgebra of $B(X)$ that is closed under taking adjoints. Suppose $f \in B(X)$ and $\epsilon > 0$ are given, then $f(X) \subset \mathbb{C}$ is bounded. Hence, $f(X)$ is pre-compact. Thus, $\exists$ disjoint Borel sets $\{V_1, V_2, \ldots, V_n\}$ of $\mathbb{C}$ such that

$$\text{diam}(V_i) < \epsilon \quad \forall i \text{ and } f(X) \subset \bigcup_{i=1}^n V_i$$

Assume WLOG that $f(X) \cap V_i \neq \emptyset$, so for $1 \leq i \leq n$, choose $\alpha_i \in f(X) \cap V_i$, and define

$$g := \sum_{i=1}^n \alpha_i \chi_{f^{-1}(V_i)}$$

then $g \in S(X)$ and $\|g - f\| \leq \epsilon$ [Check!] $\square$

Lemma 3.6.10. *Let $\widehat{\pi} : B(X) \to \mathcal{B}(H)$ be a σ-representation, then for any $x, y \in H$, define*

$$\mu_{x,y}(E) := \langle \widehat{\pi}(\chi_E)x, y\rangle \quad \forall \text{ Borel sets } E$$

Then

1. *$\mu_{x,y}$ is a complex measure on X*
2. *For all $f \in B(X)$,*

$$\int_X f d\mu_{x,y} = \langle \widehat{\pi}(f)x, y\rangle \qquad (*)$$

Proof. 1. We need to check countable additivity. Since $\widehat{\pi}$ is linear, $\mu_{x,y}$ is clearly finitely additive. So suppose $E \in \mathcal{M}$ has a partition $\{E_n\}$, then consider

$$F_n := E_1 \sqcup E_2 \sqcup \ldots \sqcup E_n$$

Then

$$\mu_{x,y}(F_n) = \sum_{i=1}^n \mu_{x,y}(E_i)$$

However, χ_{F_n} is a sequence of uniformly bounded functions such that

$$\chi_{F_n} \to \chi_E$$

Since $\widehat{\pi}$ is a σ-representation, it follows that

$$\widehat{\pi}(\chi_{F_n}) \xrightarrow{w} \widehat{\pi}(\chi_E)$$

Thus, $\mu_{x,y}(F_n) \to \mu_{x,y}(E)$ which means that

$$\mu_{x,y}(E) = \sum_{i=1}^\infty \mu_{x,y}(E_i)$$

Hence $\mu_{x,y}$ is a complex measure.

2. Note that $(*)$ holds if f is a characteristic function by definition. Hence, it holds for all $f \in S(X)$ by linearity. Now if $f \in B(X)$, then $\exists (f_n) \in S(X)$ such that $f_n \to f$ uniformly (by Lemma 3.6.9) and so

$$\widehat{\pi}(f_n) \xrightarrow{w} \widehat{\pi}(f)$$

since $\widehat{\pi}$ is a σ-representation. Hence,

$$\langle \widehat{\pi}(f_n)x, y\rangle \to \langle \widehat{\pi}(f)x, y\rangle$$

But by the dominated convergence theorem,

$$\int_X f_n d\mu_{x,y} \to \int_X f d\mu_{x,y}$$

and hence the result.

□

Theorem 3.6.11 (Uniqueness part of Theorem 3.6.8)**.** *Let $\pi : C(X) \to \mathcal{B}(H)$ be a non-degenerate representation, and suppose π_1 and $\pi_2 : B(X) \to \mathcal{B}(H)$ are two σ-representations extending π. We WTS that $\pi_1(f) = \pi_2(f)$ for all $f \in \mathcal{B}(X)$.*

Proof. For any $x, y \in H$, it suffices to prove that

$$\langle \pi_1(f)x, y\rangle = \langle \pi_2(f)x, y\rangle$$

Let $\mu_{x,y}$ and $\lambda_{x,y}$ be the associated complex measures from Lemma 3.6.10, then we want to show that

$$\int_X f d\mu_{x,y} = \int_X f d\lambda_{x,y} \quad \forall f \in B(X)$$

Since $\pi_1(g) = \pi_2(g)$ for all $g \in C(X)$, we know that this equality holds in $C(X)$. Thus, $\mu_{x,y}$ and $\lambda_{x,y}$ define the same linear functional on $C(X)$. By the uniqueness part of the Riesz Representation theorem (Theorem 3.5.12), it follows that

$$\mu_{x,y}(E) = \lambda_{x,y}(E) \quad \forall E \in \mathcal{M}$$

Hence, the required equality holds for all $f \in S(X)$. Using the fact that both π_1 and π_2 are σ-representations, it follows that

$$\pi_1(f) = \pi_2(f) \quad \forall f \in B(X)$$

□

Note: We have just prove that if μ and λ are two complex measures such that

$$\int_X f d\mu = \int_X f d\lambda \quad \forall f \in C(X)$$

Then the same equality holds for all $f \in B(X)$. We will use this fact repeatedly in the following arguments.

Lemma 3.6.12. *Let $g \in B(X)$, and $\{\mu_1, \mu_2, \ldots, \mu_n\}$ be a finite collection of complex Borel regular measures on X. Then for all $\epsilon > 0, \exists f \in C(X)$ such that*

$$\int_X |f - g| d\mu_i < \epsilon \quad \forall 1 \leq i \leq n$$

Proof. Let $\nu := |\mu_1| + |\mu_2| + \ldots + |\mu_n|$, then ν is a positive Borel measure. By Lusin's theorem, $\exists f \in C(X)$ such that $\|f\|_\infty \leq \|g\|_\infty$ and if

$$N := \{x \in X : f(x) \neq g(x)\} \Rightarrow \mu(N) < \epsilon$$

Hence, $|\mu_i(N)| \leq |\mu_i|(N) \leq \mu(N) < \epsilon$ for all $1 \leq i \leq n$. Hence,

$$\int_X |f - g| d\mu_i = \int_N |f - g| d\mu_i \leq 2\epsilon \|g\|_\infty$$

This is true for all $1 \leq i \leq n$, proving the result. $\square$

(End of Day 28)

Theorem 3.6.13 (Existence part of Theorem 3.6.8)**.** *Let $\pi : C(X) \to \mathcal{B}(H)$ be a non-degenerate representation, we want to define a σ-representation $\widehat{\pi} : B(X) \to \mathcal{B}(H)$ which extends π.*

Proof.

1. Once again, fix $x, y \in H$ and consider

$$\Lambda_{x,y} : C(X) \to \mathbb{C} \text{ given by } f \mapsto \langle \pi(f)x, y\rangle$$

This is clearly a linear functional. Also, since $\|\pi(f)\| \leq \|f\|$, it follows that it is bounded and

$$\|\Lambda_{x,y}\| \leq \|x\|\|y\|$$

By the Riesz Representation theorem, $\exists$ a complex Borel measure $\mu_{x,y}$ on X such that

$$\int_X f d\mu_{x,y} = \langle \pi(f)x, y\rangle \quad \forall f \in C(X)$$

and

$$\|\mu_{x,y}\| \leq \|x\|\|y\|$$

Now since $\mu_{x,y}$ is a Borel measure, we may define

$$\int_X f d\mu_{x,y} \quad \forall f \in B(X)$$

2. For any $f \in B(X)$ fixed, define a map

$$\eta_f : H \times H \to \mathbb{C} \text{ given by } (x, y) \mapsto \int_X f d\mu_{x,y}$$

Then we claim: η_f is a sesqui-linear form

Proof. Given $x_1, x_2, y \in H$ and $\epsilon > 0$, then by Lemma 3.6.12, $\exists g \in C(X)$ such that
$$\int_X |f - g| d\mu < \epsilon \quad \forall \mu \in \{\mu_{x_1,y}, \mu_{x_2,y}, \mu_{x_1+x_2,y}\}$$
Hence,

$$|\eta_f(x_1 + x_2, y) - \eta_f(x_1, y) - \eta_f(x_2, y)| \leq 3\epsilon + |\eta_g(x_1 + x_2, y) - \eta_g(x_1, y) - \eta_g(x_2, y)|$$

But
$$\eta_g(x, y) = \langle \pi(g)x, y\rangle$$
and so the last term is zero, whence

$$|\eta_f(x_1 + x_2, y) - \eta_f(x_1, y) - \eta_f(x_2, y)| \leq 3\epsilon$$

This is true for all $\epsilon > 0$ and so

$$\eta_f(x_1 + x_2, y) = \eta_f(x_1, y) + \eta_f(x_2, y)$$

Similarly, one can prove the other conditions to ensure that η_f is a sesqui-linear form. □

3. Also, since $\|\mu_{x,y}\| \leq \|x\|\|y\|$, it follows that

$$|\eta_f(x, y)| = \left|\int_X f d\mu_{x,y}\right| \leq \|f\|\|x\|\|y\|$$

So by Theorem 2.1.2, $\exists T_f \in \mathcal{B}(H)$ such that

$$\eta_f(x, y) = \int_X f d\mu_{x,y} = \langle T_f(x), y\rangle$$

and
$$\|T_f\| \leq \|f\|$$
So we define
$$\widehat{\pi} : B(X) \to \mathcal{B}(H) \text{ by } f \mapsto T_f$$
and we claim that $\widehat{\pi}$ is a σ-representation. Suppose we prove this, then it is clear that $\widehat{\pi}$ extends π since for all $f \in C(X)$ and $x, y \in H$, we have

$$\langle \widehat{\pi}(f)x, y\rangle = \int_X f d\mu_{x,y} = \langle \pi(f)x, y\rangle$$

4. Note that by construction
$$\|\widehat{\pi}(f)\| \leq \|f\|_\infty$$

5. Claim : $\widehat{\pi}$ is linear

Proof. Given $f_1, f_2 \in B(X)$, we have

$$\eta_{f_1+f_2}(x,y) = \int_X (f_1+f_2)d\mu_{x,y} = \int_X f_1 d\mu_{x,y} + \int_X f_2 d\mu_{x,y} = \eta_{f_1}(x,y) + \eta_{f_2}(x,y)$$

Hence,

$$\langle T_{f_1+f_2}x, y\rangle = \langle T_{f_1}x, y\rangle + \langle T_{f_2}x, y\rangle$$

and so $\widehat{\pi}(f_1+f_2) = \widehat{\pi}(f_1) + \widehat{\pi}(f_2)$

Similarly,

$$\widehat{\pi}(\alpha f) = \alpha\widehat{\pi}(f) \quad \forall f \in B(X), \alpha \in \mathbb{C}$$

□

6. Claim: $\widehat{\pi}(\overline{f}) = \widehat{\pi}(f)^*$

   Proof. a) If $f \in C(X)$ is a positive function, then $\exists h \in C(X)$ such that $h\overline{h} = f$, and so for any $x \in H$

   $$\langle \pi(f)x, x\rangle = \langle \pi(h)x, \pi(h)x\rangle \geq 0$$

   Hence,

   $$\int_X f d\mu_{x,x} \geq 0$$

   and so $\mu_{x,x}$ is a positive measure (by the Riesz Representation theorem).

   b) Thus, if $f = \overline{f}$, then

   $$\langle T_f x, x\rangle = \int_X f d\mu_{x,x} \in \mathbb{R} \quad \forall x \in H$$

   By Theorem 2.1.8, $T_f = T_f^*$.

   c) Now for any $f \in B(X)$, write $f = g + ih$ where g, h are real-valued. Hence,

   $$T_f = T_g + iT_h$$

   so

   $$T_f^* = T_g^* - iT_h^* = T_g - iT_h = T_{\overline{f}}$$

   Hence, $\widehat{\pi}(\overline{f}) = \widehat{\pi}(f)^*$

□

7. Claim:

   $$\widehat{\pi}(fg) = \widehat{\pi}(f)\widehat{\pi}(g) \quad \forall f, g \in B(X)$$

Proof. a) Note that if $f, g \in C(X)$, then $\widehat{\pi}(fg) = \widehat{\pi}(f)\widehat{\pi}(g)$ holds since $\widehat{\pi}$ is an extension of π. Now recall that: if μ, λ are two complex measures on X such that

$$\int_X f d\mu = \int_X f d\lambda \quad \forall f \in C(X)$$

Then the same equality is true for all $f \in B(X)$.

b) For any $f, h \in C(X)$, and $x, y \in H$ fixed

$$\int_X fh d\mu_{x,y} = \langle \pi(fh)x, y\rangle = \langle \pi(f)\pi(h)x, y\rangle = \int_X f d\mu_{\pi(h)x,y}$$

Thus,

$$\int_X fh d\mu_{x,y} = \int_X f d\mu_{\pi(h)x,y} \quad \forall f \in B(X)$$

In other words, $\forall f \in B(X), h \in C(X)$

$$\langle \widehat{\pi}(fh)x, y\rangle = \langle \widehat{\pi}(f)\widehat{\pi}(h)x, y\rangle$$

and so $\widehat{\pi}(fh) = \widehat{\pi}(f)\widehat{\pi}(h)$.

c) Now for $f \in B(X), g \in C(X)$, and $x, y \in H$ fixed

$$\begin{aligned}
\int_X gf d\mu_{x,y} = \int_X fg d\mu_{x,y} &= \langle \widehat{\pi}(fg)x, y\rangle \\
&= \langle \widehat{\pi}(f)\widehat{\pi}(g)x, y\rangle \quad \text{by (b)} \\
&= \langle \widehat{\pi}(g)x, \widehat{\pi}(f)^* y\rangle \\
&= \langle \widehat{\pi}(g)x, \widehat{\pi}(\overline{f})y\rangle \quad \text{by (6)} \\
&= \int_X g d\mu_{x,\widehat{\pi}(\overline{f})y}
\end{aligned}$$

Again by the uniqueness part of the Riesz Representation Theorem, it follows that

$$\int_X gf d\mu_{x,y} = \int_X g d\mu_{x,\widehat{\pi}(\overline{f})y} \quad \forall g \in B(X)$$

In other words, for all $f, g \in B(X)$

$$\langle \widehat{\pi}(fg)x, y\rangle = \langle \widehat{\pi}(gf)x, y\rangle = \langle \widehat{\pi}(g)x, \widehat{\pi}(\overline{f})y\rangle = \langle \widehat{\pi}(f)\widehat{\pi}(g)x, y\rangle$$

Hence, $\widehat{\pi}(fg) = \widehat{\pi}(f)\widehat{\pi}(g)$ as required.

□

8. Claim: $\widehat{\pi}$ is a σ-representation

Proof. Suppose $(f_n) \in B(X)$ is a uniformly bounded sequence such that $f_n \to 0$ pointwise. By Lemma 3.5, it suffices to prove that

$$\widehat{\pi}(f_n) \xrightarrow{w} 0$$

So for any $x, y \in H$, the dominated convergence theorem implies that

$$\langle \widehat{\pi}(f_n)x, y \rangle = \int_X f_n d\mu_{x,y} \to 0$$

Hence the result.

□

□

Corollary 3.6.14. *Let $T \in \mathcal{B}(H)$ be a normal operator, then there is a unique σ-representation*

$$\widehat{\Theta} : B(\sigma(T)) \to \mathcal{B}(H)$$

which extends the continuous functional calculus

$$\Theta : C(\sigma(T)) \to \mathcal{B}(H)$$

This is called the <u>*Borel Functional Calculus*</u> *and we again write*

$$f(T) := \widehat{\Theta}(f) \quad \forall f \in B(\sigma(T))$$

(End of Day 29)

Remark 3.6.15. If $T \in \mathcal{B}(H)$ is normal and $A \subset \mathcal{B}(H)$ is a C*-algebra containing T, then the continuous functional calculus gives a map

$$\Theta : C(\sigma(T)) \to C^*(T) \subset A$$

However, the range of the Borel functional calculus

$$\widehat{\Theta} : B(\sigma(T)) \to \mathcal{B}(H)$$

may not lie in A. For instance, suppose $H = L^2[0,1]$ and $T \in \mathcal{B}(H)$ given by

$$T(f)(x) := xf(x)$$

Then $\sigma(T) = [0,1]$, so if $A = C^*(T)$, then we get an isomorphism

$$C[0,1] \to A$$

In particular, A has no non-trivial projections. However, $B([0,1])$ has many projections (in fact, the linear span of projections is dense in $B([0,1])$). Thus, the range of $\widehat{\Theta}$ must be strictly larger than A.

3.7 Spectral Measures

Definition 3.7.1. Let $(X, \mathcal{M})$ be a measurable space and H a Hilbert space. A spectral measure (or a resolution of the identity) on X is a map

$$E : \mathcal{M} \to \mathcal{B}(H)$$

such that

1. $E(\emptyset) = 0, E(X) = I$
2. $E(\omega) = E(\omega)^2 = E(\omega)^*$ for all $\omega \in \mathcal{M}$
3. $E(\omega_1 \cap \omega_2) = E(\omega_1)E(\omega_2)$ for all $\omega_1, \omega_2 \in \mathcal{M}$
4. If $\{\omega_n\}$ are disjoint sets in $\mathcal{M}$, then

   $$E(\bigcup_{n=1}^{\infty} \omega_n) = \sum_{n=1}^{\infty} E(\omega_n)$$

   where the series converges strongly (See Definition 3.6.4). In other words, for all $x \in H$

   $$E(\bigcup_{n=1}^{\infty} \omega_n)(x) = \lim_{k\to\infty} \sum_{n=1}^{k} E(\omega_n)(x)$$

Remark 3.7.2. Let $E : \mathcal{M} \to \mathcal{B}(H)$ be a spectral measure.

1. If $\{\omega_n\}$ are disjoint sets in $\mathcal{M}$, then by condition (iii)

   $$E(\omega_i)E(\omega_j) = 0 \quad \forall i \neq j$$

   Hence, the $E(\omega_i)$'s are a family of mutually orthogonal projections. Condition (iv) implies that

   $$E(\bigcup_{n=1}^{\infty} \omega_n)(H) = \bigoplus_{n=1}^{\infty} E(\omega_n)(H)$$

2. If $x, y \in H$, then the map

   $$E_{x,y}(\omega) := \langle E(\omega)x, y\rangle$$

   defines a complex measure on X [Check!]

3. If $x = y$ above, then

   $$E_{x,x}(\omega) = \langle E(\omega)x, x\rangle = \langle E(\omega)^2 x, x\rangle = \|E(\omega)x\|^2 \geq 0$$

   and so $E_{x,x}$ is a positive measure.

4. Each measure $E_{x,y}$ is automatically regular [Rudin, Theorem 2.18]

Example 3.7.3. 1. Let $H = \mathbb{C}^n$ and $T \in \mathcal{B}(H)$ a normal operator. Write $X := \sigma(T) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and let $\mathcal{M} = \mathcal{P}(X)$. Define $E : \mathcal{M} \to \mathcal{B}(H)$ by

$$E(\{\lambda_i\}) := P_i$$

where P_i is the projection onto $\ker(T - \lambda_i I)$ (Exercise). Now, for any $x, y \in H$, $E_{x,y}$ is a measure. We consider

$$\int_X \lambda dE_{x,y} = \sum_{i=1}^n \langle \lambda_i E(\{\lambda_i\})(x), y\rangle$$

But

$$\sum_{i=1}^n \lambda_i E(\{\lambda_i\}) = \sum_{i=1}^n \lambda_i P_i = T$$

by Theorem 3.1.8. Hence

$$\int_X \lambda dE_{x,y} = \langle Tx, y\rangle$$

2. Let $(X, \mathcal{M})$ be a measurable space and μ be a positive Radon measure on X. Let $H := L^2(X, \mu)$ and define

$$E : \mathcal{M} \to \mathcal{B}(H) \text{ by } E(\omega) := M_{\chi_\omega}$$

Now for any $x, y \in H$, we have

$$E_{x,y}(\omega) = \langle M_{\chi_\omega} x, y\rangle = \int_X \chi_\omega x\overline{y} d\mu$$

Hence,

$$\int_X \chi_\omega dE_{x,y} = E_{x,y}(\omega) = \int_X \chi_\omega x\overline{y} d\mu$$

Hence, for any $f \in B(X)$,

$$\int_X f dE_{x,y} = \int_X f x\overline{y} d\mu = \langle M_f x, y\rangle$$

In particular,

$$\int_X \lambda dE_{x,y} = \langle M_\zeta x, y\rangle$$

where $\zeta(z) = z$

Theorem 3.7.4. *Let $X \subset \mathbb{C}$ compact, H a Hilbert space, and $\widehat{\pi} : B(X) \to \mathcal{B}(H)$ a non-degenerate σ-representation. Then the map*

$$E(\omega) := \widehat{\pi}(\chi_\omega)$$

defines a spectral measure on X. Furthermore, for any $f \in B(X)$ and any $x, y \in H$, we have

$$\int_X f dE_{x,y} = \langle \widehat{\pi}(f)x, y\rangle$$

Proof. Exercise. Use the same ideas as in Lemma 3.6.10. ☐

Definition 3.7.5. Suppose $\widehat{\pi}$ is a σ-represention of $B(X)$ and E is the corresponding spectral measure, then, for any $x, y \in H$ and any $f \in B(X)$, we have

$$\int_X f dE_{x,y} = \langle \widehat{\pi}(f)x, y\rangle \qquad (*)$$

Therefore, we write

$$\int_X f dE := \widehat{\pi}(f) \quad \forall f \in B(X)$$

Note: The symbol

$$\int_X f dE$$

does not have any intrinsic meaning since E is not a complex measure. It only means that $(*)$ holds for any $x, y \in H$.

Theorem 3.7.6. *Let $X \subset \mathbb{C}$ be compact and $\pi : C(X) \to \mathcal{B}(H)$ be a non-degenerate representation, then $\exists$ a unique spectral measure E on the Borel σ-algebra of X such that*

$$\pi(f) = \int_X f dE \quad \forall f \in C(X)$$

Proof. Theorem 3.6.8+Theorem 3.7.4 ☐

Theorem 3.7.7 (Spectral Theorem). *Let $T \in \mathcal{B}(H)$ be a normal operator, then $\exists$ a unique spectral measure E on $\sigma(T)$ such that*

$$T = \int_{\sigma(T)} \lambda dE$$

Proof. Apply Theorem 3.7.6 to the continuous functional calculus of T. ☐

(End of Day 30)

3.8 Compact Normal Operators

Lemma 3.8.1. *Let $T \in \mathcal{B}(H)$ be a normal operator with spectral measure E. For any $\lambda \in \mathbb{C}, \lambda$ is an eigen-value of T iff $E(\{\lambda\}) \neq 0$. Furthermore, in that case, $E(\{\lambda\})$ is the projection onto* $\ker(T - \lambda I)$.

Proof. Let $X := \sigma(T), \widehat{\pi} : B(X) \to \mathcal{B}(H)$ be the Borel functional calculus. Then, if $\zeta \in B(X)$ is the function $\zeta(z) = z$, then

$$T = \widehat{\pi}(\zeta)$$

Also if $\omega \subset X$ is a Borel set, then

$$E(\omega) = \widehat{\pi}(\chi_\omega)$$

Furthermore, for any $f \in B(X)$ and $x, y \in H$,

$$\int_X f dE_{x,y} = \langle \widehat{\pi}(f)x, y \rangle$$

1. Hence, if $P = E(\{\lambda\})$, then

$$TP = \widehat{\pi}(\zeta\chi_{\{\lambda\}})) = \widehat{\pi}(\lambda\chi_{\{\lambda\}}) = \lambda P$$

   Thus, if $P \neq 0$, then any element of $P(H)$ is an eigen-vector with eigen-value λ.

2. Conversely, suppose λ is an eigen-value with eigen-vector x, then we claim that

$$E(\{\lambda\})(x) = x$$

   which would imply that $E(\{\lambda\}) \neq 0$.

   a) Define

$$\Delta_n := \{z \in \mathbb{C} : |z - \lambda| \geq \frac{1}{n}\}$$

   Write $E_n := E(\Delta_n)$, then

$$E_n T = \widehat{\pi}(\chi_{\Delta_n}\zeta) = \widehat{\pi}(\zeta\chi_{\Delta_n}) = TE_n$$

   Hence,

$$(T - \lambda I)E_n(x) = E_n(T - \lambda I)x = 0$$

   But

$$\begin{aligned}
0 = \|(T - \lambda I)E_n(x)\|^2 &= \langle (T - \lambda I)E_n(x), (T - \lambda I)E_n(x) \rangle \\
&= \langle E_n^*(T - \lambda I)^*(T - \lambda I)E_n(x), x \rangle \\
&= \langle \widehat{\pi}(\chi_{\Delta_n}\overline{(\zeta - \lambda I)}(\zeta - \lambda I)\chi_{\Delta_n})x, x \rangle \\
&= \int_X \chi_{\Delta_n}|z - \lambda|^2 dE_{x,x} \\
&\geq \frac{1}{n^2}\int_X \chi_{\Delta_n} dE_{x,x} \\
&= \frac{1}{n^2}\langle \widehat{\pi}(\chi_{\Delta_n})x, x \rangle \\
&= \frac{1}{n^2}\langle E_n(x), x \rangle \\
&= \frac{1}{n^2}\langle E_n(x), E_n(x) \rangle = \frac{1}{n^2}\|E_n(x)\|^2
\end{aligned}$$

   Hence, $E(\Delta_n)(x) = E_n(x) = 0$ for all $n \in \mathbb{N}$

b) Now observe that if

$$\Delta_n \subset \Delta_{n+1}$$

and

$$\Delta := \sigma(T) \setminus \{\lambda\} = \bigcup_{n=1}^{\infty} \Delta_n$$

Since $\widehat{\pi}$ is a σ-representation, it follows that

$$E(\Delta)(x) = \lim E(\Delta_n)(x) = 0$$

But then

$$x = E(X)(x) = E(\{\lambda\})(x) + E(\Delta)(x) = E(\{\lambda\})(x)$$

Hence, $E(\{\lambda\}) \neq 0$ as required.

3. Finally, we observe from the proof that $E(\{\lambda\})(x) = x$ iff $x \in \ker(T - \lambda I)$ as required.

□

Lemma 3.8.2. *Let $T \in \mathcal{B}(H)$ be a normal operator with spectral measure E. Then T is compact iff*

$$P_\epsilon := E(\{z \in \sigma(T) : |z| > \epsilon\})$$

is a finite rank projection for each $\epsilon > 0$

Proof. 1. Let $X := \sigma(T)$, and set

$$B_\epsilon := \{z \in X : |z| > \epsilon\} \text{ and } F_\epsilon = X \setminus B_\epsilon$$

then

$$T - TP_\epsilon = \int_X \lambda dE - \int_X \lambda \chi_{B_\epsilon}(\lambda) dE = \int_X \lambda \chi_{F_\epsilon} dE = f(T)$$

where $f(z) = z\chi_{F_\epsilon}(z)$. Hence,

$$\|T - TP_\epsilon\| = \|f\|_\infty = \sup\{|z| : z \in F_\epsilon\} \leq \epsilon$$

So, if P_ϵ has finite rank for all $\epsilon > 0$, then $TP_\epsilon \in \mathcal{K}(H)$, so T is a limit of finite rank operators. Hence, $T \in \mathcal{K}(H)$

2. Suppose T is compact, then define

$$g(z) := \frac{1}{z}\chi_{B_\epsilon}(z)$$

Then $g \in B(X)$ and $g(z)z = \chi_{B_\epsilon}(z)$. Hence,

$$g(T)T = P_\epsilon \in \mathcal{K}(H)$$

But P_ϵ is a projection, so P_ϵ has finite rank (Why?)

□

Theorem 3.8.3 (Spectral Theorem for Compact Normal Operators)**.** *Let $T \in \mathcal{K}(H)$ be a compact normal operator. Then*

1. *$\sigma(N)$ is either finite, or is a countable set with 0 as the only limit point.*

   Write $\sigma(N) \setminus \{0\} = \{\lambda_1, \lambda_2, \ldots\}$, and set

   $$H_k := \ker(T - \lambda_k I)$$

   and let E_k be the projection onto H_k. Then

2. *Each H_k is non-zero, finite dimensional, and mutually orthogonal.*

3. $$T = \sum_{k=1}^{\infty} \lambda_k E_k$$

   where the series converges in the operator norm.

Proof. 1. Fix $\epsilon > 0$, and consider

$$B_\epsilon := \{z \in \sigma(T) : |z| > \epsilon\}$$

By Exercise 1 of section 3.4, every element of B_ϵ is an eigen-value. Furthermore, if $\lambda, \mu \in B_\epsilon$ are distinct, then by Lemma 3.8.1 and Lemma 3.1.7,

$$E(\{\lambda\}) \perp E(\{\mu\})$$

Hence, if $P_\epsilon = E(B_\epsilon)$ as above, then P_ϵ has finite rank by Lemma 3.8.2. Together, these facts imply (why?) that B_ϵ is a finite set (In fact, $|B_\epsilon| \leq \text{rank}(P_\epsilon)$).

2. Fix λ_k, then each H_k is non-zero because each such λ_k is an eigen-value. For $\epsilon > |\lambda_k|$, P_ϵ has finite rank. But

$$P_\epsilon E_k = E(B_\epsilon)E(\{\lambda_k\}) = E(B_\epsilon \cap \{\lambda_k\}) = E(\{\lambda_k\}) = E_k$$

So $H_k = E_k(H) = P_\epsilon(E_k(H)) \subset P_\epsilon(H)$ is finite dimensional. Finally, the H_k are mutually orthogonal by Lemma 3.1.7.

3. We WTS that

$$T = \sum_{k=1}^{\infty} \lambda_k E_k$$

Suppose that $\sigma(T)$ is infinite, as the finite case is similar (easier). By part (1),

$$\sigma(T) = \{0, \lambda_1, \lambda_2, \ldots\}$$

where λ_k are a sequence of non-zero complex numbers converging to 0. Since each B_ϵ is finite, each λ_k is an isolated point of $\sigma(T)$. Hence,

$$\chi_{\{\lambda_k\}} \in C(\sigma(T))$$

and furthermore,

$$E_k = \chi_{\{\lambda_k\}}(T)$$

Define

$$s_n := \sum_{k=1}^n \lambda_k \chi_{\{\lambda_k\}}$$

Then $s_n(z) = z$ for all $z \in \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, so

$$\|\zeta - s_n\|_\infty \leq \sup_{k>n} |\lambda_k|$$

But this term converges to 0 by hypothesis. Hence, $s_n \to \zeta$ in the sup norm. Therefore,

$$T = \lim s_n(T) = \sum_{k=1}^\infty \lambda_k E_k$$

and the sum converges in the norm topology.

□

Remark 3.8.4. If $T \in \mathcal{B}(H)$ is a normal operator with countable spectrum $\sigma(T) = \{\lambda_1, \lambda_2, \ldots\}$, then for any $x, y \in H$, we have

$$\langle Tx, y\rangle = \int_{\sigma(T)} \lambda dE_{x,y} = \sum_{i=1}^\infty \lambda_i E_{x,y}(\{\lambda_i\}) = \sum_{i=1}^\infty \lambda_i \langle E(\{\lambda_i)(x), y\rangle$$

Hence,

$$T = \sum_{i=1}^\infty \lambda_i E(\{\lambda_i\})$$

where the convergence is in the weak operator topology. However, in the above theorem, because T is compact, we get norm convergence because the spectral values converge to 0.

(End of Day 31)

Theorem 3.8.5. *Let H be a separable Hilbert space and $J \lhd \mathcal{B}(H)$ be a two-sided ideal that contains a non-compact operator, then $J = \mathcal{B}(H)$*

Proof. Let $A \in I$ be non-compact, then $T := A^*A \in J$ is normal. Furthermore, T is not compact by Corollary 2.5.5. By Lemma 3.8.2, $\exists \epsilon > 0$ such that P_ϵ has infinite rank. Furthermore, in the proof, we saw that $\exists S \in \mathcal{B}(H)$ such that

$$P_\epsilon = ST$$

Hence, $P_\epsilon \in J$. Let $M := P_\epsilon(H)$, then M is a closed subspace of H and

$$\dim(H) = \dim(H) = \aleph_0$$

Hence, there is a surjective isometry $U : H \to M$, so that

$$I = U^* P_\epsilon U$$

Since $P_\epsilon \in J$, it follows that $I \in J$, so $J = \mathcal{B}(H)$. □

Definition 3.8.6. Let H be a Hilbert space and $x, y \in H$. Define $\Theta_{x,y} \in \mathcal{B}(H)$ by

$$\Theta_{x,y}(z) := \langle z, x\rangle y$$

Then $\Theta_{x,y}$ is a rank one operator.

Lemma 3.8.7. *Every finite rank operator is a linear combination of these* $\Theta_{x,y}$

Proof. Exercise $\square$

Theorem 3.8.8. *If H is a separable Hilbert space, then the only non-trivial closed, two-sided ideal of $\mathcal{B}(H)$ is $\mathcal{K}(H)$*

Proof. Let $J \neq \{0\}$ be a closed ideal, then by Theorem 3.8.5, it suffices to show that $\mathcal{K}(H) \subset J$. Choose $T \in J$ non-zero, then $\exists x_0, x_1 \in H$ such that $T(x_0) = x_1 \neq 0$. For any $y_0, y_1 \in H$ of norm 1, consider two operators

$$A := \Theta_{y_0,x_0} \text{ and } B := \Theta_{x_1,y_1}$$

Then for any $z \in H$,

$$\begin{aligned} BTA(z) &= BT(\langle z, y_0\rangle x_0) = \langle z, y_0\rangle BT(x_0) \\ &= \langle z, y_0\rangle B(x_1) = \langle z, y_0\rangle y_1 \\ &= \Theta_{y_0,y_1}(z) \end{aligned}$$

Hence, every $\Theta_{x,y}$ belongs to J. By the previous lemma, all finite rank operators belong to J. Since J is closed, it follows that $\mathcal{K}(H) \subset J$. $\square$

3.9 Exercises

1. Let X be a compact Hausdorff space, and H a Hilbert space, and suppose $\pi : C(X) \to \mathcal{B}(H)$ is a non-degenerate representation. Let $\mathbf{1}$ denote the constant function 1, then prove that
   a) $\pi(\mathbf{1})$ is a projection in $\mathcal{B}(H)$
   b) Conclude that $\pi(\mathbf{1}) = I$
2. Let $T \in \mathcal{B}(H)$ be a normal operator.
   a) If $\sigma(T)$ is finite, then show that T is a linear combination of projections.
   b) If $\sigma(T)$ is a singleton, then show that T is a scalar multiple of the identity.
3. Show that an element $U \in \mathcal{B}(H)$ is unitary iff there is a self-adjoint operator $T \in \mathcal{B}(H)$ such that $U = e^{iT}$
4. Show that the space $U(H)$ of unitary operators on H is path connected.

5. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $H = L^2(X, \mu)$ and let $\varphi \in L^\infty(X, \mu)$ be fixed. We define $E : \mathcal{M} \to \mathcal{B}(H)$ by

$$E(\omega) := M_{\chi_{\varphi^{-1}(\omega)}}$$

Then prove that

a) E is a spectral measure on X

b)

$$M_\varphi = \int_X \lambda dE$$

6. Let $U : H_0 \to H$ be a unitary, and $(X, \mathcal{M})$ a measurable space. Suppose

$$\tilde{E} : \mathcal{M} \to \mathcal{B}(H_0)$$

is a spectral measure, then define

$$E : \mathcal{M} \to \mathcal{B}(H) \text{ by } \omega \mapsto U\tilde{E}(\omega)U^{-1}$$

a) Prove that E defines a spectral measure on X

b) If $x, y \in H$, set $\tilde{x} = U^{-1}(x), \tilde{y} = U^{-1}(y)$, then prove that for any $f \in B(X)$,

$$\int_X f dE_{x,y} = \int_X f d\tilde{E}_{\tilde{x},\tilde{y}}$$

7. Let $\pi : C(X) \to \mathcal{B}(H)$ be a non-degenerate representation and E be the associated spectral measure from Theorem 3.7.6. Prove that π is injective iff $E(\omega) \neq 0$ for all open $\omega \subset X$.

[*Hint:* Use Remark 3.7.2 and Corollary 2.1.10]

8. Let $T \in \mathcal{B}(H)$ be a normal operator with spectral measure E on $\sigma(T)$. Show that, for any Borel set $\omega \subset \sigma(T)$,
$$E(\omega)T = TE(\omega)$$
Conclude that, if $\dim(H) > 1$, then T has a non-trivial invariant subspace.

9. Let $T \in \mathcal{B}(H)$ be a normal operator with spectral measure E. Suppose that T satisfies the following properties:

a) $\sigma(T)$ is a countable set with 0 as the unique limit point.

b) For each $\lambda \in \sigma(T) \setminus \{0\}, E(\{\lambda\})$ is a finite rank projection.

Show that T is a compact operator.

4 Additional Topics

4.1 Quotients of C^* algebras

Remark 4.1.1. Recall that if A is a C*-algebra and $I \lhd A$ is a closed two-sided ideal, then A/I is a Banach algebra (Theorem 1.1.6) with the quotient norm

$$\|a + I\| = \inf\{\|a + b\| : b \in I\}$$

We now wish to define an involution on A/I by

$$(a + I)^* := a^* + I$$

and show that A/I is a C*-algebra with this involution and norm.

Lemma 4.1.2. *Let A be a C^*-algebra and $I \lhd A$ be a closed ideal in A. For any $a \in I, \exists$ a sequence of self-adjoint $e_n \in I$ such that $\sigma(e_n) \subset [0, 1]$ and*

$$\lim_{n\to\infty} \|a - ae_n\| = 0$$

Proof. By adjoining a unit to A if need be, we assume WLOG that A is unital.

1. Suppose $a = a^*$, then $\sigma(a) \subset \mathbb{R}$ so if

$$f_n(t) := \frac{nt^2}{1 + nt^2}$$

then $f_n \in C(\sigma(a))$. Hence, we may define $e_n := f_n(a)$.

a) Since $f_n = \overline{f_n}$, $e_n = e_n^*$

b) Now $f_n \in C(\sigma(a))$ is a limit of polynomials in $p_{n,k}(a)$. Furthermore, since $f_n(0) = 0$, we may choose these polynomials $p_{n,k}$ such that $p_{n,k}(0) = 0$ (See also Corollary 2.5.5). Hence, $p_{n,k}(a) \in I$ for all k, so that $e_n \in I$ because I is closed. Since the RHS is a limit of polynomials in a, $e_n \in I$ since I is closed.

c) Since $f_n(t) \in [0, 1]$ for all $t \in \sigma(a)$, it follows from the spectral mapping theorem, that $\sigma(e_n) \subset [0, 1]$. In particular,

$$\sigma(1 - e_n) \subset [0, 1]$$

and so $\|1 - e_n\|$ and $\|e_n\| \leq 1$

d) Hence

$$\|a - ae_n\|^2 = \|a(1-e_n)\|^2 = \|(1-e_n)a^2(1-e_n)\| \leq \|a^2(1-e_n)\|$$

Now, $g_n(t) := 1 + nt^2 \in C(\sigma(a))$ is an invertible function because $\sigma(a) \subset \mathbb{R}$. Hence, $(1+na^2)$ is invertible in A, and

$$\|a^2(1-e_n)\| = \|a^2(1+na^2)^{-1}\| = \frac{1}{n}\|na^2(1+na^2)^{-1}\| = \frac{1}{n}\|1-e_n\| \leq \frac{1}{n}$$

and so $\|ae_n - a\| \to 0$

2. Now if a is not self-adjoint, let $b := a^*a \in I$. By part (i), there is sequence e_n of self-adjoint elements such that $\sigma(e_n) \subset [0,1]$ and

$$\|a^*a(1-e_n)\| \to 0$$

Hence,

$$\|ae_n - a\|^2 = \|(1-e_n)a^*a(1-e_n)\| \leq \|a^*a(1-e_n)\| \to 0$$

□

(End of Day 32)

Example 4.1.3. 1. Let $A = \mathcal{B}(\ell^2)$ and $I = \mathcal{K}(\ell^2)$, and let E_n be the projection onto the subspace spanned by $\{e_1, e_2, \ldots, e_n\}$. Then for any $T \in I$,

$$\|T - TE_n\| \to 0$$

2. Let $A = C[0,1]$ and $I = C_0((0,1/2))$, then we may choose $e_n \in I$ such that $e_n(x) = 1$ if $1/n \leq x \leq 1/2 - 1/n$ and $0 \leq e_n \leq 1$. Then for any $f \in I$,

$$\|f - fe_n\| \to 0$$

Proof. Exercise. □

Corollary 4.1.4. *Let A be a C^*-algebra and $I \lhd A$ a closed ideal. Then for any $a \in A$*

$$\|a + I\| = \inf\{\|a - ax\| : x \in I, x = x^*, \sigma(x) \subset [0,1]\}$$

Proof. Fix $a \in A$ and recall that

$$\|a + I\| = \inf\{\|a - b\| : b \in I\}$$

Let $E = \{x \in I, x = x^*, \sigma(x) \subset [0,1]\}$, then since $ax \in I$ for each $x \in E$, it follows that

$$\|a + I\| \leq \beta := \inf\{\|a - ax\| : x \in E\}$$

Now suppose $b \in I$, then choose $e_n \in I$ such that $\|b - be_n\| \to 0$, then since $\|(1-e_n)\| \leq 1$, we have

$$\|a+b\| \geq \|(a+b)(1-e_n)\| = \|(a-ae_n)-(be_n-b)\| \geq \|a-ae_n\| - \|be_n-b\| \geq \beta - \|be_n - b\|$$

Taking limit, we see that

$$\|a + b\| \geq \beta$$

This is true for all $b \in I$, so $\|a + I\| = \beta$ as required. □

Corollary 4.1.5. *Let A be a C^*-algebra and $I \lhd A$ a closed ideal in A. For any $a \in I, a^* \in I$*

Note: This shows that if I is closed, then the second half of Definition 2.1.21 is redundant.

Proof. Let $a \in I$, choose $e_n \in I$ self-adjoint such that $\|ae_n - a\| \to 0$, then

$$(ae_n)^* = e_n a^* \to a^*$$

since the map $a \mapsto a^*$ is continuous (Lemma 2.1.16). Since $e_n \in I$, it follows that $e_n a^* \in I$ for all n. Since I is closed, $a^* \in I$. □

Theorem 4.1.6. *Let A be a C^*-algebra and $I \lhd A$ a closed ideal in A. The involution*

$$(a + I)^* := a^* + I$$

is well-defined, and A/I is a C^*-algebra with respect to this involution and the norm as above.

Proof. By Remark 2.1.14, it suffices to check that

$$\|a + I\|^2 \leq \|a^*a + I\|$$

But by Corollary 4.1.4,

$$\|a + I\|^2 = \inf\{\|a - ax\|^2 : x \in I, x = x^*, \sigma(x) \subset [0, 1]\}$$

and for any x as above $\|1 - x\| \leq 1$, so

$$\|a - ax\|^2 = \|(a - ax)^*(a - ax)\| = \|(1 - x)a^*a(1 - x)\| \leq \|a^*a(1 - x)\|$$

Taking infimum, we get the required inequality. □

Example 4.1.7. The Calkin algebra is defined as

$$\mathcal{Q}(H) := \mathcal{B}(H)/\mathcal{K}(H)$$

By Theorem 4.1.6, $\mathcal{Q}(H)$ is a C^*-algebra.

Theorem 4.1.8. *Let $\varphi : A \to B$ be a $*$-homomorphism, then*

1. $\ker(\varphi) \lhd A$ *is a closed ideal*
2. *Consider the quotient map $\pi : A \to A/\ker(\varphi)$, then the induced map*

   $$\overline{\varphi} : A/\ker(\varphi) \to B$$

   given by Theorem 1.1.9 is an isometric $*$-isomorphism from $A/\ker(\varphi)$ to $\varphi(A)$
3. *$\varphi(A)$ is a C^*-subalgebra of B*

Proof. 1. $\ker(\varphi)$ is clearly an ideal, and $\ker(\varphi) = \varphi^{-1}(\{0\})$ is closed since φ continuous.

2. Consider the map $\overline{\varphi} : A/\ker(\varphi) \to B$ given by

$$a + I \mapsto \varphi(a)$$

Then it clearly a $*$-homomorphism that must be injective by Theorem 1.1.9. By Theorem 2.2.11, this implies that $\overline{\varphi}$ is isometric.

3. Hence, $\varphi(A) = \overline{\varphi}(A/I)$ must be a closed C^*-subalgebra of B since it is the isometric image of a C^*-algebra [Check!]

□

4.2 Positive Linear Functionals

Let A be a C*-algebra.

Remark 4.2.1. 1. Define

$$A_+ := \{a \in A : a = a^*, \sigma(a) \subset [0, \infty)\}$$

Note that, by Theorem 2.3.11, $A_+ \subset \{b^*b : b \in A\}$. We will soon show that, in fact, these two sets are equal. Elements of A_+ are called positive elements in A.

2. Write A_{sa} for the set of all self-adjoint elements of A.

Lemma 4.2.2. *Let $a, b \in A_+$, then $a + b \in A_+$*

Proof. Note that $c \in A_+$ iff $\lambda c \in A_+$ for all $\lambda \in [0, \infty)$. Therefore, we may assume WLOG that $\|a\| \leq 1$ and $\|b\| \leq 1$.If $z := \frac{a+b}{2}$, then WTS: $z \in A_+$. Note that z is self-adjoint. Since $a \in A_+$ and $\|a\| \leq 1$, it follows by functional calculus that

$$\sigma(1 - a) \subset [-1, 1] \Rightarrow \|1 - a\| = r(1 - a) \leq 1$$

Similarly, $\|1 - b\| \leq 1$, so

$$\|1 - z\| = \left\| \frac{(1 - a) + (1 - b)}{2} \right\| \leq \frac{1}{2} + \frac{1}{2} = 1$$

Hence, if $t \in \sigma(z)$, then $|1 - t| \leq 1$. Since $t \in \mathbb{R}$, this implies $t \geq 0$, so

$$\sigma(z) \subset [0, \infty)$$

as required. □

Lemma 4.2.3. *Let $a \in A$, such that $-a^*a \in A_+$, then $a = 0$*

Proof. 1. We first show that if $a, b \in A$, then $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$: Suppose $\lambda \in\notin \sigma(ab) \cup \{0\}$, then by rescaling, we may assume that $\lambda = 1$. Now, $1 - ab$ is invertible, with inverse c, say. Then $c - 1 = cab$. Define $x := 1 + bca$, then

$$\begin{aligned} x(ba-1) = (1+bca)ba - x = ba + b(cab)a - x &= ba + b(c-1)a - x \\ &= ba + bca - ba - x = bca - x = 1 \end{aligned}$$

Similarly, $(ba - 1)x = 1$, and we are done.

2. Now suppose $-a^*a \in A_+$, then $\sigma(a^*a) \subset (-\infty, 0]$. By part (i),

$$\sigma(aa^*) \subset (-\infty, 0]$$

Hence, $-aa^* \in A_+$, so $-(a^*a + aa^*) \in A_+$, whence

$$\sigma(a^*a + aa^*) \subset (-\infty, 0]$$

Write $a = b + ic$, where $b, c \in A_{sa}$, then

$$a^*a + aa^* = (b - ic)(b + ic) + (b + ic)(b - ic) = 2b^2 + 2c^2$$

Hence, $-(b^2 + c^2) \in A_+$. But $c^2 \in A_+$, so by the previous lemma,

$$-b^2 \in A_+ \Rightarrow \sigma(b^2) \subset (-\infty, 0]$$

But b is self-adjoint, so $\sigma(b^2) \subset [0, \infty)$, so this implies

$$\sigma(b^2) \subset \{0\}$$

But this implies $\|b\| = r(b) = 0$, so $b = 0$. Similarly, $c = 0$, so $a = 0$ as required.

□

Theorem 4.2.4. *For $a \in A$, TFAE:*

1. $a = a^*$ *and* $\sigma(a) \subset [0, \infty)$
2. $\exists b \in A$ *such that* $a = b^*b$
3. $\exists c \in A_{sa}$ *such that* $a = c^2$

Proof. (i)⇒ (ii) follows by Theorem 2.3.11, and (iii)⇒ (i) follows by the spectral mapping theorem, so it suffices to prove (ii)⇒ (iii): If $a = b^*b$, then $a = a^*$, so $\sigma(a) \subset \mathbb{R}$. Define $f, g : \sigma(a) \to \mathbb{R}$ by

$$f(t) := \begin{cases} \sqrt{t} & : t \geq 0 \\ 0 & : t < 0 \end{cases} \text{ and } g(t) := \begin{cases} 0 & : t \geq 0 \\ \sqrt{-t} & : t < 0 \end{cases}$$

Let $x := f(a), y := g(a)$, then x and y are self-adjoint. Furthermore, by the functional calculus,

$$a = x^2 - y^2$$

Furthermore $f(t)g(t) = 0$, so $xy = yx = 0$, so that

$$(by)^*(by) = yb^*by = yay = yx^2y - y^4 = -y^4$$

By Lemma 4.2.3, it follows that $y^4 = 0$. By the continuous functional calculus, this implies $y = (y^4)^{1/4} = 0$. Hence, $a = x^2$ as required. $\square$

Definition 4.2.5. A linear functional $\tau : A \to \mathbb{C}$ is said to be positive if $\tau(a) \geq 0$ for all $a \in A_+$.

Example 4.2.6.
1. Let X be a compact Hausdorff space and $A = C(X)$. If μ is a positive Borel measure on X, then $\tau : A \to \mathbb{C}$ given by

$$f \mapsto \int_X f d\mu$$

is a positive linear functional. By the Riesz representation theorem, these are all the positive linear functionals on $C(X)$.

2. For instance, if $x_0 \in X$, then $f \mapsto f(x_0)$ is a positive linear functional.
3. If $A = \mathcal{B}(H)$ and $x \in H$, then $\tau : A \to \mathbb{C}$ given by

$$\tau(T) := \langle Tx, x\rangle$$

is a positive linear functional.

4. If $A = M_n(\mathbb{C})$, then the trace is a positive linear functional on A because

$$Tr(T) = \sum_{i=1}^n \langle T(e_i), e_i\rangle$$

(End of Day 33)

Definition 4.2.7. Let $a \in A_{sa}$, then $\sigma(a) \subset \mathbb{R}$, so define $f : \sigma(a) \to \mathbb{R}$ by

$$f(t) = \begin{cases} t & : t \geq 0 \\ 0 & : t \leq 0 \end{cases}$$

Then $f \in C(\sigma(a))$, so we define

$$a_+ := f(a)$$

Similarly, we define $a_- := g(a)$, where

$$g(t) = \begin{cases} 0 & : t \geq 0 \\ -t & : t \leq 0 \end{cases}$$

Note that $a_+, a_- \in A_+$ and

$$a = a_+ - a_-$$

Furthermore, $\|a_+\| = \|f(a)\| = \|f\|_\infty \leq \|a\|$, and similarly, $\|a_-\| \leq \|a\|$.

Lemma 4.2.8. *Let $S := \{a \in A_+ : \|a\| \leq 1\}$. Suppose τ is a linear functional such that τ is bounded on S, then τ is bounded.*

Proof. For any $a \in A$ with $\|a\| \leq 1$, consider

$$b := \frac{a + a^*}{2} \text{ and } c := \frac{a - a^*}{2i}$$

Then $b, c \in A_{sa}$ and $a = b + ic$. Furthermore, $\|b\|, \|c\| \leq 1$. Then $b_+, b_-, c_+, c_- \in S$, so if $M \geq 0$ such that

$$|\tau(x)| \leq M \quad \forall x \in S$$

we have

$$|\tau(a)| \leq |\tau(b)| + |\tau(c)| \leq |\tau(b_+)| + |\tau(b_-)| + |\tau(c_+)| + |\tau(c_-)| \leq 4M$$

Hence, τ is bounded and $\|\tau\| \leq 4M$. $\square$

Theorem 4.2.9. *Every positive linear functional on A is bounded.*

Proof. By the above lemma, it suffices to show that τ is bounded on $S := \{a \in A_+ : \|a\| \leq 1\}$. Suppose not, then for each $n \in \mathbb{N}, \exists a_n \in S$ such that

$$|\tau(a_n)| = \tau(a_n) \geq 4^n$$

Consider the series

$$\sum_{n=1}^{\infty} \frac{a_n}{2^n}$$

Then the series converges absolutely, so it converges to a point a. Now, note that

$$a - \frac{a_n}{2^n} = \lim_{\ell \to \infty} \sum_{k \neq n}^{\ell} \frac{a_k}{2^k}$$

Each term of the limit is in A_+ by Lemma 4.2.2. Thus, $a - a_n/2^n \in A_+$, so

$$a \geq a_n/2^n \Rightarrow \tau(a) \geq \tau(a_n)/2^n \geq 2^n$$

This implies that $\tau(a) \notin \mathbb{R}$, which is impossible. $\square$

Theorem 4.2.10. *Let A be a unital C^*-algebra and τ a positive linear functional on A. Then*

1. *For any $a, b \in A$,*
$$|\tau(b^* a)| \leq \tau(a^* a)^{1/2} \tau(b^* b)^{1/2}$$
2. $\tau(a^*) = \overline{\tau(a)}$
3. $|\tau(a)|^2 \leq \|\tau\| \tau(a^* a)$

4. The set

$$N_\tau := \{a \in A : \tau(a^*a) = 0\}$$

is a closed left-ideal of A.

Proof. 1. The map $u : A \times A \to \mathbb{C}$ given by

$$(a, b) \mapsto \tau(b^*a)$$

is a bounded sesqui-linear form. So the result follows from the Cauchy-Schwartz inequality.

2. If $a \in A_{sa}$, then $a = a_+ - a_-$, so $\tau(a) = \tau(a_+) - \tau(a_-) \in \mathbb{R}$. Hence, if $a \in A$, we write $a = b + ic$ for $b, c \in A_{sa}$. Then

$$a^* = b - ic$$

so $\tau(a^*) = \tau(b) - i\tau(c) = \overline{\tau(b) + i\tau(c)} = \overline{\tau(a)}$

3. If $a \in A$, then by part (i)

$$|\tau(a)|^2 = \tau(1^*a)^2 \leq \tau(a^*a)\tau(1^*1)$$

But $\tau(1^*1) = \tau(1) \leq \|\tau\|$ since $\|1\| = 1$

4. If $a \in N_\tau$, then for any $x \in A$,

$$|\tau(xa)| \leq \tau(xx^*)^{1/2}\tau(a^*a) = 0$$

Hence, $\tau(xa) = 0$. Also, $\tau(a^*x) = \overline{\tau(x^*a)} = 0$. Hence, if $a, b \in N_\tau$, then

$$\tau((a + b)^*(a + b)) = \tau(a^*a + a^*b + b^*a + b^*b) = 0$$

Hence, $a+b \in N_\tau$. Hence, N_τ is a vector subspace of A. Furthermore, if $c \in A, a \in N_\tau$, then

$$\tau((ca)^*(ca)) = \tau(a^*c^*ca) = 0$$

Finally, observe that if $a_n \to a$, then $a_n^*a_n \to a^*a$. Since τ is continuous, we can conclude that N_τ is closed.

□

Example 4.2.11. 1. Let $\tau : \mathcal{B}(H) \to \mathbb{C}$ be given by $\tau(T) := \langle Te_1, e_1\rangle$. Then

$$N_\tau = \{T \in \mathcal{B}(H) : Te_1 = 0\}$$

This is a left-ideal, but not a right-ideal (Why?).

2. Let μ be a positive Borel measure on a compact Hausdorff space X and let τ be the positive linear functional

$$f \mapsto \int_X f d\mu$$

Then

$$N_\tau = \{f \in C(X) : f \equiv 0 \text{ a.e.}[\mu]\}$$

Theorem 4.2.12. *If τ is a bounded linear functional on a unital C^*-algebra A, then τ is positive iff $\|\tau\| = \tau(1)$*

Proof. 1. Suppose τ is positive, then for any $a \in A$, then for any $a \in A$,

$$|\tau(a)|^2 \leq \tau(a^*a)\tau(1)$$

If $\|a\| \leq 1$, then $\|a^*a\| = \|a\|^2 \leq 1$. Hence,

$$\sigma(a^*a) \subset [0,1] \Rightarrow a^*a \leq 1$$

Thus, $\tau(a^*a) \leq \tau(1)$. Hence,

$$|\tau(a)|^2 \leq \tau(1)^2 \Rightarrow |\tau(a)| \leq \tau(1)$$

This is true for all $a \in A$ such that $\|a\| \leq 1$, so

$$\|\tau\| \leq \tau(1)$$

But $\|1\| = 1$, so the reverse inequality holds as well.

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2. Conversely, suppose τ is a bounded linear functional such that $\|\tau\| = \tau(1)$, then WTS: τ is positive. We prove this in two steps. By scaling, we may assume that $\|\tau\| = \tau(1) = 1$.

   a) For any $a \in A_{sa}$, we show that $\tau(a) \in \mathbb{R}$. We may assume that $\|a\| \leq 1$. First write

   $$\tau(a) = \alpha + i\beta$$

   WTS: $\beta = 0$. Suppose not, then replacing a by $-a$ if necessary, we may assume that $\beta < 0$. For $n \in \mathbb{N}$,

   $$\begin{aligned} \|a - in\|^2 &= \|(a-in)^*(a-in)\| \\ &= \|(a+in)(a-in)\| = \|a^2+n^2\| \\ &\leq \|a^2\| + n^2 \end{aligned}$$

   Since $\|a\| \leq 1$, we have

   $$\|a - in\|^2 \leq 1 + n^2$$

Since $\tau(1) = 1$,

$$\begin{aligned}|\alpha + i\beta - in|^2 = |\tau(a - in)|^2 &\leq \|\tau\|\tau((a-in)^*(a-in)) \\ &\leq \|\tau\|^2\|a - in\|^2 \leq 1 + n^2\end{aligned}$$

Hence,

$$\alpha^2 + \beta^2 - 2n\beta + n^2 \leq 1 + n^2 \Rightarrow 2n\beta + 1 \geq \alpha^2 + \beta^2$$

This cannot happen if $\beta < 0$. This contradicts our assumption. Hence, $\beta = 0$ must hold, so $\tau(a) \in \mathbb{R}$ if $a \in A_{sa}$.

b) Now suppose $a \in A_+$, WTS: $\tau(a) \in \mathbb{R}_+$. Assume WLOG that $\|a\| \leq 1$, then by the previous lemma, $1 - a \leq 1$, so since $\tau(a) \in \mathbb{R}$,

$$1 - \tau(a) = \tau(1 - a) = |\tau(1 - a)| \leq \|\tau\|\|1 - a\| \leq 1$$

Hence, $\tau(a) \geq 0$

□

Definition 4.2.13. Let A be a C*-algebra. A state on A is a positive linear functional of norm 1. We write $S(A)$ for the set of all states on A.

Lemma 4.2.14. *Let A be a unital C*-algebra and $B \subset A$ be a sub-algebra such that $1_A \in B$. If $\tau : B \to \mathbb{C}$ a positive linear functional on A, then τ extends to a positive linear functional $\widetilde{\tau} : A \to \mathbb{C}$ such that $\|\widetilde{\tau}\| = \|\tau\|$*

Proof. By Hahn-Banach, there is an extension $\widetilde{\tau} : A \to \mathbb{C}$ such that $\|\tau\| = \|\widetilde{\tau}\|$. However,

$$\widetilde{\tau}(1_A) = \tau(1_A) = \|\tau\| = \|\widetilde{\tau}\|$$

so $\widetilde{\tau}$ is positive by Theorem 4.2.12. □

Theorem 4.2.15. *Let A be a C*-algebra and $a \in A$ be a normal element. Then $\exists \tau \in S(A)$ such that $|\tau(a)| = \|a\|$*

Proof. Consider the unitization $\widetilde{A}$ and think of A as an ideal of $\widetilde{A}$. Then a is normal in $\widetilde{A}$ and $\widetilde{A}$ is unital, so define

$$B := C^*(1_{\widetilde{A}}, a)$$

Then B is a commutative C*-algebra, so if $X := \Omega(B)$, there is an isometric $*$-isomorphism

$$\Gamma_B : B \to C(X) \text{ given by } b \mapsto \widehat{b}$$

In particular,

$$\|a\| = \|\widehat{a}\|_\infty = \sup_{\tau \in \Omega(B)} |\tau(a)|$$

Since $\Omega(B)$ is compact, $\exists \tau_1 \in \Omega(B)$ such that

$$|\tau_1(a)| = \|a\|$$

By Lemma 1.4.2,

$$\tau_1(1_{\widetilde{A}}) = \|\tau_1\| = 1$$

Therefore, τ_1 is positive. By Lemma 4.2.14, τ_1 extends to a state τ_2 on $\widetilde{A}$. Clearly,

$$|\tau_2(a)| = |\tau_2(a)| = \|a\|$$

Now we may restrict τ_2 to a linear functional τ on A. Clearly, τ is positive because τ_2 is positive. Furthermore, $\|\tau\| \leq \|\tau_2\| = 1$. However, $|\tau(a)| = \|a\|$, so $\|\tau\| \geq 1$, so $\tau \in S(A)$. $\square$

(End of Day 35)

4.3 The Gelfand-Naimark-Segal Construction

Remark 4.3.1. 1. Given a C*-algebra A and a representation $\varphi : A \to \mathcal{B}(H)$, we may use this to construct positive linear functionals on A: If $\zeta \in H$, define $\tau : A \to \mathbb{C}$ by

$$a \mapsto \langle \varphi(a)\zeta, \zeta\rangle \qquad (\dagger)$$

The Gelfand-Naimark-Segal (GNS) construction is a converse to this - given a state $\tau \in S(A)$, we use it to construct a representation such that ($\dagger$) holds for some $\zeta \in H$. Furthermore, the triple (H, φ, ζ) will be uniquely associated to τ in a certain sense.

2. The idea is similar to the following : Let X be a compact Hausdorff space and μ a positive Borel measure on X. Let $\tau : C(X) \to \mathbb{C}$ be the positive linear functional

$$f \mapsto \int_X f d\mu$$

We set $H := L^2(X, \mu)$, which is the completion of $C(X)$ in the norm induced by the inner product

$$\langle f, g\rangle := \int_X f\overline{g} d\mu = \tau(f\overline{g})$$

For every $f \in C(X)$, we define $M_f \in \mathcal{B}(H)$ by

$$M_f(g) := fg$$

Then the map $\varphi : f \mapsto M_f$ defines a representation of $C(X)$. Furthermore, if $\zeta := 1 \in C(X)$, then for any $f \in C(X)$,

$$\langle \varphi(f)\zeta, \zeta\rangle = \langle f, \zeta\rangle = \int_X f d\mu = \tau(f)$$

Throughout this section, fix a unital C*-algebra A. What follows can be done in the non-unital case, but needs a little more work.

Lemma 4.3.2. *If $\tau \in S(A)$, define $N_\tau := \{a \in A : \tau(a^*a) = 0\}$.*

1. *If $a \in N_\tau$, then for any $b \in A$,*
$$\tau(ba) = 0 \text{ and } \tau(a^*b) = 0$$

2. *For any $a, b \in A$,*
$$\tau(b^*a^*ab) \leq \|a^*a\|\tau(b^*b)$$

Proof. 1. The first statement follows from Cauchy-Schwartz. The second follows from the fact that
$$\tau(a^*b) = \tau((b^*a)^*) = \overline{\tau(b^*a)}$$

2. Fix $b \in A$. If $\tau(b^*b) = 0$, then the inequality is true by Cauchy-Schwartz. So suppose $\tau(b^*b) > 0$. Define $\rho : A \to \mathbb{C}$ by
$$c \mapsto \frac{\tau(b^*cb)}{\tau(b^*b)}$$
Note that if $c \in A_+$, then $\exists d \in A$ such that $c = d^*d$, so
$$b^*cb = b^*d^*db = (db)^*db \in A_+$$
Hence, ρ is a positive linear functional. Furthermore,
$$\rho(1) = 1 = \|\rho\|$$
Hence, $\rho \in S(A)$, and
$$\rho(a^*a) \leq \|a^*a\|$$
which gives the required result.

□

Lemma 4.3.3. *If $\tau \in S(A)$, define*
$$K := A/N_\tau$$
Then K is a vector space. Define $u : K \times K \to \mathbb{C}$ by
$$u(a + N_\tau, b + N_\tau) \mapsto \tau(b^*a)$$
Then u is a well-defined inner product on K.

Proof. 1. Well-defined: If $a + N_\tau = c + N_\tau$, then
$$\tau(b^*a) - \tau(b^*c) = \tau(b^*(a - c)) = 0$$
Similarly, if $b + N_\tau = d + N_\tau$ as well.

2. Bounded sesqui-linear form on K: because for any $x, y \in N_\tau$,

$$\tau((b+y)^*(a+x)) = \tau(b^*a + b^*x + y^*a + y^*x) = \tau(b^*a)$$

Hence by Cauchy-Schwartz,

$$|\tau(b^*a)| \leq \|(a+x)^*(a+x)\|^{1/2}\|(b+y)^*(b+y)\|^{1/2} = \|a+x\|\|b+y\|$$

Taking infimum, we see that

$$|\tau(b^*a)| \leq \|a+N_\tau\|\|b+N_\tau\|$$

3. Positive definite: If $a + N_\tau \in K$ is such that

$$\tau(a^*a) = 0 \Rightarrow a + N_\tau = 0$$

$\square$

We define H_τ to be the Hilbert space completion of K.

Theorem 4.3.4 (Gelfand-Naimark-Segal)**.** *Let K as above, and $a \in A$. Define $M_a : K \to K$ by*

$$M_a(b + N_\tau) := ab + N_\tau$$

Then

1. *M_a uniquely defines a bounded linear operator on H_τ.*
2. *The map $\varphi_\tau : A \to \mathcal{B}(H_\tau)$ given by*

$$a \mapsto M_a$$

   is a unital representation of A.
3. *If $\zeta := 1_A + N_\tau \in H_\tau$, then ζ is a cyclic vector for the representation.*
4. *For each $a \in A$, we have*
$$\tau(a) = \langle \varphi_\tau(a)\zeta, \zeta\rangle$$
5. *(Uniqueness) Suppose (L, ψ, η) is a triple such that*
   a) $\psi : A \to \mathcal{B}(L)$ is a representation
   b) η is a cyclic vector for the representation
   c) For all $a \in A$,
$$\tau(a) = \langle \psi(a)\eta, \eta\rangle$$
   Then there is a unitary $U : H_\tau \to L$ such that

$$U^{-1}\psi(a)U = \varphi_\tau(a) \quad \forall a \in A$$

The triple $(H_\tau, \varphi_\tau, \zeta)$ is called the GNS-representation associated to τ.

(End of Day 36)

Proof. 1. If $a \in A$, then by Lemma 4.3.2,

$$\begin{aligned}\|M_a(b+N_\tau)\|^2 &= \|ab+N_\tau\|^2 = \tau((ab)^*ab)\\ &= \tau(b^*a^*ab) \leq \|a^*a\|\tau(b^*b)\\ &= \|a\|^2\|b+N_\tau\|^2\end{aligned}$$

Hence, M_a defines a bounded operator on K with $\|M_a\| \leq \|a\|$. Thus, M_a extends uniquely to a bounded operator M_a on H_τ.

2. If $a, b \in A$, then

$$M_aM_b(x+N_\tau) = M_a(bx+N_\tau) = abx+N_\tau = M_{ab}(x+N_\tau)$$

Furthermore, if $a \in A$, then

$$\begin{aligned}\langle M_a(b+N_\tau), c+N_\tau\rangle &= \langle ab+N_\tau, c+N_\tau\rangle\\ &= \tau(c^*ab) = \tau((a^*c)^*b)\\ &= \langle b+N_\tau, a^*c+N_\tau = \langle b+N_\tau, M_{a^*}(c+N_\tau)\rangle\end{aligned}$$

Hence,

$$(M_a)^* = M_{a^*}$$

so φ_τ is a $*$-homomorphism. Note that $M_1 = \mathrm{id}_K$, so φ_τ is unital as well.

3. If $\zeta = 1_A + N_\tau$, then

$$\varphi_\tau(A)(\zeta) = \{a+N_\tau : a \in A\} = K \Rightarrow \overline{\varphi_\tau(A)(\zeta)} = H_\tau$$

as required.

4. Finally, if $a \in A$,

$$\langle \varphi_\tau(a)\zeta, \zeta\rangle = \langle a+N_\tau, 1_A+N_\tau\rangle = \tau(1_A^*a) = \tau(a)$$

5. For uniqueness, suppose (K, ψ, η) is a triple as above, note that $K = A/N_\tau$ is a dense subspace of H_τ, so define $U : K \to L$ by

$$U(a+N_\tau) := \psi(a)\eta$$

Then

a) U is well-defined: If $a, b \in A$ are such that $c := b-a \in N_\tau$, then

$$\|\psi(c)\eta\|^2 = \langle \psi(c)\eta, \psi(c)\eta\rangle = \langle \psi(c^*c)\eta, \eta\rangle = \tau(c^*c) = 0$$

Hence $\psi(c)\eta = 0$ whence $\psi(a)\eta = \psi(b)\eta$ as required.

b) U preserves the inner product: If $a, b \in A$, then

$$\langle a + N_\tau, b + N_\tau \rangle = \tau(b^* a) = \langle \psi(b^* a)\eta, \eta \rangle = \langle \psi(a)\eta, \psi(b)\eta \rangle$$

c) Hence U extends to an isometry $U : H_\tau \to L$. Note that U is surjective because the range contains $\{\psi(a)\eta : a \in A\}$ which is dense in L. Hence, U is a unitary.

d) Finally, note that for all $a, b \in A$

$$U^{-1}\psi(a)U(b+N_\tau) = U^{-1}\psi(a)(\psi(b)\eta) = U^{-1}\psi(ab)(\eta) = ab+N_\tau = \varphi_\tau(a)(b+N_\tau)$$

Hence, $U^{-1}\psi(a)U = \varphi_\tau(a)$

□

Example 4.3.5. Let μ be a positive Borel measure on a compact Hausdorff space X and let $\tau : C(X) \to \mathbb{C}$ be the positive linear functional

$$f \mapsto \int_X f d\mu$$

Then

$$N_\tau = \{f \in C(X) : f \equiv 0 \text{ a.e.}[\mu]\}$$

Now H_τ is the completion of

$$K = C(X)/N_\tau$$

Hence, $H_\tau \cong L^2(X, \mu)$. Furthermore, the GNS representation associated to τ is precisely the map

$$\varphi : C(X) \to \mathcal{B}(L^2(X, \mu)) \text{ given by } f \mapsto M_f$$

Definition 4.3.6. 1. We say that φ is faithful if it is injective.

2. Let $\{H_\lambda : \lambda \in I\}$ be a possibly uncountable family of Hilbert spaces. Define

$$K := \{(x_\lambda) : x_\lambda \neq 0 \text{ for only finitely many } \lambda \in I\}$$

Then K is an inner product space with the usual inner product. The completion of K w.r.t this inner product is a Hilbert space, denoted by

$$H := \bigoplus_{\lambda \in I} H_\lambda$$

3. For each $\lambda \in I$, if $A_\lambda \in \mathcal{B}(H_\lambda)$, then as in Remark 3.3.7, we may define

$$A := \bigoplus_{\lambda \in I} A_\lambda \in \mathcal{B}(H)$$

provided $\sup_\lambda \|A_\lambda\| < \infty$.

4. For each $\lambda \in I$, if $(H_\lambda, \varphi_\lambda)$ is a representation of A, then for each $a \in A$, by Theorem 2.2.11, $\|\varphi_\lambda(a)\| \leq \|a\|$. Hence, we may define $\varphi : A \to \mathcal{B}(H)$ by

$$\varphi(a) := \bigoplus_\lambda \varphi_\lambda(a)$$

This is a representation of A, and is denoted by

$$\varphi = \bigoplus_{\lambda \in I} \varphi_\lambda$$

Definition 4.3.7. Consider all GNS-representations $\{(H_\tau, \varphi_\tau) : \tau \in S(A)\}$. Define

$$H := \bigoplus H_\tau \text{ and } \varphi := \bigoplus \varphi_\tau$$

The pair (H, φ) is called the universal representation of A.

Theorem 4.3.8. *The universal representation is injective (faithful).*

Proof. Suppose $a \in A$ such that $\varphi(a) = 0$, then for any $\tau \in S(A), \varphi_\tau(a) = 0$. Hence,

$$\begin{aligned}
\varphi_\tau(a)(1_A + N_\tau) &= a + N_\tau = 0 + N_\tau \\
&\Rightarrow a \in N_\tau \\
&\Rightarrow \tau(a^*a) = 0 \quad \forall \tau \in S(A) \\
&\Rightarrow \|a^*a\| = 0 \qquad \text{(by Theorem 4.2.15)} \\
&\Rightarrow a = 0
\end{aligned}$$

□

Corollary 4.3.9. *Every C^*-algebra is isometrically isomorphic to a subalgebra of $\mathcal{B}(H)$ for some Hilbert space H.*

Proof. Every injective $*$-homomorphism is isometric by Theorem 2.2.11, so the universal representation sets up the required isomorphism. □

(End of Day 37)

Review for the Final Exam

(End of Day 38)

5 Instructor Notes

1. The course went well, and the students seem interested and responsive, which was good.

2. The goal of the course was as before, to do the spectral theorem in the spirit of [Arveson]. On advice from other faculty, I decided to add the GNS construction at the end, which was nice.

3. The only thing that I left out that would be nice to include is many theorems in the context of non-unital C*-algebras, starting from Gelfand-Naimark, all the way to the GNS construction. One needs to include approximate units, but other than that, it should be an easy change for the next time.

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