MTH 410/514/620: Representation Theory

Semester 2, 2016-2017

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Contents

0.1 Course Structure

2/1: Section 3.1 of [BS] until 3.1.5

4/1: Until Definition 3.1.14

5/1: Until Definition 3.2.1 (avoiding Definition 3.1.16)

(End of Week 1)

9/1: Completed Chapter 3.

11/1: Started Chapter 4. Completed until Corollary 4.1.9.

12/1: Until Prop 4.2.3.

(End of Week 2)

16/1: Until Prop 4.2.10 (including examples of $\widehat{G}$ for $\mathbb{Z}_n, \mathbb{Z}_n \times \mathbb{Z}_m$ and S_3)

18/1: Computed $\widehat{G}$ for D_4. Then started Section 4.3, and completed until Theorem 4.3.9

19/1: Completed Section 4.3

(End of Week 3)

23/1: Completed until Theorem 4.4.6. Then defined the Fourier coefficient of a function w.r.t. a representation as in [BS, Definition 5.5.2]. Then proved [T, Lemma 9.4].

25/1: Completed until Theorem 4.4.12, following [T, Theorem 9.3] for the proof of Theorem 4.4.7.

26/1: Completed Chapter 4. Discussed the character table of $\mathbb{Z}_n$, direct product of two Abelian groups. Also discussed the group structure on $\widehat{G} = \mathrm{Hom}(G, S^1)$, and Pontrjagin duality for a finite Abelian group.

(End of Week 4)

30/1: Discussed linear characters (see additional notes below)

1/2: Discussed a way of counting conjugacy classes, and then determined the character table for S_3.

2/2: Calculated the character tables for non-abelian groups of order 8, and for A_4.

(End of Week 5)

No classes. Quiz on 9/2/17.

(End of Week 6)

16/2: Started Chapter 5. Completed until Theorem 5.3.5.

(End of Week 7)

20/2: Skipped Section 5.4, and completed Chapter 5.

22/2: Started Chapter 6. Completed until Remark 6.2.2.

23/2: Completed until Corollary 6.2.5. Included [JL, Examples 22.12(i),(ii)].

24/2: Completed until Theorem 6.3.9.

(End of Week 8)

27/2: Completed Chapter 6.

1/3: Started Chapter 7. Completed until Proposition 7.2.7, skipping parts of Section 7.1

2/3: Completed Section 7.1, and until Theorem 7.2.8.

(End of Week 9)

20/3: Completed Chapter 7.

22/3: Tensor products of vector spaces (see additional notes below for the remainder of the course)

23/3: Direct product of groups

25/3: Inner tensor product of representations from

(End of Week 10)

27/3: Character table of S_5, and started restriction to a subgroup from

29/3: Continued restriction to a subgroup, and started the Character table of A_5

30/3: Completed the character table of A_5

(End of Week 11)

3/4: Started Induced representations

5/4: Proved the Frobenius Character formula

6/4: Proved Frobenius reciprocity

(End of Week 12)

10/4: Example of group of order 21

12/4: Example of group of order $p(p-1)$

13/4: Review.

(End of Week 13)

0.2 Instructor Notes

Given below are some additional notes meant to supplement the material from the textbook.

1 Character Tables

The goal of these notes is to supplement the discussion at the end of [BS, Chapter 4] by computing the character tables for some non-abelian groups of small order.

1.1 Linear Characters

Remark. [BS, Exercise 4.6] Let G be a group, $H \lhd G$, and $\pi : G \to G/H$ be the natural quotient map. Observe that

1. If $\rho : G/H \to GL(V)$ is a representation, then $\rho \circ \pi : G \to GL(V)$ is a representation.
2. If $\rho : G/H \to GL(V)$ and $\psi : G \to GL(W)$ are two representations, then $\rho \sim \psi$ iff $\rho \circ \pi \sim \psi \circ \pi$.
3. ρ is irreducible if and only if $\rho \circ \pi$ is irreducible.

Hence, we get a well-defined map

$$\mu : \widehat{G/H} \to \widehat{G}$$

This is injective by (2) above, but not surjective in general.

Theorem 1.1.1. *Let G be a group, $H \lhd G$, and $\pi : G \to G/H$ be the natural quotient map. If $\varphi : G \to GL(V)$ is a representation such that $H \subset \ker(\varphi)$, then $\exists$ a unique representation $\rho : G/H \to GL(V)$ such that*

$$\rho \circ \pi = \varphi$$

Proof. If $\varphi : G \to GL(V)$ such that $H \subset \ker(\varphi)$, then define

$$\rho : G/H \to GL(V) \text{ by } gH \mapsto \varphi(g)$$

1. This is well-defined because if $g_1H = g_2H$, then $g_2^{-1}g_1 \in H$, so $g_2^{-1}g_1 \in \ker(\varphi)$ and hence
$$\varphi(g_1) = \varphi(g_2)$$
2. ρ is a homomorphism because if $g_1H, g_2H \in G/H$, then
$$\rho(g_1H \cdot g_2H) = \rho(g_1g_2H) = \varphi(g_1g_2) = \varphi(g_1)\varphi(g_2) = \rho(g_1H)\rho(g_2H)$$
3. It is clear that $\rho \circ \pi = \varphi$ by definition

4. As for uniqueness, suppose ψ is another function such that $\psi \circ \pi = \varphi = \rho \circ \pi$, then $\psi(gH) = \varphi(g) = \rho(gH)$ for all $g \in G$.

$\square$

Definition 1.1.2. A linear character is a representation of degree 1. Write $\widehat{G}^{lin}$ for the set of all linear characters of G.

Observe that if $\varphi : G \to \mathbb{C}^*$ is a linear character, then

$$G/\ker(\varphi) \cong \text{Image}(\varphi) < \mathbb{C}^*$$

so $G/\ker(\varphi)$ is an Abelian group.

Definition 1.1.3. If G is any group, and $x, y \in G$, the commutator of x and y is given by

$$[x, y] := xyx^{-1}y^{-1}$$

The commutator subgroup of G, denoted by $[G, G]$, is the smallest subgroup of G containing the set

$$S = \{[x, y] : x, y \in G\}$$

Equivalently,

$$[G, G] = \{u_1^{\epsilon_k} u_1^{\epsilon_k} \dots u_k^{\epsilon_k} : u_i \in S, \epsilon_i \in \{\pm 1\}\}$$

In fact, we can refine this further. If $u = [x, y]$, then $u^{-1} = [y, x] \in S$, so

$$[G, G] = \{u_1 u_2 \dots u_k : u_i \in S\}$$

In other words, $[G, G]$ is the set of all products of commutators in G.

Theorem 1.1.4. *Let G be a group, and $[G, G]$ its commutator subgroup.*

1. $[G, G] \lhd G$
2. *If $H \lhd G$ such that G/H is Abelian, then $[G, G] \subset H$*
3. *In particular, $G/[G, G]$ is Abelian.*
4. *G is Abelian iff $[G, G] = \{e\}$*

Proof. 1. Note that if $x, y \in G$ and $g \in G$, then

$$g[x, y]g^{-1} = [gxg^{-1}, gyg^{-1}]$$

Hence, $gSg^{-1} \subset S$, and so $[G, G] \lhd G$ by the description of elements of $[G, G]$ given above.

2. G/H is abelian if and only if

$$(xH)(yH) = (yH)(xH) \quad \forall x, y \in H \Leftrightarrow (xy)H = (yx)H \quad \forall x, y \in G$$

This is equivalent to $[x, y] \in H$ for all $x, y \in H$, and so $[G, G] \subset H$

3. Follows from (1) and (2).

4. Trivial.

$\square$

Theorem 1.1.5. *Let $\overline{G} := G/[G,G]$, and let $\pi : G \to \overline{G}$ denote the natural quotient map.*

1. *If $\varphi : \overline{G} \to \mathbb{C}^*$ is a representation, then $\varphi \circ \pi$ is a representation of G*

2. *If $\rho : G \to \mathbb{C}^*$ is a linear character, then $\exists \varphi : \overline{G} \to \mathbb{C}^*$ such that $\rho = \varphi \circ \pi$*

3. *Consider the injective map*
$$\mu : \widehat{\overline{G}} \to \widehat{G}$$
as described above. Then Image(μ) $= \widehat{G}^{lin}$.

Proof. 1. By definition

2. If $\rho : G \to \mathbb{C}^*$ is a linear character, then $G/\ker(\rho)$ is abelian as mentioned above. Hence, $[G,G] \subset \ker(\rho)$ by the previous theorem. Hence, $\exists$ unique $\overline{\rho} : \overline{G} \to \mathbb{C}^*$ such that $\rho = \overline{\rho} \circ \pi$.

3. The map $\widehat{\overline{G}} \to \widehat{G}$ is well-defined and injective as before. Furthermore, if $\varphi \in \widehat{\overline{G}}$, then $d_\varphi = 1$ since $\overline{G}$ is abelian, so
$$\varphi : \overline{G} \to \mathbb{C}^*$$
Hence, $\varphi \circ \pi : G \to \mathbb{C}^*$ is a degree one representation. Equivalently,
$$\mu(\varphi) \in \widehat{G}^{lin}$$
Conversely, if $\rho \in \widehat{G}^{lin}$, then $\rho = \mu(\overline{\rho})$, where $\overline{\rho}$ is as in part (2). Hence, $\rho \in$ Image(μ).

$\square$

Corollary 1.1.6. *The number of linear characters of G is equal to the index of of $[G,G]$ in G. In particular, this number divides $|G|$.*

Proof. This follows from the above statement and the fact that $\overline{G}$ is abelian, and so
$$|\widehat{\overline{G}}| = |\overline{G}| = [G : [G,G]]$$

$\square$

1.2 Counting Conjugacy Classes

Lemma 1.2.1. *Let $H \lhd G$, then H is a disjoint union of conjugacy classes in G.*

Lemma 1.2.2. *Let $H \lhd G$ and $\pi : G \to G/H$ the quotient map. If $D \subset G/H$ is a conjugacy class, then*

$$\pi^{-1}(D)$$

is a disjoint union of conjugacy classes in G. Furthermore, if

1. *If $D \neq \{\pi(e)\}$, then $\pi^{-1}(D) \cap H = \emptyset$*
2. *If D_1 and D_2 are two disjoint conjugacy classes of G/H, then $\pi^{-1}(D_1) \cap \pi^{-1}(D_2) = \emptyset$.*

Proof. We wish to show that, if C is a conjugacy class in G, then either

$$C \cap \pi^{-1}(D) = \emptyset \text{ or } C \subset \pi^{-1}(D)$$

□

By the previous lemma, if $H \lhd G$, we may write

$$H = \sqcup_{i=1}^{k} C_i$$

where C_i are conjugacy classes in G, and suppose

$$G/H = \sqcup_{j=1}^{\ell} D_j$$

where D_j are the conjugacy classes in G/H, then for each $1 \leq j \leq \ell$. Suppose $D_1 = \{\pi(e)\}$, we write

$$\pi^{-1}(D_j) = B_{j,1} \sqcup B_{j,2} \sqcup \ldots \sqcup B_{j,s_j}$$

where $B_{j,t}$ are conjugacy classes in G. Hence, we get

Lemma 1.2.3. *The collection*

$$\mathcal{F} = \{C_1, C_2, \ldots, C_k, B_{2,1}, B_{3,1}, \ldots, B_{\ell,1}\}$$

are disjoint conjugacy classes in G. Hence,

$$|Cl(G)| \geq k + \ell - 1$$

Note: A strict inequality may hold above.

1.3 Examples

We now construct the character tables for some non-Abelian groups. Given a non-abelian group G, we will follow these steps:

1. Determine $[G, G]$ by examining normal subgroups H such that G/H is abelian.
2. Determine all linear characters on G by using information from $\overline{G} = G/[G, G]$
3. Use the degree formula to enumerate the number and degrees of all irreducible representations of G.
4. Determine the number of conjugacy classes of G using the previous section, and also their representatives.
5. Use this to build a partial character table, with some unknown entries.
6. Determine the unknown entries by using the orthogonality relations.

1.3.1 The symmetric group S_3

Let $G = S_3$.

1. Recall that $A_3 \lhd S_3$ and $S_3/A_3 \cong \mathbb{Z}_2$. Hence,
$$[G, G] \subset A_3$$
Since G is non-abelian, $[G, G] \neq \{e\}$. Since A_3 is cyclic of prime order, we have
$$[G, G] = A_3$$
2. Since $\overline{G} = G/[G, G] \cong \mathbb{Z}_2$, G has two linear characters obtained by lifting the two irreducible representations of $\mathbb{Z}_2$.
$$\begin{aligned} \rho_1 &: 1 \mapsto 1 \\ \rho_2 &: 1 \mapsto -1 \end{aligned}$$
write $\varphi_i : G \to \mathbb{C}^*$ to be maps, $\varphi_i = \rho_i \circ \pi$
3. The degree formula now reads
$$6 = |G| = 2 + \sum_{n_i > 1} n_i^2$$
Hence, it follows that G has exactly one irreducible representation of degree 2, and no other representations of higher degree. We denote this representation by ρ.

4. By the previous step, G has 3 conjugacy classes. Notice that $H = A_3$ has is the union of two conjugacy classes of G.

$$C_1 = \{e\}, C_2 = \{(123), (132)\}$$

Also, G/H is abelian, so it has conjugacy classes

$$D_1 = \{\pi(e)\}, D_2 = \{\pi((12))\}$$

Hence, if $\mathcal{F}$ is as in the previous section, then

$$\mathcal{F} = \{(e), ((123)), ((12))\}$$

Since $|Cl(G)| = 3$, it follows that $Cl(G) = \mathcal{F}$.

5. Note that if $\rho_i : \mathbb{Z}_2 \to \mathbb{C}^*$ is a representation, then

$$\varphi_i = \rho_i \circ \pi : G \to \mathbb{C}^*$$

is a one-dimensional representation such that

$$\chi_{\varphi_i}(g) = \chi_{\rho_i}(\pi(g))$$

So we obtain a partial character table as follows

	e	(123)	(12)
φ_1	1	1	1
φ_2	1	1	-1
ρ	2	a	b

6. The orthogonality of columns now gives two equations

$$1 + 1 + 2a = 0 \Rightarrow a = -1$$
$$1 - 1 + 2b = 0 \Rightarrow b = 0$$

So the character table of S_3 is

	e	(123)	(12)
φ_1	1	1	1
φ_2	1	1	-1
χ_ρ	2	-1	0

Note that this agrees with what he had obtained earlier.

1.3.2 Non-Abelian groups of order 8

1. If G is non-Abelian and $|G| = 8$, then $Z(G) \neq \{e\}$, and so $|Z(G)| \in \{2, 4, 8\}$. Since G is non-abelian, and
Proposition 1.3.1. *If $G/Z(G)$ is cyclic, then G is abelian.*

It follows that $|Z(G)| = 2$ and $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In particular, since $G/Z(G)$ is abelian, it follows that $[G, G] \subset Z(G)$. Since $[G, G] \neq \{e\}$ (since G is non-Abelian), we have

$$[G, G] = Z(G)$$

2. Since $\overline{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we have 4 irreducible representations of $\overline{G}$ given by

$$\begin{aligned} \rho_1 &: \{(1,0),(0,1)\} \mapsto 1 \\ \rho_2 &: (1,0) \mapsto 1 \text{ and } (0,1) \mapsto -1 \\ \rho_3 &: (1,0) \mapsto -1 \text{ and } (0,1) \mapsto 1 \\ \rho_4 &: (1,0) \mapsto -1 \text{ and } (0,1) \mapsto -1 \end{aligned}$$

We write $\varphi_i := \rho_i \circ \pi : G \to \mathbb{C}^*$.

3. The degree formula gives

$$8 = 4 + \sum_{n_i > 1} n_i^2$$

Once again, we see that G has exactly one irreducible of representation of degree > 1. We denote this by ρ, and note that $d_\rho = 2$.

4. Since G has 5 irreducible representations, $|Cl(G)| = 5$. Note that $H = Z(G)$ has 2 conjugacy classes of G, we denote them by

$$C_1 = \{e\}, C_2 = \{x\}$$

Since $G/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we write

$$G/H = \{\pi(e), \pi(g_1), \pi(g_2), \pi(g_3)\}$$

Each singleton forms a conjugacy class in G/H, so we obtain

$$\mathcal{F} = \{\{e\}, \{x\}, (g_1), (g_2), (g_3)\}$$

Since $|Cl(G)| = 5$, it follows that $Cl(G) = \mathcal{F}$.

5. Once again, if $\varphi_i = \rho_i \circ \pi$, then

$$\chi_{\varphi_i}(g) = \chi_{\rho_i}(\pi(g))$$

So we obtain a partial character table as

g	1	x	g_1	g_2	g_3
φ_1	1	1	1	1	1
φ_2	1	1	-1	1	-1
φ_3	1	1	1	-1	-1
φ_4	1	1	-1	-1	1
ρ	2	a	b	c	d

6. Using the orthogonality of columns, we get 4 equations

$$\begin{aligned}
1+1+1+1+2a=0 &\Rightarrow a=-2\\
2-2+2-2+2b=0 &\Rightarrow b=0\\
2+2-2-2+2c=0 &\Rightarrow c=0\\
2-2-2+2+2d=0 &\Rightarrow d=0
\end{aligned}$$

   Hence, any two non-Abelian groups of order 8 have the same character table, given by

g	1	x	g_1	g_2	g_3
φ_1	1	1	1	1	1
φ_2	1	1	-1	1	-1
φ_3	1	1	1	-1	-1
φ_4	1	1	-1	-1	1
χ_ρ	2	-2	0	0	0

In particular, the groups D_4 and Q_8 are two non-isomorphic groups which have the same character table.

In fact, more is true: If p is a prime, then any two non-Abelian groups of order p^3 have the same character table. We will prove this later in the course.

1.3.3 The Alternating Group A_4

Let $G = A_4$,

1. Set $H = \{e, (12)(34), (13)(24), (14)(23)\}$. Then $H \lhd S_4$ since it consists of precisely two conjugacy classes. Hence, $H \lhd A_4$. Furthermore, G/H is a group of order 4, and hence is Abelian. By the earlier section,

$$[G, G] \subset H$$

   Since A_4 is non-Abelian, $[G, G] \neq \{e\}$. However, the non-identity elements in H form a single conjugacy class in A_4, so since $[G, G] \lhd A_4$ (it must be a union of conjugacy classes), it follows that $[G, G] = H$

2. Now $\overline{G} = G/H \cong \mathbb{Z}_3$, so G has 3 linear characters given by

$$\rho_i : 1 \to \omega^{i-1}, i = 1, 2, 3$$

   where $\omega = e^{2\pi i/3}$. Let $\varphi_i = \rho_i \circ \pi$

3. Now the degree formula gives $12 = |G| = 3 + \sum_{d_i>1} d_i^2$. Hence, G has exactly one more irreducible representation, ρ such that $d_\rho = 3$.

4. By the previous step, $|Cl(G)| = 4$. Notice that H is a union of two conjugacy classes
$$C_1 = \{e\}, C_2 = \{(12)(34), (13)(24), (14)(23)\}$$
Also, write
$$G/H = \{\pi(e), \pi((123)), \pi((132)\}$$
then these yield singleton conjugacy classes in G/H. Hence we get
$$\mathcal{F} = \{\{e\}, ((12)(34)), ((123)), ((132))\}$$
Since $|Cl(G)| = 4$, it follows that $Cl(G) = \mathcal{F}$.

5. As before, the character table now looks like:

g	1	(12)(34)	g_1	g_2
χ_{φ_1}	1	1	1	1
χ_{φ_2}	1	1	ω	ω^2
χ_{φ_3}	1	1	ω^2	ω
χ_ρ	3	a	b	c

where $\omega = e^{2\pi i/3}$.

6. Now the orthogonality of the columns yields
$$3 + 3a = 0 \Rightarrow a = -1$$
$$1 + \omega + \omega^2 + 3b = 0 \Rightarrow b = 0$$
$$1 + \omega^2 + \omega + 3c = 0 \Rightarrow c = 0$$
because $1 + \omega + \omega^2 = 0$. This gives the character table of A_4 as

g	1	(12)(34)	g_1	g_2
1_G	1	1	1	1
χ_w	1	1	ω	ω^2
χ_{w^2}	1	1	ω^2	ω
ρ	3	-1	0	0

2 Tensor Products of Representations

Towards the end of the course, we veered away from the textbook completely. I wanted to cover tensor products, restriction and induction - all topics which, I felt, were covered poorly in the textbook.

2.1 Tensor Products of Vector Spaces

Let U, V, W, X, etc. denote finite dimensional vector spaces over a field k

Definition 2.1.1. A map $f : V \times W \to X$ is said to be <u>bilinear</u> if for all $\alpha_i, \beta_j \in k, v_i \in V, w_j \in W$, we have

$$f\left(\sum_i \alpha_i v_i, \sum_j \beta_j w_j\right) = \sum_{i,j} \alpha_i \beta_j f(v_i, w_j)$$

Example 2.1.2.
1. If V is an inner product space over $\mathbb{R}$, then the inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ is bilinear.
2. Cross product $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$
3. If V is a vector space, and V^* its dual, then $B : V \times V^* \to k$ defined by $B(v, f) := f(v)$ is bilinear.
4. $\psi : \mathbb{C} \times \mathbb{R}^n \to \mathbb{C}^n$ given by $(z, \overline{v}) \mapsto (zv_1, zv_2, \ldots, zv_n)$

Definition 2.1.3.
1. $B_k(V, W)$ is the vector space of all bilinear maps $f : V \times W \to k$
2. For $v \in V, w \in W$, define $v \otimes w : B_k(V, W) \to k$ by $v \otimes w(f) := f(v, w)$. Notice that $v \otimes w \in B_k(V, W)^*$, the dual space of $B_k(V, W)$
3. Define $V \otimes W := \text{span}\{v \otimes w : v \in V, w \in W\}$

Lemma 2.1.4. *The map $\varphi : V \times W \to V \otimes W$ given by $\varphi(v, w) := v \otimes w$ is bilinear.*

Proof. We prove linearity in the first variable as the other variable is similar. So fix $v_1, v_2 \in V, w \in W$, and $\alpha \in k$, and we WTS:

$$\varphi(\alpha v_1 + v_2, w) = \alpha\varphi(v_1, w) + \varphi(v_2, w)$$

So fix $f \in B_k(V, W)$, then

$$\begin{aligned}\varphi(\alpha v_1 + v_2, w)(f) &= f(\alpha v_1 + v_2, w) \\ &= \alpha f(v_1, w) + f(v_2, w) \\ &= \alpha\varphi(v_1, w)(f) + \varphi(v_2, w)(f) \\ &= [\alpha\varphi(v_1, w) + \varphi(v_2, w)](f)\end{aligned}$$

□

Theorem 2.1.5. *If $\{v_i\}$ and $\{w_j\}$ are bases for V and W respectively, then $\{v_i \otimes w_j\}$ is a basis for $V \otimes W$. In particular,* $\dim(V \otimes W) = \dim(V) \times \dim(W)$

Proof. Let $S = \{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$.

1. S is linearly independent: If $\alpha_{i,j} \in k$ such that

   $$\sum_{i,j} \alpha_{i,j} v_i \otimes w_j = 0 \qquad (*)$$

   Fix i, j and let $f_{i,j} : V \times W \to k$ be given by

   $$f_{i,j}(v_k, w_\ell) = \delta_{i,k}\delta_{k,\ell}$$

   extended to a bilinear map on $V \times W$. Then $f_{i,j} \in B_k(V, W)$, and

   $$(v_k \otimes w_\ell)(f_{i,j}) = f_{i,j}(v_k, w_\ell) = \delta_{i,k}\delta_{k,\ell}$$

   Hence, applying $(*)$ to $f_{i,j}$ gives
   $$\alpha_{i,j} = 0$$
   This is true for all $1 \leq i \leq n, 1 \leq j \leq m$, so S is linearly independent.

2. S spans $V \times W$: By definition,

   $$V \otimes W := \text{span}\{v \otimes w : v \in V, w \in W\}$$

   so it suffices to show that $v \otimes w \in \text{span}(S)$ for any $v \in V, w \in W$. So fix $v \in V, w \in W$, then write
   $$v = \sum_i \alpha_i v_i \text{ and } w = \sum_j \beta_j w_j$$
   Then since the map $(v, w) \mapsto v \otimes w$ is bilinear, we get

   $$v \otimes w = \sum_{i,j} \alpha_i \beta_j v_i \otimes w_j \in \text{span}(S)$$

□

Proposition 2.1.6 (Universal Property - I)**.** *If X is a finite dimensional vector space, and $g : V \times W \to X$ is a bilinear map, then $\exists! T : V \otimes W \to X$ linear such that $T \circ \varphi = g$. In other words, there is an isomorphism*

$$B_X(V, W) \cong Hom_k(V \otimes W, X)$$

Proof. If $g : V \times W \to X$ is bilinear, define

$$T : V \otimes W \to X \text{ given by } T(v_i \otimes w_j) = g(v_i, w_j)$$

extended linearly to $V \otimes W$. This is well-defined by the previous theorem. Furthermore, T is linear and

$$T \circ \varphi(v_i, w_j) = g(v_i, w_j)$$

Since both sides are bilinear, they must agree on $V \times W$.

For uniqueness, note that if $S : V \otimes W \to X$ is a linear map such that

$$S \circ \varphi = g$$

Then

$$S(v_i \otimes w_j) = g(v_i, w_j) = T(v_i \otimes w_j) \quad \forall i, j$$

Since S and T are linear, it follows that $S = T$ by the previous theorem. □

Theorem 2.1.7 (Universal Property - II)**.** *Let U be a finite dimensional vector space and $\psi : V \times W \to U$ is a bilinear map such that, for any bilinear map $h : V \times W \to X$, $\exists! S : U \to X$ such that $S \circ \psi = h$, then there is an isomorphism $\mu : U \to V \otimes W$ such that $\mu \circ \psi = \varphi$*

Proof. Let (U, ψ) be a pair as above. By the previous theorem $(V \otimes W, \varphi)$ is another pair that satisfies the same property. By the previous theorem, $\exists T : V \otimes W \to U$ such that

$$T \circ \varphi = \psi$$

Similarly, $\exists S : U \to V \otimes W$ such that

$$S \circ \psi = \varphi$$

Hence,

$$S \circ T \circ \varphi = \varphi \qquad \text{and} \qquad T \circ S \circ \psi = \psi$$

By the uniqueness, it follows that $S \circ T = \mathrm{id}_{V \otimes W}$. Similarly,

$$T \circ S = \mathrm{id}_U$$

and hence S is the required isomorphism. □

Example 2.1.8. 1. $\mathbb{C} \otimes \mathbb{R}^n \cong \mathbb{C}^n$

Proof. Define $\psi : \mathbb{C} \times \mathbb{R}^n \to \mathbb{C}^n$ by

$$(z, \overline{v}) = (zv_1, zv_2, \ldots, zv_n)$$

This is a bilinear map. Hence, $\exists T : \mathbb{C} \otimes \mathbb{R}^n \to \mathbb{C}^n$ such that

$$T(z \otimes \overline{v}) = \psi(z, \overline{v})$$

Now note that

$$\psi(1, e_i) = e_i$$

so ψ is surjective. Hence, T is surjective. However,

$$\dim(\mathbb{C}^n) = 2n = \dim(\mathbb{C}) \times \dim(\mathbb{R}^n) = \dim(\mathbb{C} \otimes \mathbb{R}^n)$$

and so T must be injective and hence an isomorphism. $\square$

2. $\mathbb{C}^n \otimes \mathbb{C}^m \cong \mathbb{C}^{nm}$

Proof. Define $\psi : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^{nm}$ by

$$\psi(\overline{x}, \overline{y}) = (x_1y_1, x_1y_2, \ldots, x_1y_m, x_2y_1, \ldots, x_2y_m, \ldots, x_ny_m)$$

Then follow the argument as above. $\square$

3. $V \otimes V^* \cong \mathrm{End}_k(V)$

Proof. Define $\psi : V \times V^* \to \mathrm{End}_k(V)$ by

$$\psi(v, f)(w) = f(w)v$$

Then ψ is bilinear, so follow a similar argument as above. $\square$

Definition 2.1.9. Let $T : V_1 \to V_2$ and $S : W_1 \to W_2$ be two linear maps. Then define

$$\psi : V_1 \times W_1 \to V_2 \otimes W_2 \text{ by } \psi(v, w) = T(v) \otimes S(w)$$

Then ψ is clearly bilinear. So $\exists! R : V_1 \otimes W_1 \to V_2 \otimes W_2$ such that

$$R(v \otimes w) = T(v) \otimes S(w) \quad \forall v \in V_1, w \in W_1$$

We write $R = T \otimes S$

2.2 Direct Product of Groups

Theorem 2.2.1. *Let $\rho : G \to GL(V)$ and $\pi : H \to GL(W)$ be two representations. Then $\exists$ a unique representations*

$$\psi : G \times H \to GL(V \otimes W)$$

such that

$$\psi(g, h)(v \otimes w) = \rho_g(v) \otimes \pi_h(w)$$

This is called the <u>outer tensor product of ρ and π</u> and we write $\psi = \rho \widehat{\otimes} \pi$

Proof. 1. For each $(g, h) \in G \times H$ fixed, define

$$\varphi : V \times W \to V \otimes W \text{ given by } \varphi(v, w) = \rho_g(v) \otimes \pi_h(w)$$

This map is clearly bilinear, so $\exists$ a unique linear map

$$R_{(g,h)} : V \otimes W \to V \otimes W \text{ such that } R_{(g,h)}(v \otimes w) = \rho_g(v) \otimes \pi_h(w)$$

So we define $\psi(g, h) := R_{(g,h)}$

2. We first check that ψ is well-defined: To see this, note that

$$R_{(g^{-1},h^{-1})}(v \otimes w) = \rho_{g^{-1}}(v) \otimes \pi_{h^{-1}}(w)$$

Hence, for any $v \in V, w \in W$, we have

$$R_{(g,h)} \circ R_{(g^{-1},h^{-1})}(v \otimes w) = v \otimes w = R_{(g^{-1},h^{-1})} \circ R_{(g,h)}(v \otimes w)$$

But $V \otimes W = \text{span}\{v \otimes w : v \in V, w \in W\}$, so since both sides are linear maps, we see that

$$R_{(g,h)} \circ R_{(g^{-1},h^{-1})} = I = R_{(g^{-1},h^{-1})} \circ R_{(g,h)}$$

Hence, $R_{(g,h)} \in GL(V \otimes W)$

3. Now we check that ψ is a homomorphism: As above, it suffices to show that

$$R_{(g_1,h_1)} \circ R_{(g_2,h_2)}(v \otimes w) = R_{(g_1g_2,h_1h_2)}(v \otimes w) \quad \forall v \in V, w \in W$$

This follows from the definition and the fact that ρ and π are representations.

4. Uniqueness follows from the uniqueness of the previous definition.

□

Theorem 2.2.2. *With the notation as above,*

$$\chi_{\rho \widehat{\otimes} \pi}(g, h) = \chi_\rho(g)\chi_\pi(h)$$

Proof. Fix $(g, h) \in G \times H$. Since ρ_g is diagonalizable, $\exists$ a basis $S = \{v_1, v_2, \ldots, v_n\}$ of V such that

$$\rho_g(v_i) = \lambda_i v_i \quad \forall 1 \leq i \leq n$$

Similarly, $\exists$ a basis $T = \{w_1, w_2, \ldots, w_m\}$ of W such that

$$\pi_h(w_j) = \mu_j w_j \quad \forall 1 \leq j \leq m$$

Let $\mathcal{B} = \{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$, then $\mathcal{B}$ is a basis for $V \otimes W$. Furthermore, if $\psi = \rho \otimes \pi$, then

$$\psi_{(g,h)}(v_i \otimes w_j) = \lambda_i \mu_j (v_i \otimes w_j)$$

Taking a trace, we get

$$\begin{aligned}
\chi_\psi(g,h) &= \sum_{i,j} \lambda_i \mu_j \\
&= \left(\sum_i \lambda_i\right)\left(\sum_j \mu_j\right) \\
&= \chi_\rho(g)\chi_\pi(h)
\end{aligned}$$

□

Theorem 2.2.3. *Let $\rho_i : G \to GL(V_i)$ and $\pi_i : H \to GL(W_i)$ for $i = 1, 2$. If $\psi_i = \rho_i \widehat{\otimes} \pi_i$, then*

$$\langle \chi_{\psi_1}, \chi_{\psi_2} \rangle_{L(G\times H)} = \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle_{L(G)} \langle \chi_{\pi_1}, \chi_{\pi_2} \rangle_{L(H)}$$

Proof. We compute

$$\begin{aligned}
\langle \chi_{\psi_1}, \chi_{\psi_2} \rangle_{L(G\times H)} &= \frac{1}{|G \times H|} \sum_{(g,h)\in G\times H} \chi_{\psi_1}(g,h)\overline{\chi_{\psi_2}(g,h)} \\
&= \frac{1}{|G||H|} \sum_{g\in G, h\in H} \chi_{\rho_1}(g)\chi_{\pi_1}(h)\overline{\chi_{\rho_2}(g)\chi_{\pi_2}(h)} \\
&= \left(\frac{1}{|G|}\sum_{g\in G} \chi_{\rho_1}(g)\overline{\chi_{\rho_2}(g)}\right)\left(\frac{1}{|H|}\sum_{h\in H} \chi_{\pi_1}(h)\overline{\chi_{\pi_2}(h)}\right) \\
&= \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle_{L(G)} \langle \chi_{\pi_1}, \chi_{\pi_2} \rangle_{L(H)}
\end{aligned}$$

□

Corollary 2.2.4. *1. Let $\rho : G \to GL(V)$ and $\pi : H \to GL(W)$. Then $\rho \widehat{\otimes} \pi$ is irreducible if and only if both ρ and π are irreducible.*

2. Let $\rho_i : G \to GL(V_i)$ and $\pi_i : H \to GL(W_i)$ be irreducible. Then $\rho_1 \sim \rho_2$ and $\pi_1 \sim \pi_2$ if and only if

$$\rho_1 \widehat{\otimes} \pi_1 \sim \rho_2 \widehat{\otimes} \pi_2$$

Proof. 1. Recall that if φ is any representation of a group, then

$$\langle \chi_\varphi, \chi_\varphi \rangle \geq 1$$

and equality holds if and only if φ is irreducible. Now simply apply the previous theorem.

2. Note that for any $(g,h) \in G \times H$

$$\chi_{\psi_i}((g,h)) = \chi_{\rho_i}(g)\chi_{\pi_i}(h)$$

Hence, if $\rho_1 \sim \rho_2$ and $\pi_1 \sim \pi_2$, it follows that

$$\chi_{\psi_1} = \chi_{\psi_2}$$

and so $\psi_1 \sim \psi_2$.

3. Conversely, if $\psi_1 \sim \psi_2$, then by part (1)

$$\langle \chi_{\psi_1}, \chi_{\psi_2} \rangle = 1$$

From this it follows that

$$\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = \langle \chi_{\pi_1}, \chi_{\pi_2} \rangle = 1$$

By Schur orthogonality, it follows that $\rho_1 \sim \rho_2$ and $\pi_1 \sim \pi_2$.

$\square$

Theorem 2.2.5. *The map*

$$\alpha : \widehat{G} \times \widehat{H} \to \widehat{G \times H} \text{ given by } ([\rho], [\pi]) \mapsto [\rho \widehat{\otimes} \pi]$$

is a well-defined bijection.

Proof. 1. α is well-defined by the previous Corollary

2. To see that α is injective by the previous corollary, part 2.

3. To see that α is surjective, we show that

$$|Cl(G)||Cl(H)| = |Cl(G \times H)|$$

If $(g, h), (x, y) \in G \times H$, then

$$(x, y)^{-1}(g, h)(x, y) = (x^{-1}gx, y^{-1}hy)$$

Hence, $(g, h) \sim (g', h')$ if and only if $g \sim g'$ and $h \sim h'$. Hence the map

$$\alpha : Cl(G) \times Cl(H) \to Cl(G \times H) \text{ given by } ([g], [h]) \mapsto [(g, h)]$$

is a well-defined bijection.

$\square$

Example 2.2.6. We determine the character table of $S_3 \times \mathbb{Z}_2$. We have the character table of $G = S_3$ as

	e	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

and that of $\mathbb{Z}_2$ is given by

	0	1
χ_1	1	1
χ_2	1	-1

Now the representatives of the conjugacy classes of $S_3 \times \mathbb{Z}_2$ are

$$\{(e,0),(e,1),((12),0),((12),1),((123),0),((123),1)\}$$

We multiply characters to get the character table of $S_3 \times \mathbb{Z}_2$ to be

	$(e,0)$	$(e,1)$	$((12),0)$	$((12),1)$	$((123),0)$	$((123),1))$
$\chi_1 \times \chi_1$	1	1	1	1	1	1
$\chi_1 \times \chi_2$	1	-1	1	-1	1	-1
$\chi_2 \times \chi_1$	1	1	-1	-1	1	1
$\chi_2 \times \chi_2$	1	-1	-1	1	1	-1
$\chi_3 \times \chi_1$	2	2	0	0	-1	-1
$\chi_3 \times \chi_2$	2	-2	0	0	-1	1

Compare this with the discussion in [BS, Section 4.5]. This is, in fact, the tensor product of two square matrices representing the character tables of S_3 and $\mathbb{Z}_2$.

2.3 Inner Tensor Products of Representations

Theorem 2.3.1. *Let $\rho : G \to GL(V)$ and $\pi : G \to GL(W)$ be two representations of a group G. Then $\exists!$ representation $\varphi : G \to GL(V \otimes W)$ such that*

$$\varphi_g(v \otimes w) = \rho_g(v) \otimes \pi_g(w)$$

This is called the inner tensor product of ρ and π and is denote by by $\rho \otimes \pi$.

Proof. Consider the outer tensor product

$$\rho \widehat{\otimes} \pi : G \times G \to GL(V \otimes W)$$

and the diagonal homomorphism $\Delta : G \to G \times G$ given by $g \mapsto (g,g)$. Then define

$$\varphi = (\rho \widehat{\otimes} \pi) \circ \Delta$$

Then φ satisfies the required condition. Uniqueness also holds as before. $\square$

Theorem 2.3.2. *If ρ, π as above, then*

$$\chi_{\rho \otimes \pi}(g) = \chi_\rho(g)\chi_\pi(g) \quad \forall g \in G$$

In particular, the product of two characters is a character.

Proof. By the earlier theorem,

$$\chi_{\rho \otimes \pi}(g) = \chi_\psi(g,g) = \chi_\rho(g)\chi_\pi(g)$$

$\square$

Example 2.3.3. The character table of S_4 described in [BS, Example 7.2.13] is given below. Let π denote the augmentation representation of S_4 and ρ the irreducible representation of degree 2.

	1	(12)	(123)	(1234)	(12)(34)
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
$\chi_3 = \chi_\pi$	3	1	0	-1	-1
$\chi_4 = \chi_2\chi_3$	3	-1	0	1	-1
$\chi_5 = \chi_\rho$	2	0	-1	0	2
$\chi_\rho\chi_\pi$	6	0	0	0	-2
$\chi_2\chi_\rho$	2	0	-1	0	2

Hence,

1. If $\eta = \rho \otimes \pi$, then η has degree 6. In particular, η is not irreducible, so the inner tensor product of irreducible representations need not be irreducible.

2. Also, if $\mu = \chi_2 \otimes \rho$, then

$$\chi_\mu(g) = \mathrm{sgn}(g)\chi_\rho(g) = \chi_\rho(g)$$

since $\chi_\rho(g) = 0$ for all $g \notin A_4$. Hence, $\mu \sim \rho$. In particular,

$$\chi_2 \otimes \rho \sim \chi_1 \otimes \rho$$

but χ_2 is not equivalent to χ_1.

Compare these examples with Corollary 2.2.4.

2.3.1 Symmetric and Alternating Squares

Definition 2.3.4. Let V be a vector space, then $\exists!$ linear map $T : V \otimes V \to V \otimes V$ such that

$$T(v \otimes w) = w \otimes v$$

Write

$$S^2(V) = \{x \in V \otimes V : Tx = x\}$$
$$A^2(V) = \{x \in V \otimes V : Tx = -x\}$$

Lemma 2.3.5. *1. $V \otimes V = S^2(V) \oplus A^2(V)$*

2. Let $\rho : G \to GL(V)$ be a representation. Write $\varphi = \rho \otimes \rho$. If T as above, then

$$T\varphi_g = \varphi_g T \quad \forall g \in G$$

Proof. HW ∎

Definition 2.3.6. Let $\rho : G \to GL(V)$ be a representation of G. Then by the previous two lemmas, we may define

$$\rho_S = (\rho \otimes \rho)|_{S^2(V)} \text{ and } \rho_A = (\rho \otimes \rho)|_{A^2(V)}$$

Then

$$\rho \otimes \rho \sim \rho_S \oplus \rho_A$$

These are called the symmetric square and alternating square of ρ respectively.

Lemma 2.3.7. *Let $\{v_1, v_2, \dots, v_n\}$ be a basis for V. THen*

1. $\{v_i \otimes v_j + v_j \otimes v_i : 1 \leq i \leq j \leq n\}$ *is a basis for* $S^2(V)$
2. $\dim(S^2(V)) = n(n+1)/2$
3. $\{v_i \otimes v_j - v_j \otimes v_i : 1 \leq i \leq n\}$ *is a basis for* $A^2(V)$
4. $\dim(A^2(V)) = n(n-1)/2$

Proof. Let $S = \{v_i \otimes v_j + v_j \otimes v_i : 1 \leq i \leq j \leq n\}$, then $S \subset S^2(V)$. Similarly, if $T = \{v_i \otimes v_j - v_j \otimes v_i : 1 \leq i < j \leq n\}$, then $T \subset A^2(V)$. Furthermore, S and T are linearly independent since the set $\{v_i \otimes v_j : 1 \leq i, j \leq n\}$ is linearly independent. Hence,

$$\dim(S^2(V)) \geq n(n+1)/2 \text{ and } \dim(A^2(V)) \geq n(n-1)/2$$

However,

$$\dim(S^2(V)) + \dim(A^2(V)) = \dim(V \otimes V) = n^2$$

So both the above inequalities are equalities and the results follow. □

Proposition 2.3.8. *Let $\rho : G \to GL(V)$ be a representation with character χ. Suppose χ_S and χ_A denote the characters of ρ_S and ρ_A respectively, then*

$$\chi_S(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2)) \quad \forall g \in G$$
$$\chi_A(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2)) \quad \forall g \in G$$

Proof. Fix $g \in G$, then ρ_g is diagonalizable. So choose a basis $\{v_1, v_2, \dots, v_n\}$ of V such that

$$\rho_g(v_i) = \lambda_i v_i \quad \forall 1 \leq i \leq n$$

Hence,

$$\chi(g) = \sum_{i=1}^{n} \lambda_i \text{ and } \chi(g^2) = \sum_{i=1}^{n} \lambda_i^2$$

If $w_{i,j} = v_i \otimes v_j + v_j \otimes v_i$, then

$$\rho_S(g)(w_{i,j}) = \rho_g(v_i) \otimes \rho_g(v_j) + \rho_g(v_j) \otimes \rho_g(v_i) = \lambda_i \lambda_j w_{i,j}$$

Similarly, taking $t_{i,j} = v_i \otimes v_j - v_j \otimes v_i$, then

$$\rho_A(g)(t_{i,j}) = \lambda_i \lambda_j t_{i,j}$$

Hence,

$$\begin{aligned}
\chi_S(g) &= \sum_{1 \le i \le j \le n} \lambda_i \lambda_j \\
\chi_A(g) &= \sum_{1 \le i < j \le n} \lambda_i \lambda_j \\
\Rightarrow \chi_S(g) &= \sum_{i=1}^{n} \lambda_i^2 + \chi_A(g) \\
&= \chi(g^2) + \chi_A(g) \\
\Rightarrow \chi(g^2) &= \chi_S(g) - \chi_A(g) \qquad (*)
\end{aligned}$$

Also,

$$\begin{aligned}
\chi(g)^2 &= \left(\sum_{i=1}^{n} \lambda_i \right)^2 = \sum_{i=1}^{n} \lambda_i^2 + 2 \sum_{i<j} \lambda_i \lambda_j \\
&= \chi(g^2) + 2\chi_A(g) \\
\Rightarrow \chi(g)^2 &= \chi_S(g) + \chi_A(g) \qquad (**)
\end{aligned}$$

Solving $(*)$ and $(**)$ gives the required result. □

2.3.2 Character Table of S_5

We now determine the character table of S_5. Let $G = S_5$

1. As done for S_4, we see that $[G, G] = A_5$. Hence, G has two linear characters

$$\chi_1 \text{ and } \chi_2 = \operatorname{sgn}$$

2. The augmentation representation ρ is a degree 4 irreducible representation with character

$$\chi_3(g) = |\mathrm{Fix}(g)| - 1$$

3. Let $\varphi_g = \chi_2(g)\rho_g$ is another irreducible degree 4 representation with character

$$\chi_4(g) = \operatorname{sgn}(g)(|\mathrm{Fix}(g)| - 1)$$

4. The conjugacy classes of S_5 are given as

e	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
1	10	20	15	30	20	24

Hence, S_5 has 7 irreducible representations. We have determined 4 so far, so we have a partial character table as below

	e	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1
χ_3	4	2	1	0	0	-1	-1
χ_4	4	-2	1	0	0	1	-1

5. Let ρ be as above, then if χ_S and χ_A are the characters of the symmetric and alternating squares of ρ, then we can obtain their values by the previous theorem. For instance,

$$\chi_S((123)) = \frac{1}{2}(\chi((123))^2 + \chi((123)^2)) = \frac{1}{2}(1^2 + \chi((132))) = \frac{1}{2}(1+1) = 1$$

Similarly, we obtain the values of χ_S and χ_A as below

	e	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
χ_S	10	4	1	2	0	1	0
χ_A	6	0	0	-2	0	0	1

6. Now,

$$\langle \chi_A, \chi_A \rangle = \frac{1}{120}[(1 \cdot 36) + (20 \cdot 0) + (15 \cdot 4) + (30 \cdot 0) + (20 \cdot 0) + (24 \cdot 1)] = 1$$

So χ_A is the character of an irreducible representation. This must necessarily be different from the ones already obtained since it has degree 6. We write $\chi_5 = \chi_A$.

7. Now,

$$\langle \chi_S, \chi_S \rangle = 3$$

so it does not correspond to an irreducible representations, but calculating inner products gives

$$\langle \chi_S, \chi_i \rangle = 1 \quad i \in \{1, 3\}, \text{ and}$$
$$\langle \chi_S, \chi_j \rangle = 0 \quad j \in \{2, 4, 5\}$$

Hence, $\exists$ a sixth irreducible representation ψ such that

$$\chi_S = \chi_1 + \chi_3 + \chi_\psi$$

We write $\chi_6 = \chi_\psi$ and note that

$$\chi_6 = \chi_S - \chi_1 - \chi_3 \qquad (*)$$

In particular, $\chi_6(1) = 10 - 1 - 4 = 5$.

8. Now if $\varphi_g = \chi_2(g)\psi_g$, then φ is an irreducible representation of degree 5 such that

$$\chi_\varphi(g) = \chi_2(g)\chi_\psi(g) \qquad (**)$$

In this case, using equation $(*)$, we see that

$$\chi_\psi((12)) = \chi_S((12)) - \chi_1((12)) - \chi_3((12)) = 4 - 1 - 2 = 1 \neq \chi_\varphi((12))$$

Hence, φ is not equivalent to ψ. We write $\chi_7 = \chi_\varphi$, so equations $(*)$ and $(**)$ allow us to complete the character table of S_5.

	e	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1
χ_3	4	2	1	0	0	-1	-1
χ_4	4	-2	1	0	0	1	-1
χ_5	6	0	0	-2	0	0	1
χ_6	5	1	-1	1	-1	1	0
χ_7	5	-1	-1	1	1	-1	0

3 Restriction to a Subgroup

Definition 3.0.1. Let G be a group, $H < G$ and $\rho : G \to GL(V)$ be a representation. We may restrict ρ to obtain a representation

$$\rho|_H\colon H \to GL(V)$$

This is called the restriction of ρ to H.

Note that even if ρ is irreducible, $\rho|_H$ may not be.

Proposition 3.0.2. *Suppose $\exists H < G$ such that H is Abelian, then*

$$d_\rho \leq [G : H] \quad \forall \rho \in \widehat{G}$$

Proof. Let $\rho : G \to GL(V)$ be irreducible and $d = d_\rho$, then $\rho|_H\colon H \to GL(V)$ is a representation. Hence, $\exists$ one dimensional representations $\varphi_1, \varphi_2, \ldots, \varphi_d : H \to \mathbb{C}^*$ such that

$$\rho \sim \varphi_1 \oplus \varphi_2 \oplus \ldots \oplus \varphi_d$$

In particular, $\exists W < V$ such that $\dim(W) = 1$, which is invariant under $\rho|_H$. Write $W = \operatorname{span}\{v\}$, and set

$$W' = \operatorname{span}\{\rho_g(v) : g \in G\}$$

Then, $W' < V$ is $\rho(G)$-invariant. Since ρ is irreducible,

$$V = W' = \operatorname{span}\{\rho_g(v) : g \in G\}$$

Now suppose $g \in G, h \in H$, then

$$\rho(gh)(v) = \rho(g)\rho(h)(v) = \lambda\rho(g)(v)$$

and so $\rho(gh)(v) \in \operatorname{span}\{\rho(g)(v)\}$. Hence, if $G/H = \{g_1H, g_2H, \ldots, g_\ell H\}$, with $\ell = [G : H]$, then

$$V = \operatorname{span}\{\rho(g_i)v : 1 \leq i \leq \ell\}$$

In particular, $d = \dim(V) \leq \ell = [G : H]$. $\square$

Example 3.0.3. Let $G = D_n$ be the dihedral group of order 2n. Then any irreducible representation of G has degree 1 or 2. If $n = p$, prime, we describe all the irreducible representations of G.

1. Write $G = D_p = \langle a, b : a^p = b^2 = 1, bab = a^{p-1}\rangle$, and $H = \langle a\rangle$. Then $H \lhd G$ and $G/H \cong \mathbb{Z}_2$, so $[G, G] \subset H$. However, $|H| = p$ and G is non-abelian, so $[G, G] = H$. Hence, G has exactly two linear characters given by

$$\chi_1 : a \mapsto 1, b \mapsto 1$$
$$\chi_2 : a \mapsto 1, b \mapsto -1$$

2. Since every other irreducible representation has degree 2, the degree formula gives

$$2p = 2 + 4k \Rightarrow k = (p-1)/2$$

and so G has exactly $(p-1)/2$ irreducible representations of degree 2.

3. For $1 \leq j \leq (p-1)/2$, define $\psi_j : G \to GL_2(\mathbb{C})$ by

$$a \mapsto \begin{pmatrix} \zeta^j & 0 \\ 0 & \zeta^{-j} \end{pmatrix} \text{ and } b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

As in the earlier HW, ψ_j is irreducible. Furthermore, if $\zeta^j = \zeta^{\pm i}$, then

$$p \mid (j \pm i)$$

This is impossible if $1 \leq i, j \leq (p-1)/2$, and so for such i, j, we have that $\psi_j(a)$ and $\psi_i(a)$ have different eigen-values. In particular, ψ_j is not equivalent to ψ_i.

4. Thus, the irreducible representations of G are

$$\widehat{G} = \{\chi_1, \chi_2, \psi_j : 1 \leq j \leq (p-1)/2\}$$

Definition 3.0.4. As observed above, even if $\rho : G \to GL(V)$ is irreducible, its restriction $\rho|_H : H \to GL(V)$ may not be irreducible. Write $\widehat{H} = \{\psi_1, \psi_2, \ldots, \psi_r\}$, and set

$$s_i = \langle \chi_{\rho|_H}, \chi_{\psi_i} \rangle_{L(H)}$$

Then s_i are the multiplicities of ψ_i in $\rho|_H$. We say that ψ_i is a constituent of $\rho|_H$ if $s_i \neq 0$. Note that

$$\chi_{\rho|_H} = \sum_{i=1}^{r} s_i \chi_{\psi_i}$$

Theorem 3.0.5. *Let $H < G$, and let $\psi : H \to GL(W)$ be a non-zero representation of H. Then $\exists$ an irreducible representation $\rho : G \to GL(V)$ such that*

$$\langle \chi_{\rho|_H}, \chi_\psi \rangle_{L(H)} \neq 0$$

In particular, every irreducible representation of H occurs as a constituent of an irreducible representation of G.

Proof. Write $\widehat{G} = \{\varphi^{(1)}, \varphi^{(2)}, \ldots, \varphi^{(s)}\}, d_i = d_{\varphi^{(i)}}$, and let $\chi_i = \chi_{\varphi^{(1)}}$. Let $L : G \to GL(L(G))$ denote the left regular representation, then

$$\chi_L = \sum_{i=1}^{s} d_i \chi_i$$

Let $\psi : H \to GL(W)$ as above, then

$$\begin{aligned}\sum_{i=1}^{s} d_i \langle \chi_{\varphi^{(i)}|_H}, \chi_\psi \rangle &= \langle \chi_L|_H, \chi_\psi \rangle_{L(H)} \\ &= \frac{1}{|H|} \sum_{h \in H} \chi_L(h) \overline{\chi_\psi(h)} \\ &= \frac{1}{|H|} \chi_L(e) \overline{\chi_\psi(e)} = \frac{|G|}{|H|} d_\psi \neq 0\end{aligned}$$

Hence, $\exists 1 \leq i \leq s$ such that

$$\langle \chi_{\varphi^{(i)}|_H}, \chi_\psi \rangle_{L(H)} \neq 0$$

□

Proposition 3.0.6. *Let $H < G, \rho : G \to GL(V)$ be an irreducible representation of G. Let $\widehat{H} = \{\psi_1, \psi_2, \ldots, \psi_r\}$, and write*

$$s_i = \langle \chi_{\rho|_H}, \chi_{\psi_i} \rangle_{L(H)}$$

Then

$$\sum_{i=1}^{r} s_i^2 \leq [G : H]$$

and equality holds if and only if

$$\chi_\rho(g) = 0 \quad \forall g \in G \setminus H$$

Proof. We know that

$$\sum_{i=1}^{r} s_i^2 = \langle \chi_{\rho|_H}, \chi_{\rho|_H} \rangle_{L(H)} = \frac{1}{|H|} \sum_{h \in H} \chi_\rho(h) \overline{\chi_\rho(h)}$$

Since ρ is irreducible on G, we have

$$\begin{aligned}1 = \langle \chi_\rho, \chi_\rho \rangle_{L(G)} &= \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) \overline{\chi_\rho(g)} \\ &= \frac{1}{|G|} \sum_{h \in H} \chi_\rho(h) \overline{\chi_\rho(h)} + \frac{1}{|G|} \sum_{g \in G \setminus H} \chi_\rho(g) \overline{\chi_\rho(g)} \\ &= \frac{|H|}{|G|} \sum_{i=1}^{r} s_i^2 + K\end{aligned}$$

Hence,

$$\sum_{i=1}^{r} s_i^2 = \frac{(1-K)|G|}{|H|}$$

Note that $K \geq 0$ and $K = 0$ if and only if $\chi_\rho(g) = 0$ for all $g \in G \setminus H$. This gives the result. □

Corollary 3.0.7. *Let $H < G$ be a subgroup of index 2, and let $\rho : G \to GL(V)$ be an irreducible representation of G. Then one of the following happens:*

1. $\rho|_H$ *is an irreducible representation of* H.
2. $\exists \psi_1, \psi_2 \in \widehat{H}$ *such that* $\rho|_H = \psi_1 \oplus \psi_2$.

Furthermore, part (2) occurs if and only if $\chi_\rho(g) = 0$ *for all* $g \in G \setminus H$.

3.1 Character Table of A_5

3.1.1 Conjugacy classes in A_5

Definition 3.1.1. Let G be a group and $x \in G$.

1. The conjugacy class of x in G is denoted by $x^G = \{yxy^{-1} : y \in G\}$
2. The centralizer of x in G is

$$C_G(x) = \{y \in G : yx = xy\} = \{y \in G : yxy^{-1} = x\}$$

Note that if we let G act on itself by conjugation, then the conjugacy class of G is the orbit of x, while the centralizer of x is the stabilizer of x. So by the orbit-stabilizer theorem,

$$|x^G| = [G : C_G(x)]$$

Now, for any $\sigma \in A_n$, write

$$\sigma^{S_n} \text{ and } \sigma^{A_n}$$

to denote the conjugacy classes of σ in S_n and A_n respectively. Clearly,

$$\sigma^{A_n} \subset \sigma^{S_n}$$

Note that since $A_n \lhd S_n$, we have $\sigma^{S_n} \subset A_n$

Proposition 3.1.2. *For* $\sigma \in A_n$ *with* $n > 1$, *we have*

1. *If* σ *commutes with an odd permutation, then* $\sigma^{A_n} = \sigma^{S_n}$
2. *If* σ *does not commute with some odd permutation, then*

$$\sigma^{S_n} = \sigma^{A_n} \sqcup ((12)\sigma(12))^{A_n}$$

and

$$|\sigma^{A_n}| = |((12)\sigma(12))^{A_n}| = \frac{|\sigma^{S_n}|}{2}$$

Proof. 1. Suppose $\tau \in S_n$ is an odd permutation which commutes with σ, then we WTS: $\sigma^{S_n} \subset \sigma^{A_n}$. So fix $\eta \in \sigma^{S_n}$ and $\delta \in S_n$ such that

$$\eta = \delta\sigma\delta^{-1}$$

If $\delta \in A_n$, then $\eta \in \sigma^{A_n}$. If not, then $\delta' = \delta\tau \in A_n$ and

$$\delta'\sigma\delta'^{-1} = \delta\sigma\delta^{-1} = \eta \Rightarrow \eta \in \sigma^{A_n}$$

2. Suppose σ does not commute with any odd permutation. Then, by definition,

$$C_{S_n}(\sigma) = C_{A_n}(\sigma)$$

Hence,

$$|\sigma^{A_n}| = [A_n : C_{A_n}(\sigma)] = \frac{|A_n|}{|C_{A_n}(\sigma)|} = \frac{|S_n|}{2|C_{S_n}(\sigma)|} = \frac{|\sigma^{S_n}|}{2}$$

Now observe that

$$\sigma^{S_n} = \{\delta\sigma\delta^{-1} : \delta \in A_n\} \sqcup \{\delta\sigma\delta^{-1} : \delta \in S_n \setminus A_n\}$$

Now δ is odd if and only if $\eta = \delta(12) \in A_n$. Hence,

$$\{\delta\sigma\delta^{-1} : \delta \in S_n \setminus A_n\} = \{\eta(12)\sigma(12)\eta^{-1} : \eta \in A_n\} = ((12)\sigma(12))^{A_n}$$

The theorem now follows.

□

We now examine the conjugacy classes in S_5

e	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
1	10	20	15	30	20	24

Of these, $(12) \notin A_5, (1234) \notin A_5, (123)(45) \notin A_5$. Also,

$$(45)(123)(45) = (123) \Rightarrow (123)^{S_5} = (123)^{A_5}$$
$$(12)(12)(34)(12) = (12)(34) \Rightarrow ((12)(34))^{S_5} = ((12)(34))^{A_5}$$

and

$$C_{S_n}((12345)) = \frac{120}{24} = 5$$

is not divisible by two. Hence, $(12345)^{A_n} \neq (12345)^{S_n}$. Hence,

$$(12)(12345)(12) = (13452)$$

is another representative of a conjugacy class in A_5. So we get the conjugacy classes in A_5 are

e	(123)	(12)(34)	(12345)	(13452)
1	20	15	12	12

3.1.2 Real Character Values

Lemma 3.1.3. *If $\rho : G \to GL(V)$ is a unitary representation and $g \in G$, then*

$$\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$$

Proof. Since ρ_g is diagonalizable, write

$$[\rho_g]_{\mathcal{B}} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

where each $\lambda_i \in S^1$. Hence,

$$[\rho_g^{-1}]_{\mathcal{B}} = \text{diag}(\lambda_1^{-1}, \ldots, \lambda_n^{-1})$$

But $\lambda_i^{-1} = \overline{\lambda_1}$, so the result follows by taking traces. □

Theorem 3.1.4. *Let G be a group and $g \in G$. If g is conjugate to g^{-1} if and only if $\chi_\rho(g) \in \mathbb{R}$ for all $\rho \in \widehat{G}$*

Proof. By the previous lemma

$$\chi_\rho(g^{-1}) = \chi_\rho(g) \Leftrightarrow \chi_\rho(g) \in \mathbb{R}$$

So the corollary follows from Mid-Sem Exam, Problem 2. □

Corollary 3.1.5. *For very representation ρ of A_5, $\chi_\rho(g) \in \mathbb{R}$*

Proof. It suffices to show that every element in $\{e, (123), (12)(34), (12345), (13452)\}$ is conjugate to its own inverse. This is evident for elements in $\{e, (123), (12)(34)\}$. For the other two, check that

$$(12345)^{-1} = (54321) = (15)(24)(12345)(15(24)$$

and

$$(13452)^{-1} = (25431) = (12)(35)(13452)(12)(35)$$

□

3.1.3 Character Table of A_5

Now consider the character table of S_5 obtained in the previous section.

	e	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1
$\chi_3 = \chi_\rho$	4	2	1	0	0	-1	-1
$\chi_4 = \chi_{\chi_2 \otimes \rho}$	4	-2	1	0	0	1	-1
$\chi_5 = \chi_{\rho_A}$	6	0	0	-2	0	0	1
$\chi_6 = \chi_\psi$	5	1	-1	1	-1	1	0
$\chi_7 = \chi_{\chi_2 \otimes \psi}$	5	-1	-1	1	1	-1	0

Restricting to $H = A_5$, we see that

1. $\chi_1((12)) \neq 0$, so $\chi_1|_H$ is irreducible.

   $\chi_2((12)) \neq 0$, so $\chi_2|_H$ is irreducible. However, $\chi_2|_H = \chi_1|_H$

2. $\chi_3((12)) \neq 0$, so $\rho|_H$ is irreducible.

   $\chi_4((12)) \neq 0$, so $(\chi_2 \otimes \rho)|_H$ is irreducible. However,

$$\chi_4(g) = \chi_3(g) \quad \forall g \in A_5$$

   so $\rho|_H \sim (\chi_2 \otimes \rho)|_H$

3. $\chi_5(g) = 0$ for all $g \in S_5 \setminus A_5$, so $\rho_A = \psi_1 \oplus \psi_2$ for two irreducible representations ψ_1 and ψ_2 of A_5

4. $\chi_6((12)) \neq 0$, so $\psi|_H$ is irreducible.

   As above, $\psi|_H \sim (\chi_2 \otimes \psi)|_H$.

So we obtain a partial character table

	e	(123)	(12)(34)	(12345)	(13452)
	1	20	15	12	12
$\varphi_1 = \chi_1\|_H$	1	1	1	1	1
$\varphi_2 = \chi_3\|_H$	4	1	0	-1	-1
$\varphi_3 = \chi_6\|_H$	5	-1	1	0	0
$\varphi_4 = \chi_{\psi_1}$	n_1	a_1	a_2	a_3	a_4
$\varphi_5 = \chi_{\psi_2}$	n_2	b_1	b_2	b_3	b_4

Note that if $n_i = d_{\psi_i}, i = 1, 2$, then

$$1 + 16 + 25 + n_1^2 + n_2^2 = 60 \Rightarrow n_1^2 + n_2^2 = 18 \Rightarrow n_1 = n_2 = 3$$

Furthermore,

$$\chi_{\psi_1} + \chi_{\psi_2} = \chi_{\rho_A}$$

Hence, we get

$$\begin{aligned} a_1 + b_1 = \chi_{\rho_A}((123)) = 0 &\Rightarrow b_1 = -a_1 \\ a_2 + b_2 = \chi_{\rho_A}((12)(34)) = -2 &\Rightarrow b_2 = -2 - a_2 \\ a_3 + b_3 = \chi_{\rho_A}((12345)) = 1 &\Rightarrow b_3 = 1 - a_3 \\ &\Rightarrow b_4 = 1 - a_4 \end{aligned}$$

So we get an incomplete table as

	e	(123)	(12)(34)	(12345)	(13452)
	1	20	15	12	12
$\varphi_1 = \chi_1\|_H$	1	1	1	1	1
$\varphi_2 = \chi_3\|_H$	4	1	0	-1	-1
$\varphi_3 = \chi_6\|_H$	5	-1	1	0	0
$\varphi_4 = \chi_{\psi_1}$	3	a_1	a_2	a_3	a_4
$\varphi_5 = \chi_{\psi_2}$	3	$-a_1$	$-2-a_2$	$1-a_3$	$1-a_4$

Orthonormality of columns gives

$$\frac{20}{60}[1+1+1+a_1^2+a_1^2] = 1 \Rightarrow a_1 = 0$$
$$\frac{15}{60}[1+1+a_2^2+b_2^2] = 1 \Rightarrow a_2^2+b_2^2 = 2$$
$$\frac{12}{60}[1+1+a_3^2+b_3^2] = 1 \Rightarrow a_3^2+b_3^2 = 3$$
$$\frac{12}{60}[1+1+a_4^2+b_4^2] = 1 \Rightarrow a_4^2+b_4^2 = 3$$

Since $b_2 = -2 - a_2$ and $a_2^2 + b_2^2 = 2$, it follows that

$$a_2 = b_2 = -1$$

Now since $b_3 = (1 - a_3)$, we see that a_3 and a_4 are both solutions to the equation

$$x^2 - x - 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$$

Since $\varphi_4 \neq \varphi_5$, the character table of A_5 is

	e	(123)	(12)(34)	(12345)	(13452)
	1	20	15	12	12
$\varphi_1 = \chi_1\|_H$	1	1	1	1	1
$\varphi_2 = \chi_3\|_H$	4	1	0	-1	-1
$\varphi_3 = \chi_6\|_H$	5	-1	1	0	0
$\varphi_4 = \chi_{\psi_1}$	3	0	-1	x	y
$\varphi_5 = \chi_{\psi_2}$	3	0	-1	y	x

where $x = \frac{1+\sqrt{5}}{2}$ and $y = \frac{1-\sqrt{5}}{2}$

4 Induced Representations

4.1 Definition and Examples

Definition 4.1.1. Let G be a group and $H < G$. Let $\rho : H \to GL(W)$ be a representation.

1. Define $X = \{f : G \to W\}$. Note that X is a vector space under the pointwise operations. Define

$$I(W) := \{f \in X : f(gh) = \rho_{h^{-1}}(f(g)) \quad \forall g \in G, h \in H\}$$

   Note that $I(W)$ is a vector subspace of X.

2. For $g \in G$, define

$$T_g : I(W) \to I(W) \text{ given by } T_g(f)(x) := f(g^{-1}x)$$

   Then T_g is well-defined

   Proof. If $f \in I(W)$, then for any $h \in H, x \in G$,

$$T_g(f)(xh) = f(g^{-1}xh) = \rho_{h^{-1}}(f(g^{-1}x)) = \rho_{h^{-1}}T_g(f)(x)$$

   Hence, $T_g(f) \in I(W)$ □

3. Moreover, $T_g \in GL(I(W))$

   Proof. Simply check that

$$T_g \circ T_{g^{-1}}(f)(x) = T_{g^{-1}}(f)(g^{-1}x) = f(gg^{-1}x) = f(x) \quad \forall x \in G, f \in V$$

   Hence, $T_g \circ T_{g^{-1}} = \mathrm{id}_{I(W)}$. Similarly, $T_{g^{-1}} \circ T_g = \mathrm{id}_{I(W)}$ □

4. Finally, the map $\varphi : G \to GL(I(W))$ given by

$$\varphi(g) = T_g$$

   is a representation of G.

   Proof. For $g_1, g_2 \in G, f \in I(W)$, and $x \in G$, we have

$$\begin{aligned}(T_{g_1} \circ T_{g_2})(f)(x) &= T_{g_2}(f)(g_1^{-1}x) \\ &= f(g_2^{-1}g_1^{-1}x) \\ &= f((g_1g_2)^{-1}x) \\ &= T_{g_1g_2}(f)(x)\end{aligned}$$

   □

The representation $\varphi : G \to GL(I(W))$ is called the induced representation of $\rho : H \to GL(W)$, and is denoted by $\varphi = \mathrm{Ind}_H^G(\rho)$.

Proposition 4.1.2. $\dim(I(W)) = \dim(W)[G : H]$

Proof. Write $G/H = \{x_1H, x_2H, \ldots, x_\ell H\}$, so that $\ell = [G : H]$. Define a map

$$T : I(W) \to \oplus_{i=1}^{\ell} W \text{ given by } f \mapsto (f(x_1), f(x_2), \ldots, f(x_\ell))$$

T is clearly linear. We claim that T is bijective, which proves the theorem.

1. T is injective: Suppose $T(f) = 0$, then $f(x_i) = 0$ for all $1 \leq i \leq \ell$. Then if $g \in G, \exists 1 \leq i \leq \ell$ and $h \in H$ such that $g = x_ih$. Hence,

   $$f(g) = f(x_ih) = \rho_{h^{-1}} f(x_i) = 0$$

   Hence, $f = 0$

2. T is surjective: Given $(w_1, w_2, \ldots, w_\ell) \in \oplus_{i=1}^{\ell} W$, define $f : G \to W$ such that

   $$f(x_ih) = \rho_{h^{-1}}(w_i) \quad \forall h \in H, 1 \leq i \leq \ell$$

   This is well-defined since $G = \sqcup_{i=1}^{\ell} x_iH$. Furthermore, for any $g \in G, h \in H$, write $g = x_ih'$, so that $h'h \in H$, and

   $$f(gh) = f(x_ih'h) = \rho_{(h'h)^{-1}}(w_i) = \rho_{h^{-1}}\rho_{(h')^{-1}}(w_i) = \rho_{h^{-1}} f(g)$$

   Hence, $f \in I(W)$. Now clearly, $T(f) = (w_1, w_2, \ldots, w_\ell)$ holds.

□

Example 4.1.3. 1. Let $H = \{e\} < G$ and $\chi_1 : H \to \mathbb{C}^*$ be the trivial representation. Then, by the above definition, $W = \mathbb{C}$,

   $$X = \{f : G \to \mathbb{C}\} = L(G) \text{ and } I(W) = X = L(G)$$

   Finally,

   $$T_g(f)(x) = f(g^{-1}x)$$

   Hence, $\mathrm{Ind}_H^G(\chi_1)$ is the left regular representation of G.

2. Let $H = G$ and $\rho : G \to GL(W)$ be any representation. Then

   $$X = \{f : G \to W\} \text{ and } I(W) = \{f \in X : f(xg) = \rho_{g^{-1}}(f(x)) \quad \forall g, x \in G\}$$

   and let $\widehat{\rho} = \mathrm{Ind}_H^G(\rho)$. Define $T : I(W) \to W$ by $f \mapsto f(e)$. Then T is well-defined, and linear. Also, if $S : W \to I(W)$ given by

   $$S(w)(x) := \rho_{x^{-1}}(w) \quad \forall x \in X$$

Then, for any $f \in I(W)$, and $x \in X$

$$(S \circ T)(f)(x) = S(f(e))(x) = \rho_{x^{-1}}(f(e)) = f(ex) = f(x)$$

Hence, $S \circ T = \mathrm{id}_{I(W)}$. Also,

$$(T \circ S)(w) = T(S(w)) = S(w)(e) = \rho_{e^{-1}}(w) = w$$

and so $T \circ S = \mathrm{id}_W$. Hence, T is an isomorphism. Furthermore, for any $g \in G, f \in I(W)$,

$$T(\widehat{\rho}_g(f)) = L_g(f)(e) = f(g^{-1}e) = f(eg^{-1}) = \rho_g(f(e)) = \rho_g(T(f))$$

Hence, $T \circ \widehat{\rho}_g = \rho_g \circ T$. Hence, $T \in \mathrm{Hom}_G(\widehat{\rho}, \rho)$. Hence,

$$\mathrm{Ind}_H^G(\rho) \sim \rho$$

3. Let $G = D_{2n} = \langle a, b : a^n = b^2 = 1, bab = a^{n-1} \rangle$ and let $H = \langle a \rangle$. Let $\rho \in \widehat{H}$ be an irreducible representation of H, then $H \cong \mathbb{Z}_n$, so $\exists k \in \{0, 1, \ldots, n-1\}$ such that

$$\rho(a) = \zeta^k$$

where $\zeta = e^{2\pi i/n}$. Here, $W = \mathbb{C}$, so $X = \{f : G \to \mathbb{C}\} = L(G)$. Also, $I(W) < X$ is a space of dimension

$$\dim(\mathbb{C})[G : H] = 2$$

By the above proposition, we have an isomorphism

$$I(W) \to \mathbb{C}^2 \text{ given by } f \mapsto (f(e), f(b))$$

Let $\mathcal{B} = \{f_1, f_2\} \subset I(W)$ be functions such that

$$f_1(e) = 1, f_1(b) = 0 \text{ and } f_2(e) = 0, f_2(b) = 1$$

Write $\widehat{\rho} = \mathrm{Ind}_H^G(\rho)$. Then,

$$\begin{aligned}
\widehat{\rho}_a(f_1)(e) &= f_1(a^{-1}) = f_1(ea^{-1}) = \rho_{a^{-1}}(f_1(e)) = \rho_{a^{-1}}(1) = \zeta^{-k} \\
\widehat{\rho}_a(f_1)(b) &= f_1(a^{-1}b) = f_1(ba) = \rho_a(f_1(b)) = \rho_a(0) = 0 \\
\widehat{\rho}_a(f_2)(e) &= f_2(a^{-1}) = \rho_a(f_2(e)) = \rho_a(0) = 0 \\
\widehat{\rho}_a(f_2)(b) &= f_2(a^{-1}b) = f_2(ba) = \rho_a(f_2(b)) = \zeta^k
\end{aligned}$$

Hence,

$$[\widehat{\rho}_a]_{\mathcal{B}} = \begin{pmatrix} \zeta^{-k} & 0 \\ 0 & \zeta^k \end{pmatrix}$$

Also,

$$\begin{aligned}
\widehat{\rho}_b(f_1)(e) &= f_1(b^{-1}e) = f_1(b) = 0 \\
\widehat{\rho}_b(f_1)(b) &= f_1(b^{-1}b) = f_1(e) = 1 \\
\widehat{\rho}_b(f_2)(e) &= f_2(b^{-1}e) = f_2(b) = 1 \\
\widehat{\rho}_b(f_2)(b) &= f_2(b^{-1}b) = f_2(e) = 0
\end{aligned}$$

and so

$$[\widehat{\rho}_b]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

4.2 Frobenius Character Formula

We fix some notation for this section:

1. Let $\rho : H \to GL(W)$ be a representation and $\widehat{\rho} = \text{Ind}_H^G(\rho)$. We wish to determine the character of the induced representation. We write

$$\chi = \chi_\rho \text{ and } \text{Ind}_H^G(\chi) = \chi_{\widehat{\rho}}$$

   To do this, we assume that W has an inner product $\langle \cdot, \cdot \rangle$ and that ρ is a unitary representation of H.

2. A set $T = \{x_1, x_2, \ldots, x_\ell\} \subset G$ is called a <u>transversal</u> of H in G if

$$G = \bigsqcup_{i=1}^{\ell} x_i H$$

3. If $I(W)$ as above, we define an inner product on $I(W)$ as

$$\langle f_1, f_2 \rangle = \sum_{k=1}^{\ell} \langle f_1(x_k), f_2(x_k) \rangle$$

   Note that this defines an inner product on $I(W)$ by the proof of Proposition 4.1.2.

4. Choose an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of W and . For $1 \leq i \leq \ell, 1 \leq j \leq n$, let $f_{i,j} \in I(W)$ such that

$$f_{i,j}(x_k) = \delta_{i,k} e_j$$

   Then $\{f_{i,j} : 1 \leq i \leq \ell, 1 \leq j \leq n\}$ forms an orthonormal basis for $I(W)$ (using the isomorphism from Proposition 4.1.2)

Theorem 4.2.1 (Frobenius Character Formula)**.** *Let $\rho : H \to GL(W)$ be a representation with character χ, and let χ^G denoted the character of the induced representation $Ind_H^G(\rho)$. If $T = \{x_1, x_2, \ldots, x_\ell\}$ denotes a transversal of H in G, then*

$$Ind_H^G(\chi)(g) = \sum_{x_i^{-1} g x_i \in H} \chi(x_i^{-1} g x_i)$$

Proof. Let $f_{i,j}$ be the ONB of $I(W)$ as defined above, then we wish to determine

$$\sum_{j=1}^{n} \langle \widehat{\rho}_g(f_{i,j}), f_{i,j} \rangle$$

1. Consider each term, then by definition

$$\langle \widehat{\rho}_g(f_{i,j}), f_{i,j} \rangle = \sum_{k=1}^{\ell} \langle \widehat{\rho}_g(f_{i,j})(x_k), f_{i,j}(x_k) \rangle = \langle \widehat{\rho}_g(f_{i,j})(x_i), e_j \rangle$$

Now,
$$\widehat{\rho}_g(f_{i,j})(x_i) = f_{i,j}(g^{-1}x_i)$$
Since $g^{-1}x_i \in G = \sqcup_{m=1}^{\ell} x_m H, \exists$ unique $1 \leq m \leq \ell$ such that
$$g^{-1}x_i \in x_m H$$
and so $\exists$ unique $h \in H$ such that $g^{-1}x_i = x_m h$. Hence,
$$\begin{aligned}\widehat{\rho}_g(f_{i,j})(x_i) &= f_{i,j}(x_m h) = \rho_{h^{-1}}(f_{i,j}(x_m)) \\ &= \rho_{h^{-1}}(\delta_{i,m} e_j) = \begin{cases} 0 & : i \neq m \\ \rho_{h^{-1}}(e_j) & : i = m \end{cases}\end{aligned}$$
Now,
$$i = m \Leftrightarrow g^{-1}x_i \in x_i H \Leftrightarrow x_i^{-1} g^{-1} x_i \in H \Leftarrow x_i^{-1} g x_i \in H$$
and in this case, $h = x_i^{-1} g^{-1} x_i$, so $h^{-1} = x_i^{-1} g x_i$. Hence,
$$\widehat{\rho}_g(f_{i,j})(x_i) = \begin{cases} 0 & : x_i^{-1} g x_i \in H \\ \rho_{x_i^{-1} g x_i}(e_j) & : \text{ otherwise} \end{cases}$$
Hence,
$$\begin{aligned}\mathrm{Ind}_H^G(g) &= \sum_{i,j} \langle \widehat{\rho}_g(f_{i,j}), f_{i,j} \rangle \\ &= \sum_{x_i^{-1} g x_i \in H} \sum_{j=1}^{n} \langle \rho_{x_i^{-1} g x_i}(e_j), e_j \rangle \\ &= \sum_{x_i^{-1} g x_i \in H} \chi(x_i^{-1} g x_i)\end{aligned}$$
□

Example 4.2.2. Let $G = D_{2n} = \langle a, b : a^n = b^2 = 1, bab = a^{n-1} \rangle, H = \langle a \rangle$ and $\rho : H \to \mathbb{C}^*$ be the map
$$a \mapsto \zeta^k$$
where $\zeta = e^{2\pi i/n}$ and $0 \leq k \leq n-1$. Then $[G : H] = 2$, and a transversal of H in G is $\{e, b\}$. Also,
$$\begin{aligned} eae = a \in H, &\text{ and } bab = a^{n-1} \in H \\ ebe \notin H, &\text{ and } bbb = b \notin H \\ \Rightarrow \mathrm{Ind}_H^G(\chi)(a) &= \chi(a) + \chi(a^{n-1}) = \zeta^k + \zeta^{(n-1)k} = \zeta^k + \zeta^{-k} \\ \mathrm{Ind}_H^G(\chi)(b) &= 0 \end{aligned}$$
This agrees with the calculation in the example at the end of the previous section.

For a function $f : H \to \mathbb{C}$, we write

$$\dot{f}(g) := \begin{cases} f(g) & : g \in H \\ 0 & : \text{ otherwise} \end{cases}$$

Then the Frobenius Character formula gives

$$\text{Ind}_H^G(\chi)(g) = \sum_{i=1}^{\ell} \dot{\chi}(x_i^{-1} g x_i)$$

Proposition 4.2.3. *For any $g \in G$,*

$$Ind_H^G(\chi)(g) = \frac{1}{|H|} \sum_{x \in G} \dot{\chi}(x^{-1} g x)$$

Proof. For any $x \in G, \exists$ unique $1 \leq i \leq \ell, h \in H$ such that $x = x_i h$. Then

$$\dot{\chi}(x^{-1} g x) = \dot{\chi}(x_i g x_i)$$

Hence,

$$\sum_{x \in G} \dot{\chi}(x^{-1} g x) = \sum_{i=1}^{\ell} \sum_{x \in x_i H} \dot{\chi}(x^{-1} g x) = \sum_{i=1}^{\ell} |H| \dot{\chi}(x_i^{-1} g x_i)$$

□

Definition 4.2.4. Let $H < G$, and $Z(L(H)), Z(L(G))$ denote the spaces of class functions on H and G respectively.

1. Define $\text{Res}_H^G : Z(L(G)) \to Z(L(H))$ by

   $$a \mapsto a|_H$$

   Note that if a is a class function, then so is $a|_H$.

2. Define $\text{Ind}_H^G : Z(L(H)) \to Z(L(G))$ by

   $$\text{Ind}_H^G(b)(g) \mapsto \frac{1}{|H|} \sum_{x \in G} \dot{b}(x^{-1} g x)$$

   Then this map is well-defined

   Proof. Let $y \in G$, we wish to show that

   $$\text{Ind}_H^G(b)(ygy^{-1}) = \text{Ind}_H^G(b)(g)$$

   To see this, note that

   $$\text{Ind}_H^G(b)(ygy^{-1}) = \frac{1}{|H|} \sum_{x \in G} \dot{b}(x^{-1} y g y^{-1} x) = \frac{1}{|H|} \sum_{z \in G} \dot{b}(z^{-1} g z)$$

   since the map $x \mapsto y^{-1} x$ is a bijection on G. □

Proposition 4.2.5. *If $\rho_i : H \to GL(W_i), i = 1, 2$ are two representations of H, then $Ind_H^G(\rho_1 \oplus \rho_2) \sim Ind_H^G(\rho_1) \oplus Ind_H^G(\rho_2)$*

Proof. Let χ, φ, and ψ denote the characters of $\mathrm{Ind}_H^G(\rho_1 \oplus \rho_2), \mathrm{Ind}_H^G(\rho_1)$, and $\mathrm{Ind}_H^G(\rho_2)$ respectively. Then by the Frobenius character formula and the fact that Ind_H^G is additive, we get

$$\begin{aligned}\chi &= \mathrm{Ind}_H^G(\chi_{\rho_1 \oplus \rho_2}) \\ &= \mathrm{Ind}_H^G(\chi_{\rho_1} + \chi_{\rho_2}) \\ &= \mathrm{Ind}_H^G(\chi_{\rho_1}) + \mathrm{Ind}_H^G(\chi_{\rho_2}) \\ &= \varphi + \psi\end{aligned}$$

The result now follows from the fact that two representations of G with the same character must be equivalent. □

Note that both Res_H^G and Ind_H^G are linear maps. Now recall that both $Z(L(G))$ and $Z(L(H))$ are inner product spaces.

Theorem 4.2.6 (Frobenius Reciprocity)**.** *For any $a \in Z(L(G)), b \in Z(L(H))$*

$$\langle Res_H^G(a), b \rangle_{L(H)} = \langle a, Ind_H^G(b) \rangle_{L(G)}$$

Proof.

$$\begin{aligned}\langle a, \mathrm{Ind}_H^G(b) \rangle_{L(G)} &= \frac{1}{|G|} \sum_{g \in G} a(g) \overline{\mathrm{Ind}_H^G(b)(g)} \\ &= \frac{1}{|G|} \sum_{g \in G} a(g) \frac{1}{|H|} \sum_{x \in G} \dot{b}(x^{-1}gx)\end{aligned}$$

Now, $x^{-1}gx \in H \Leftrightarrow \exists h \in H$ such that $g = xhx^{-1}$. So rearranging, we get

$$\begin{aligned}\langle a, \mathrm{Ind}_H^G(b) \rangle_{L(G)} &= \frac{1}{|G||H|} \sum_{x \in G} \sum_{h \in H} a(xhx^{-1}) \overline{b(h)} \\ &= \frac{1}{|G||H|} \sum_{x \in G} \sum_{h \in H} a(h) \overline{b(h)} \\ &= \frac{1}{|G|} \sum_{x \in G} \langle \mathrm{Res}_H^G(a), b \rangle_{L(H)} \\ &= \langle \mathrm{Res}_H^G(a), b \rangle_{L(H)}\end{aligned}$$

□

Definition 4.2.7. Let V, W be inner product spaces and $T : V \to W, S : W \to V$. We say that S is an adjoint of T if

$$\langle Tv, w \rangle_W = \langle v, Sw \rangle_V \quad \forall v \in V, w \in W$$

Hence, Frobenius Reciprocity states that Res_H^G and Ind_H^G are adjoint to each other.

Remark. Let V, W be inner product spaces with ONB's $\mathcal{B}_1 = \{e_1, e_2, \ldots, e_n\}$ and $\mathcal{B}_2 = \{f_1, f_2, \ldots, f_m\}$ respectively. If $T : V \to W$ and $S : W \to V$ are adjoints of each other, then

$$\langle T(e_j), f_i\rangle = \langle e_j, S(f_i)\rangle = \overline{\langle S(f_i), e_j\rangle}$$

Hence, the matrix of S is the conjugate transpose of the matrix of T.

Example 4.2.8. Let $G = S_5, H = A_4$, let $\mathcal{B}_1 = \{\chi_1, \chi_2, \ldots, \chi_7\}$ denote the characters of irreducible representations of G, and let $\mathcal{B}_2 = \{\psi_1, \psi_2, \ldots, \psi_4\}$ be the irreducible characters of H. Recall the character table of G

	e	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1
χ_3	4	2	1	0	0	-1	-1
χ_4	4	-2	1	0	0	1	-1
χ_5	6	0	0	-2	0	0	1
χ_6	5	1	-1	1	-1	1	0
χ_7	5	-1	-1	1	1	-1	0

and the character table of H

g	e	(12)(34)	(123)	(132)
ψ_1	1	1	1	1
ψ_2	1	1	ω	ω^2
ψ_3	1	1	ω^2	ω
ψ_4	3	-1	0	0

Restriction gives

g	e	(12)(34)	(123)	(132)
$\chi_1\vert_H$	1	1	1	1
$\chi_2\vert_H$	1	1	1	1
$\chi_3\vert_H$	4	0	1	1
$\chi_4\vert_H$	4	0	1	1

and so on. Hence, taking inner products, we get

$$\chi_1|_H = \psi_1$$
$$\chi_2|_H = \psi_1$$
$$\chi_3|_H = \psi_1 + \psi_4$$

and so on. Hence, the matrix of Res_H^G with respect to these bases $\mathcal{B}_1$ and $\mathcal{B}_2$ can be computed to be

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Hence, the matrix of Ind_H^G is the transpose conjugate of this matrix. In particular, we can determine

$$\mathrm{Ind}_H^G(\psi_2) = \psi_6 + \psi_7$$

and other such identities.

More generally if $H < G$, write $\widehat{G} = \{\rho_1, \rho_2, \ldots, \rho_n\}$ and $\widehat{H} = \{\varphi_1, \varphi_2, \ldots, \varphi_m\}$. Then restriction of irreducible representations gives

$$\mathrm{Res}_H^G(\rho_i) \sim \sum_{j=1}^{m} r_{i,j}\varphi_j$$

for some non-negative integers $r_{i,j} \in \mathbb{Z}$. Induction gives

$$\mathrm{Ind}_H^G(\varphi_j) \sim \sum_{i=1}^{n} s_{j,i}\rho_i$$

Frobenius Reciprocity states that $r_{i,j} = s_{j,i}$ for all i, j.

Corollary 4.2.9 (Induction in stages). *Suppose $H < K < G$ and $\rho : H \to GL(W)$ is a representation. Then*

$$Ind_K^G(Ind_H^K(\rho)) \sim Ind_H^G(\rho)$$

Proof. Let $\widehat{\rho} = \mathrm{Ind}_H^K(\rho)$, then by definition

$$\mathrm{Ind}_H^K(\chi_\rho) = \chi_{\widehat{\rho}}$$

for any class function $b \in Z(L(G))$

$$\langle \mathrm{Ind}_K^G(\chi_{\widehat{\rho}}), b\rangle_{L(G)} = \langle \chi_{\widehat{\rho}}, \mathrm{Res}_K^G(b)\rangle_{L(K)}$$

Furthermore,

$$\langle \chi_{\widehat{\rho}}, \mathrm{Res}_K^G(b)\rangle_{L(K)} = \langle \chi_\rho, \mathrm{Res}_H^K(\mathrm{Res}_K^G(b))\rangle_{L(H)}$$

But $\mathrm{Res}_H^K(\mathrm{Res}_K^G(b)) = \mathrm{Res}_H^G(b)$. Hence,

$$\langle \mathrm{Ind}_K^G(\chi_{\widehat{\rho}}), b\rangle_{L(G)} = \langle \chi_\rho, \mathrm{Res}_H^G(b)\rangle_{L(H)} = \langle \mathrm{Ind}_H^G(\chi_\rho), b\rangle_{L(G)}$$

This is true for every $b \in Z(L(G))$, so

$$\chi_{\mathrm{Ind}_K^G(\mathrm{Ind}_H^K(\rho))} = \mathrm{Ind}_K^G(\mathrm{Ind}_H^K(\chi_\rho)) = \mathrm{Ind}_K^G(\chi_{\widehat{\rho}}) = \mathrm{Ind}_H^G(\chi_\rho) = \chi_{\mathrm{Ind}_H^G(\rho)}$$

Hence the result. $\square$

4.3 Examples

4.3.1 A group of order 21

In S_7, define

$$a = (1234567), b = (235)(476) \text{ and } G := \langle a, b\rangle$$

Then $a^7 = b^3 = 1, b^{-1}ab = a^2$, hence

$$G = \{a^i b^j : 0 \leq i \leq 6, 0 \leq j \leq 2\} \Rightarrow |G| = 21$$

1. Let $H = \langle a\rangle$, then $|H| = 7$ and $b^{-1}ab \in H$, so $H \lhd G$. Finally, $G/H \cong \mathbb{Z}_3$ is abelian, so

   $$[G, G] \subset H$$

   Since $|H| = 7$ and $[G, G] \neq \{e\}$, we have $[G, G] = H$. Hence, G has 3 non-trivial characters, we denote by $\{\chi_1, \chi_2, \chi_3\}$.

2. Now we determine conjugacy classes in G: Recall that if $x \in G$, then x^G denotes the conjugacy class of x, $C_G(x)$ the centralizer of x in G, and

   $$|x^G| = \frac{|G|}{|C_G(x)|}$$

   by the orbit-stabilizer theorem.

   a) Note that $e^G = \{e\} = C_1$.

   b) If $x = a$, then $a \in C_G(a)$, so $H \subset C_G(a)$, so $7 \mid |C_G(a)|$. Since $b \notin C_G(a)$, it follows that $|C_G(a)| < 21$. Since $|C_G(a)| \mid 21$, it follows that

   $$|C_G(a)| = 7 \Rightarrow C_G(a) = H$$

   Hence, $|a^G| = 3$. The relation $b^{-1}ab = a^2$ implies that $a^2 \in a^G$. Hence, $a^4 \in a^G$. Thus it follows that

   $$C_2 = a^G = \{a, a^2, a^4\}$$

   c) Similarly, $|C_G(a^3)| = H$, and so $|(a^3)^G| = 3$, and as above

   $$C_3 = (a^3)^G = \{a^3, a^5, a^6\}$$

   d) As done for a above, $|b^G| = 7$. Check that

   $$C_4 = (b)^G = \{a^i b : 0 \leq i \leq 6\}$$

   e) Similarly,

   $$C_5 = (b^2)^G = \{a^i b^2 : 0 \leq i \leq 6\}$$

   These are all the conjugacy classes of G.

3. We have a partial character table given by

x	e	a	a^3	b	b^2
$\lvert x^G \rvert$	1	3	3	7	7
χ_1	1	1	1	1	1
χ_2	1	1	1	ω	ω^2
χ_3	1	1	1	ω^2	ω

4. Now we induce representations from H. Let $\zeta = e^{2\pi i/7}$, and define

$$\rho : H \to \mathbb{C}^* \text{ given by } a \mapsto \zeta$$

and let $\psi = \chi_{\mathrm{Ind}_H^G(\rho)}$. By the Frobenius Character formula,

$$\psi(g) = \sum_{i=1}^{\ell} \dot{\rho}(x_i^{-1} g x_i)$$

where $\{x_1, x_2, \ldots, x_\ell\}$ are a set of representatives for G/H. Now $|H| = 7$, so $|G/H| = 3$, and we take

$$x_1 = e, x_2 = b, x_3 = b^2$$

Since H is normal, $x_i^{-1} g x_i \in H$ for all $g \in H$. Furthermore, if $g \notin H$, then $x_i^{-1} g x_i \notin H$ for all i. Hence,

$$\psi(g) = 0 \quad \forall g \notin H$$

Also,

$$\begin{aligned}
\psi(e) &= \rho(e) + \rho(b^{-1}eb) + \rho(b^{-2}eb^2) = 3\rho(e) = 3 \\
\psi(a) &= \rho(eae) + \rho(b^{-1}ab) + \rho(b^{-2}ab^2) = \rho(a) + \rho(a^2) + \rho(a^4) = \zeta + \zeta^2 + \zeta^4 \\
\psi(a^3) &= \zeta^3 + \zeta^5 + \zeta^6
\end{aligned}$$

So this gives us values in the table as

x	e	a	a^3	b	b^2
$\lvert x^G \rvert$	1	3	3	7	7
ψ	3	$\zeta + \zeta^2 + \zeta^4$	$\zeta^3 + \zeta^5 + \zeta^6$	0	0

Now calculate

$$\langle \psi, \psi \rangle = \frac{1}{21}[3 + 3|\zeta + \zeta^2 + \zeta^4|^2 + 3|\zeta^3 + \zeta^5 + \zeta^6|^2]$$

and check that

$$\begin{aligned}
|\zeta + \zeta^2 + \zeta^4|^2 &= (\zeta + \zeta^2 + \zeta^4)(\zeta^{-1} + \zeta^{-2} + \zeta^{-4}) \\
&= 1 + \zeta^{-1} + \zeta^{-3} + \zeta + 1 + \zeta^{-2} + \zeta^3 + \zeta^2 + 1 \\
&= 3 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6 \\
&= 3 + (-1) = 2
\end{aligned}$$

Similarly for the third term, so we get

$$\langle \psi, \psi \rangle = \frac{1}{21}[3 + 6 + 6] = 1$$

Hence, ψ is irreducible.

5. Now let $\varphi : H \to \mathbb{C}^*$ be given by

$$\rho(a) = \zeta^2$$

Then if $\eta = \mathrm{Ind}_H^G(\rho)$, we get, by a similar calculation

x	e	a	a^3	b	b^2
$\lvert x^G \rvert$	1	3	3	7	7
η	3	$\zeta^3 + \zeta^5 + \zeta^6$	$\zeta + \zeta^2 + \zeta^4$	0	0

Hence,

$$\langle \eta, \eta \rangle = 1$$

so η is also irreducible. This gives the character table of G as

x	e	a	a^3	b	b^2
$\lvert x^G \rvert$	1	3	3	7	7
χ_1	1	1	1	1	1
χ_2	1	1	1	ω	ω^2
χ_3	1	1	1	ω^2	ω
ψ	3	$\zeta + \zeta^2 + \zeta^4$	$\zeta^3 + \zeta^5 + \zeta^6$	0	0
η	3	$\zeta^3 + \zeta^5 + \zeta^6$	$\zeta + \zeta^2 + \zeta^4$	0	0

4.3.2 A group of order $p(p-1)$

Let $p \in \mathbb{N}$ prime, and let G be the group of matrices given by

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p \right\}$$

Then G is a non-abelian group with $|G| = p(p-1)$. Let H be the subgroup

$$H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z}_p \right\}$$

Note that $H \lhd G$ and

$$G/H \cong \mathbb{Z}_p^*$$

which is cyclic (and hence abelian). Hence, $[G, G] \subset H$. Since G is non-Abelian,

$$[G, G] \neq \{e\}$$

Since $|H| = p$, it follows that $[G, G] = H$. Hence, G has precisely $(p-1)$ linear characters, denoted by $\{\chi_1, \chi_2, \ldots, \chi_{p-1}\}$.

G has conjugacy classes given by

1. $C_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

2. Let $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then we have

$$\begin{aligned} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} &= \begin{pmatrix} a & a+b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/a & -b/a \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Hence,

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in C_G(x) \Leftrightarrow a = 1$$

$$C_G(x) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z}_p \right\}$$

In particular, $|x^G| = |G|/|C_G(x)| = (p-1)$.

3. Now consider an element of the form

$$z := \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, x \in \mathbb{Z}_p^*, x \neq 1$$

Then

$$\begin{aligned} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} &= \begin{pmatrix} ax & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/a & -b/a \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} x & -bx+b \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Hence,

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in C_G(z) \Leftrightarrow -bx + b = 0 \Leftrightarrow b = 0$$

Hence,

$$C_G(z) = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a \in \mathbb{Z}_p^* \right\}$$

and so $|z^G| = |G|/|C_G(z)| = p$

4. Finally, if $\pi : G \to G/H$ denotes the quotient map, then if $x_1 \neq x_2$, then

$$\begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x_2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/x_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_2/x_1 & 0 \\ 0 & 1 \end{pmatrix} \notin H$$

and so

$$\pi\left(\begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix} \right) \neq \pi\left(\begin{pmatrix} x_2 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

if $x_1 \neq x_2$. Since G/H is abelian, this implies

$$\pi\left(\begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix}\right) \sim \pi\left(\begin{pmatrix} x_2 & 0 \\ 0 & 1 \end{pmatrix}\right) \Leftrightarrow x_1 = x_2$$

and hence

$$\begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} x_2 & 0 \\ 0 & 1 \end{pmatrix} \Leftrightarrow x_1 = x_2$$

Hence, by part (3), we get $(p-1)$ conjugacy classes

$$C_x = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}^G \text{ for } x \in \mathbb{Z}_p^*, x \neq 1$$

each of which have cardinality p.

5. Now calculating cardinalities, we get

$$1 + (p-1) + \sum_{x \in \mathbb{Z}_p^*, x \neq 1} p = p + (p-2)p = p(p-1)$$

and so these are all the conjugacy classes in G. In particular, G has p conjugacy classes.

Hence, G has exactly one more irreducible representation ψ. The degree formula reads

$$p(p-1) = p - 1 + d_\psi^2 \Rightarrow d_\psi = p - 1$$

Let $\varphi : H \to \mathbb{C}^*$ be the map

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mapsto e^{2\pi i/p}$$

and let $\eta = \operatorname{Ind}_H^G(\varphi)$. We claim that $\eta = \psi$ is the required irreducible representation. Note that $d_\eta = p - 1$. Furthermore, by Frobenius reciprocity

$$\langle \chi_\eta, \chi_i \rangle = \langle \chi_\varphi, \operatorname{Res}_H^G(\chi_i) \rangle$$

Now, $\operatorname{Res}_H^G(\chi_i)$ is the trivial representation for all $1 \leq i \leq n$, and φ is a non-trivial irreducible representation. So by Schur Orthogonality,

$$\langle \chi_\varphi, \operatorname{Res}_H^G(\chi_i) \rangle = 0$$

Hence, the Maschke decomposition of η has the form

$$\chi_\eta = m\chi_\psi$$

However, $d_\eta = d_\psi$, so $m = 1$ and η is irreducible.

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