MTH 404: Measure and Integration

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Contents

I. Introduction

1. Motivation

1.1. Change in the use of integration from problems in geometry/classical mechanics to problems in differential equations/probability.

1.2. Goal: To develop a more robust integration theory, which builds on our intuition from Riemann integration, but provides some major improvements :

(i) To interchange the limit and the integral under much less stringent conditions than Riemann's theory. Example :

$$f_n(x) = \frac{e^{-nx}}{\sqrt{x}}, x > 0 \qquad \Rightarrow \qquad \lim_{n\to\infty} \int_0^\infty f_n(x)dx = 0$$

(ii) To differentiate under the integral sign. Example :

$$F(t) = \int_0^\infty x^2 e^{-tx} dx \qquad \Rightarrow \qquad F'(t) = -\int_0^\infty x^3 e^{-tx} dx$$

(iii) To enlarge the class of integrable functions. Example :

$$f(x) = \begin{cases} 1 & : x \in \mathbb{Q} \cap [0,1] \\ 0 & : x \notin \mathbb{Q} \cap [0,1] \end{cases}$$

is Lebesgue-integrable but not Riemann-integrable.

(iv) To integrate functions over more general spaces than just $\mathbb{R}^n$

(v) Lebesgue's theory simplifies the basic techniques of integration such as change of variables, double/triple integrals, etc.

2. The basic method of Lebesgue's theory

2.1. Recall Riemann Integration

(i) Open and half open intervals and their length.

(ii) Characteristic function of a set

(iii) Definition of a step function

(iv) Definition of integral of a step function

(v) Definition of lower integral for a bounded function $f : [a,b] \to \mathbb{R}$.

2.2. The definition of the Lebesgue integral is similar.

(i) Replace length by measure $m(E)$ for measurable sets $E \subset \mathbb{R}$.

(ii) Definition of simple function

(iii) Definition of integral for measurable, real-valued functions.

II. The Real Number System

1. Extended Real Number System

1.1. Adjoin two symbols $-\infty$ and $+\infty$ to $\mathbb{R}$ to form $\overline{\mathbb{R}}$

1.2. We do not define $(+\infty)+(-\infty)$, $(-\infty)+(+\infty)$ or any ratio with $(\pm\infty)$ in the denominator.

1.3. If we denote $\inf(\emptyset) = +\infty$ and $\sup(\emptyset) = -\infty$, then every subset of $\overline{\mathbb{R}}$ has a supremum or infimum.

1.4. Definition of $\limsup$ and $\liminf$ of a sequence.

(End of Day 1)

2. Open and Closed subsets of $\mathbb{R}$

2.1. Definitions :

- (i) Open subset of $\mathbb{R}$
- (ii) Open subset of E for some set $E \subset \mathbb{R}$
- (iii) Closed set
- (iv) Closure of a set $A \subset \mathbb{R}$

2.2. Proposition :

- (i) Finite intersection of open sets is open
- (ii) Finite union of closed sets is closed
- (iii) Arbitrary union of open sets is open
- (iv) Arbitrary intersection of closed sets is closed
- (v) $\overline{A}$ is closed for any $A \subset \mathbb{R}$

2.3. [HLR, Proposition 2.8]

2.4. Definitions :

- (i) Open cover of a set $A \subset \mathbb{R}$
- (ii) Compact set

2.5. Proposition : If $F \subset \mathbb{R}$ closed and bounded, then F compact. (Converse on HW)

3. Continuous functions

3.1. Definition of continuous function (mentioned [HLR, Theorem 2.18] without proof)

3.2. Proposition : Continuous image of compact set is compact

3.3. Proposition (without proof) : If $E \subset \mathbb{R}$ compact, $f : E \to \mathbb{R}$ continuous then

- (i) f is uniformly continuous on E
- (ii) f is bounded, and attains its max and min on E

(End of Day 2)

3.4. Theorem: Intermediate value theorem (without proof)

3.5. Definitions :

(i) $f_n \to f$ pointwise

(ii) $f_n \to f$ uniformly

3.6. Proposition : If $f_n : E \to \mathbb{R}$ continuous, and $f_n \to f$ uniformly on E, then f is continuous.

4. Borel σ-algebra

4.1. Definitions :

(i) Boolean algebra

(ii) σ-algebra

4.2. Note : A σ-algebra $\mathcal{A}$ is also closed under countable intersection.

4.3. [HLR, Proposition 1.2]

4.4. [HLR, Proposition 1.3]

4.5. Definition : Borel σ-algebra on a topological space X

4.6. Note : Let $\mathcal{B}$ denote the Borel σ-algebra on $\mathbb{R}$, then

(i) $\mathcal{B}$ contains all open and closed sets.

(ii) $\mathcal{B}$ contains all countable sets

(iii) An F_σ is the union of a countable family of closed sets (Example: $[a,b)$). A G_δ is the intersection of a countable family of open sets. $\mathcal{B}$ contains all F_σ's and G_δ's (It contains more than just these sets though).

III. Lebesgue Measure

1. Introduction

1.1. Definition : Positive, Countably additive measure m on a σ-algebra $\mathcal{M}$ on $\mathbb{R}$

1.2. Goal : To define such a measure on the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$ such that $m(I) =$ length of I for all intervals $I \subset \mathbb{R}$

1.3. Examples :

(i) $m(E) = +\infty$ for all $E \subset \mathbb{R}$

(ii) m is the counting measure on $\mathbb{R}$

1.4. Lemma : Suppose $m : \mathcal{M} \to [0,\infty]$ is a positive, countably additive measure on a σ-algebra $\mathcal{M}$, then

(i) [Monotonicity] If $A, B \in \mathcal{M}$ such that $A \subset B$, then $m(A) \leq m(B)$

(ii) [Countable Sub-additivity] If $\{A_n\} \subset \mathcal{M}$, then

$$m(\cup A_n) \leq \sum m(A_n)$$

(End of Day 3)

2. Outer Measure

2.1. Definition of Outer Measure

2.2. Note :

(i) If $A \subset B$, then $m^*(A) \leq m^*(B)$

(ii) If I is any interval, then $m^*(I) \leq l(I)$

2.3. [HLR, Proposition 3.1]

2.4. Corollary :

(i) [HLR, Corollary 3.3]

(ii) [HLR, Corollary 3.4]

2.5. Example : The Cantor set is uncountable, but has measure zero.

2.6. Proposition : m^* is translation invariant

2.7. [HLR, Proposition 3.2]

2.8. [HLR, § 3.4] Example of a non-measurable set

(End of Day 4)

3. Measurable Sets and Lebesgue measure

3.1. Definition of measurable set (Caratheodory) and set $\mathcal{M}$ of measurable sets.

3.2. Note :

(i) Definition means that E is 'distributed well' over subsets of $\mathbb{R}$

(ii) By definition, $E \in \mathcal{M} \Rightarrow E^c \in \mathcal{M}$

(iii) By definition, $\emptyset, \mathbb{R} \in \mathcal{M}$

(iv) In the definition, one inequality $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$ is always true. Hence, it only suffices to show that $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$

3.3. [HLR, Lemma 3.6]

3.4. [HLR, Lemma 3.7]

3.5. [HLR, Lemma 3.9]

3.6. [HLR, Theorem 3.10]

3.7. [HLR, Lemma 3.11, Theorem 3.12]

3.8. [HLR, Proposition 3.15]

3.9. Notation : m^* restricted to $\mathcal{M}$ is denoted by m and is called the Lebesgue measure on $\mathbb{R}$

3.10. [Rudin, Theorem 1.19(d),(e)]

3.11. Example: $E_n = [n, +\infty)$, then $m(E_n) = +\infty$ for all n and $\cap E_n = \emptyset$

4. Measurable functions

4.1. Remark: Suppose $E \subset \mathbb{R}$ is measurable, then the collection $\Omega = \{F \subset E : \exists G \in \mathcal{M}$ such that $F = G \cap E\}$ [Check!] is a σ-algebra. Thus, the triple (E, Ω, m) is a measure space. We are often concerned with functions $f : E \to \overline{\mathbb{R}}$ whose domain is a measurable set in $\mathbb{R}$. Hence, we consider a general *measurable space* $(X, \mathcal{M})$ and functions whose domain is X. We will later apply this to the case where $X = E$.

4.2. [Rudin, Definition 1.3(c)]

4.3. [Rudin, Theorem 1.7(b)]

4.4. [Rudin, Theorem 1.8]

4.5. Theorem: If $f, g : X \to \mathbb{R}$ measurable, then $f + g, f - g, cf, fg$ are measurable.

4.6. [Rudin, Theorem 1.12(a)]

4.7. [Rudin, Theorem 1.12(c)]

4.8. [Rudin, Theorem 1.14]

(End of Day 5)

4.9. Examples :

- (i) Characteristic function of a measurable set
- (ii) Simple function (with definition of canonical representation)
- (iii) Continuous function
- (iv) Lower/Upper semi-continuous function
- (v) Cantor Ternary function [Wheeden/Zygmund, Page 35]
- (vi) A measurable set that is not Borel

4.10. Definition: Almost everywhere.

4.11. [HLR, Proposition 3.21]

4.12. Theorem: Let $f : E \to \overline{\mathbb{R}}$ be measurable, then there exists a sequence $\{\varphi_k\}$ of simple measurable functions such that

- (i) $\varphi_k \to f$ pointwise
- (ii) If $f \geq 0$, then $\varphi_k \leq \varphi_{k+1}$ for all k
- (iii) If f is bounded, then $\varphi_k \to f$ uniformly.

5. Egoroff's theorem and Lusin's theorem

5.1. Motivation for Egoroff's theorem

5.2. [Wheeden/Zygmund, Theorem 4.17] (Egoroff's Theorem)

5.3. [Wheeden/Zygmund, Lemma 4.18]

(End of Day 6)

5.4. [Wheeden/Zygmund, Theorem 4.19]

5.5. [Wheeden/Zygmund, Theorem 4.20] (Lusin's Theorem)

IV. The Lebesgue Integral

1. The Riemann integral

1.1. Given $f : [a, b] \to \mathbb{R}$ bounded, we define

  (i) Upper and Lower Riemann sums of f w.r.t. a partition $\mathcal{P}$

  (ii) Upper and Lower Riemann integrals of f

  (iii) f is Riemann-integrable iff these two values coincide. This common value is then denoted by $R\int_a^b f(x)dx$

1.2. Definition of step function, and integral of a step function.

1.3. Proposition: Upper integral is the infimum over integrals of all step functions $\psi \geq f$, and Lower integral is the supremum over integrals of all step functions $\varphi \leq f$

1.4. Example : $f = \chi_{\mathbb{Q}\cap[0,1]}$ is not Riemann-integrable.

2. The Lebesgue Integral of a bounded function over a set of finite measure

2.1. Definition of simple function, and its integral. Note that $\varphi \geq 0 \Rightarrow \int \varphi \geq 0$

2.2. [HLR, Lemma 4.1]

(End of Day 7)

2.3. [HLR, Proposition 4.2]

2.4. Definition of Upper and Lower Lebesgue integrals for a bounded function $f : E \to \mathbb{R}$ with $m(E) < \infty$

2.5. [HLR, Proposition 4.3]

2.6. [HLR, Proposition 4.4]

2.7. [HLR, Proposition 4.5] (without proof)

(End of Day 8)

2.8. [HLR, Proposition 4.6] (Bounded Convergence Theorem)

2.9. Lemma : If $f : [a, b] \to \mathbb{R}$ bounded, and g, h are its lower and upper envelopes respectively, then g is lower semi-continuous, h is upper semi-continuous and

$$\int_a^b g = R\underline{\int_a^b} f \qquad \text{and} \qquad \int_a^b h = R\overline{\int_a^b} f$$

2.10. Suppose $f : E \to [0, \infty)$ with $m(E) < +\infty$ and $\int_E f = 0$, then $f = 0$ a.e.

2.11. [HLR, Proposition 4.7]

3. The Lebesgue Integral of a non-negative function

3.1. Remark: We consider non-negative $f : E \to \overline{\mathbb{R}}$ to avoid situations such as $\infty - \infty$ that would occur, for example, with

$$\varphi(x) = \begin{cases} 1 & : x \geq 0 \\ -1 & : x < 0 \end{cases}$$

3.2. Definition of integral of a measurable function $f : E \to [0, \infty]$

3.3. [HLR, Proposition 4.8]

3.4. [HLR, Proposition 4.9] (Fatou's Lemma)

(End of Day 9)

3.5. [HLR, Theorem 4.10] (Monotone Convergence Theorem)

3.6. [HLR, Corollary 4.11]

3.7. [HLR, Proposition 4.12]

3.8. Remark: Radon-Nikodym theorem gives a converse to the above proposition.

4. The General Lebesgue Integral

4.1. Definition

(i) f^+ and f^-. Note that $f = f^+ - f^-$ and $|f| = f^+ + f^-$

(ii) f is integrable

(iii) $L(E) = \{f : E \to \overline{\mathbb{R}} : f \text{ integrable}\}$.

4.2. Remark :

(i) If $f = f_1 - f_2$ with $f_i \geq 0$, then $\int f = \int f_1 - \int f_2$ because $f_1 + f^- = f^+ + f_2$. This is used in the next proof.

(ii) $f \in L(E) \Leftrightarrow |f| \in L(E)$ since $|f| = f^+ + f^-$

4.3. [HLR, Proposition 4.15] (without proof)

4.4. Definition: $L^1(E) = L(E)/\sim$, where $f \sim g$ iff $f = g$ a.e. This ensures that $d(f, g) = \int_E |f - g|$ is a metric on $L^1(E)$.

4.5. [HLR, Theorem 4.16] (Dominated Convergence Theorem)

4.6. Dependence on a parameter

(i) Suppose $f : E \times [a, b] \to \mathbb{R}$ is such that $x \mapsto f(x, t)$ is measurable for all $t \in [a, b]$ and $t \mapsto f(x, t)$ is continuous for all $x \in [a, b]$. Define

$$F(t) = \int_E f(x, t)dx$$

(ii) [Bartle, Corollary 5.8]

(End of Day 10)

(iii) [Bartle, Corollary 5.9]

(iv) [Bartle, Corollary 5.10]

4.7. Improper Integrals

(i) Suppose $f \geq 0$ on some interval $[a, \infty)$ and integrable over all sub-intervals $[a, b]$. Suppose that the improper integral

$$I = \lim_{b \to \infty} \int_a^b f$$

exists and is finite, then $f \in L[a, \infty)$ and the Lebesgue integral

$$\int_{[a,\infty)} f = I$$

(ii) Similarly, If $f \geq 0$ on $[a, b]$ is integrable over all sub-intervals $[\epsilon, b]$. Suppose that the improper integral

$$I = \lim_{\epsilon \to a} \int_{\epsilon}^{b} f$$

exists and is finite. Then $f \in L[a, b]$ and $\int_a^b f = I$

(iii) Example to show that $f \geq 0$ is important: Define $f : [0, \infty) \to \mathbb{R}$ by

$$f(x) = \frac{(-1)^n}{n} \qquad n - 1 \leq x < n \quad \forall n \in \mathbb{N}$$

(End of Day 11)

Quiz 1

(End of Day 12)

V. Abstract Measures

1. Abstract Integration

1.1. Definition of measure space $(X, \mathcal{M}, \mu)$

1.2. [Rudin, Theorem 1.19] (without proof)

1.3. [Rudin, Definition 1.23]

1.4. [Rudin, Proposition 1.25]

1.5. [Rudin, Theorem 1.26] (Monotone Convergence Theorem)

1.6. [Rudin, Lemma 1.28] (Fatou's Lemma)

1.7. [Rudin, Theorem 1.29]

1.8. Remark:

(i) Note that if $\varphi(E) = \int_E f d\mu$ as above, then $\mu(E) = 0 \Rightarrow \varphi(E) = 0$. When this happens we say that φ is absolutely continuous with respect to μ. We denote this by $\varphi << \mu$.

(ii) The Radon-Nikodym theorem gives a converse to the above result.

1.9. Definition

(i) $\int_E f d\mu$ for any $f : X \to \overline{\mathbb{R}}$ measurable.

(ii) $\int_E f d\mu$ for any $f : X \to \mathbb{C}$ measurable.

(iii) Definition of $L(X, \mu)$

(iv) Note that if $f : X \to \overline{\mathbb{R}}$, then $f \in L(X, \mu)$ iff $|f| \in L(X, \mu)$.

(v) If $f : X \to \mathbb{C}$, then $|\int_X f d\mu| \leq \int_X |f| d\mu$

1.10. Dominated Convergence Theorem (Proof omitted. Same as before)

(End of Day 13)

2. Outer Measures

2.1. Remark: The construction of the Lebesgue measure on $\mathbb{R}$ was done in the following steps :

(i) A length function $l : \mathcal{I} \to [0, \infty]$, where $\mathcal{I}$ is the collection of intervals.

(ii) An outer measure $m^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$

(iii) Restricting m^* to a class $\mathcal{M}$ of measurable sets to obtain a measure m

We apply this idea to an arbitrary set with a view to understand :

(i) The Stieltjes integral : Replace l by the function $[a, b] \mapsto F(b) - F(a)$ for some increasing function F

(ii) The Lebesgue measure on $\mathbb{R}^n$: Replace $\mathcal{I}$ by the set of cubes in $\mathbb{R}$ and l by the volume function.

(iii) The Riesz Representation Theorem on locally compact, Hausdorff spaces

(iv) What makes the Lebesgue measure on $\mathbb{R}^n$ so special?

2.2. Definition :

(i) Definition of a pre-measure μ_0 on $(X, \mathcal{A})$ where $\mathcal{A}$ is a Boolean algebra on X

(ii) Definition of outer measure μ^* on X

(iii) Definition of μ^*-measurable sets (Caratheodory's condition)

2.3. [Folland, Proposition 1.10, 1.13]

2.4. [Folland, Theorem 1.11] (without proof)

2.5. [Folland, Theorem 1.14]

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2.6. Definition :

(i) Complete measure

(ii) Finite measure

(iii) σ-finite measure

2.7. Remark :

(i) The measure constructed in the previous theorem is complete.

(ii) If μ_0 is σ-finite, then μ is σ-finite, and unique.

3. Lebesgue-Stieltjes Measure

3.1. Remark :

(i) The Stieltjes Integral is meant to replace the Riemann integral $\int_a^b f(x)dx$ by $\int_a^b f(x)dF(x)$ where F is an increasing function. We approach this from the point of view of measure theory.

(ii) Let F be increasing, then F can be normalized so that F is right-continuous. Define $F(\pm\infty) = \lim_{x \to \pm\infty} F(x)$, where this limit is well-defined since F is increasing.

3.2. Definition: The Boolean algebra $\mathcal{A}$ in this section is generated by sets of the form $\{(a, b], (a, \infty)\}$

3.3. [Folland, Proposition 1.15]

3.4. [Folland, Theorem 1.16]

(End of Day 15)

3.5. Definition :

(i) Lebesgue-Stieltjes measure μ_F associated to an increasing, right-continuous function F.

(ii) Given a measure μ as above, a function F such that $\mu(a,b] = F(b) - F(a)$ for all $a < b$ is called a (cumulative) distribution function of μ. Note that $\mu_F = \mu_G$ iff $F - G$ is constant. Hence, the distribution function is well-defined upto the addition of a constant.

3.6. Examples :

(i) If $F(x) = x$, then $\mu_F = m$, the usual Lebesgue measure on $\mathbb{R}$

(ii) If μ is a Borel measure on $\mathbb{R}$ such that $\mu(\mathbb{R}) = 1$, then μ is called a probability measure on $\mathbb{R}$. In this case, the function $F(x) = \mu((-\infty, x])$ is called the probability distribution function associated to μ.

(iii) If F is continuously differentiable, then

$$\mu_F(E) = \int_E F'(x)dx$$

for any Borel set E. Hence, we often write

$$\int g(x)dF(x) = \int g d\mu_F$$

(iv) If $F(x) = 1$ for $x \geq 0$ and 0 for $x < 0$, then $\mu_F = \delta_0$, the Dirac delta measure supported at 0.

(v) If F is the Cantor ternary function extended to $\mathbb{R}$ by defining $F(x) = 1$ for $x > 1$ and $F(x) = 0$ for $x < 0$, then the associated measure μ_F satisfies $\mu_F(C) = 1$ and $\mu_F(C^c) = 1$ where C is the Cantor set. (HW 6)

3.7. Remark: We say that a measure μ is supported on a set $A \subset X$ if $\mu(E) = \mu(E \cap A)$ for all measurable sets E. We say that two measures μ and ν are mutually singular if they are supported on disjoint sets (In symbols, $\mu \perp \nu$). In our case, $\mu_F \perp m$ where F is the Cantor function described above.

4. Product Measures

4.1. Definition:

(i) Product σ-algebra $\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \ldots \otimes \mathcal{M}_n$

(ii) Boolean algebra $\mathcal{A}$ of finite disjoint unions of rectangles $\prod_{j=1}^n A_j$

4.2. Proposition : $\pi(\prod_{i=1}^n A_j) = \prod_{j=1}^n \mu_j(A_j)$ defines a pre-measure on $\mathcal{A}$

4.3. Definition of $\mu_1 \times \mu_2 \times \ldots \times \mu_n$

4.4. Definition: Lebesgue measure on $\mathbb{R}^n$

(End of Day 16)

4.5. Remark: If μ and ν are σ-finite, then $\mu \times \nu$ is σ-finite, and hence the unique measure such that $\mu \times \nu(A \times B) = \mu(A)\nu(B)$

4.6. Definition :

(i) If $E \subset X \times Y$, x-section E_x and y-section E^y

(ii) If $f : X \times Y \to \mathbb{R}$, x-section f_x and y-section f^y. Note that $(\chi_E)_x = \chi_{E_x}$ and $(\chi_E)^y = \chi_{E^y}$

4.7. [Folland, Proposition 2.34]

4.8. Definition :

(i) Monotone class

(ii) Monotone class generated by a set $\mathcal{E} \subset \mathcal{P}(X)$

4.9. [Folland, Lemma 2.35] (Monotone Class Lemma)

4.10. [Folland, Theorem 2.36]

(End of Day 17)

4.11. [Folland, Theorem 2.37] (Fubini-Tonelli Theorem)

4.12. Counterexamples :

(i) σ-finiteness is necessary : [Rudin, Example 8.9(b)]

(ii) $f \in L(X \times Y)$ is necessary : [Folland, Problem 2.48] (on HW 7)

4.13. Remark: We usually omit the brackets and write

$$\int \int f d\mu d\nu = \int \int f(x,y) d\mu(x) d\nu(y)$$

5. Lebesgue Measure on $\mathbb{R}^n$

5.1. Definition of $m = m^n$ and the σ-algebra $\mathcal{M} = \mathcal{M}^n$

5.2. [Folland, Theorem 2.40] (without proof)

(End of Day 18)

5.3. [Folland, Theorem 2.42] (Translation Invariance)

5.4. Theorem: Let μ and ν be nonzero translation invariant Borel measures on $\mathbb{R}^n$ which assign finite values to compact sets. Then they are scalar multiples of one another.

(End of Day 19)

5.5. [Folland, Theorem 2.44]

5.6. [Folland, Theorem 2.46]

5.7. Remark: m is invariant under any rigid motion of $\mathbb{R}^n$

VI. L^p spaces

1. The Banach space $L^1(X, \mu)$

1.1. Definition :

(i) $L^1(X,\mu)$

(ii) Normed Linear space

1.2. Examples :

(i) $\mathbb{K}^n$ with Euclidean norm

(ii) $C(X)$ for a compact Hausdorff space X

(iii) $L^1(X,\mu)$

(iv) $C[a,b]$ with the L^1 norm is not complete.

(End of Day 20)

1.3. Definition

(i) Banach space

(ii) Convergence of a series in a NLS

(iii) Absolutely convergent series

1.4. [Folland, Theorem 5.1]

1.5. Corollary: $L^1(X,\mu)$ is a Banach space.

2. The L^p spaces

2.1. Definition :

(i) $L^p(X,\mu)$ with definition of $\|\cdot\|_p$

(ii) $l^p = L^p(X,\mu)$ where $X = \mathbb{N}$ and $\mu =$ the counting measure

2.2. Remark: L^p is not a NLS if $0 < p < 1$

2.3. [Folland, Lemma 6.1]

(End of Day 21)

2.4. [Folland, Theorem 6.2] (Holder's Inequality)

2.5. Remark:

(i) Definition of conjugate exponents

(ii) Cauchy-Schwartz inequality

(iii) L^2 is an inner-product space

2.6. [Folland, Theorem 6.5] (Minkowski's inequality)

2.7. [Folland, Theorem 6.6] (Riesz-Fischer Theorem)

2.8. Definition of L^∞ and $\|f\|_\infty$

2.9. Theorem: $\|\cdot\|_\infty$ is a norm on L^∞ and L^∞ is complete w.r.t. this norm.

(End of Day 22)

3. Approximation in L^p

3.1. Note : Convergence in $L^p \Leftrightarrow$ Convergence a.e?

(i) $X = [0,1]$ and $f_n = n\chi_{(0,1/n]}$ then $f_n \to 0$ a.e, but $\|f_n\|_1 = 1$ for all n

(ii) [Wheeden/Zygmund, Example after 4.21]

3.2. [Bartle, Theorem 7.2]

3.3. [Folland, Prop 6.7]

3.4. Definition of regular Borel measure

3.5. [Folland, Theorem 7.9]

3.6. Remark :

(i) The completion of $C[a,b]$ w.r.t. $\|\cdot\|_1$ is $L^1[a,b]$

(ii) The completion of $C_c(X)$ w.r.t. $\|\cdot\|_\infty$ is $C_0(X)$

(iii) $L^p[0,1]$ is separable if $1 \leq p < \infty$

(iv) $L^\infty[0,1]$ is not separable (on HW 9)

(End of Day 23)

3.7. Definition of convergence in measure. Note: $f_n \xrightarrow{m} f \not\Rightarrow f_n \to f$ a.e. (Example 3.1(ii))

3.8. [Folland, Theorem 6.17] (Tchebyshev's inequality)

3.9. Proposition: If $f_n \to f$ in L^p, then $f_n \xrightarrow{m} f$

3.10. Proposition: If $f_n \xrightarrow{m} f$ then there is a subsequence $f_{n_j} \to f$ pointwise.

3.11. Corollary: If $f_n \to f$ in L^p, then there is a subsequence $f_{n_j} \to f$ pointwise

3.12. Definition of almost uniform convergence.

3.13. [Folland, Theorem 2.33] (Egoroff's theorem)

3.14. [Folland, Theorem 7.10] (Lusin's theorem)

(End of Day 24)

VII. Differentiation and Integration

1. Differentiation of Monotone functions

Motivation: Let $f \in L^1[a,b]$ and define $F(x) = \displaystyle\int_a^x f(t)dt$.

(i) Is F differentiable at a point $x \in [a,b]$? If so, is $F'(x) = f(x)$?

(ii) We know this to be true if f is continuous at x. Hence, if f is Riemann-integrable, then F is differentiable and $F'(x) = f(x)$ a.e. But what about the general case for any $f \in L^1[a,b]$

(iii) Note that, if $f \geq 0$, then F is monotone increasing.

1.1. Definition of four Dini derivatives

1.2. Remark :

(i) $D^+f(x) \geq D_+f(x)$ and $D^-f(x) \geq D_-f(x)$

(ii) If $D^+f(x) = D_+f(x)$, then we say that f has a right-hand derivative at x, which we denote by $f'(x+)$. We define $f'(x-)$ similarly.

(iii) f is said to be differentiable iff $f'(x+) = f'(x-) \neq \pm\infty$

1.3. Example: If $f(x) = x\sin(1/x)$ for $x > 0$ and $f(0) = 0$, then $D^+f(0) = 1$ and $D_+f(0) = -1$

1.4. Lemma: If $f : [a, b] \to \mathbb{R}$ has a local max at a point $c \in (a, b)$, then $D_+f(c) \leq D^+f(c) \leq 0 \leq D_-f(c) \leq D^-f(c)$

1.5. [HLR, Prop 5.2] (Proof is HW)

1.6. [HLR, Lemma 5.1] (Vitali Covering Lemma) (without proof)

1.7. [HLR, Theorem 5.3 part (i)]

1.8. [HLR, Theorem 5.3 part (ii)]

1.9. Example : If f is the Cantor function, then $f' \equiv 0$ and hence $\int_0^1 f'(x)dx = 0 < 1 = f(1) - f(0)$ (given as HW)

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2. Functions of Bounded Variation

2.1. Definition : Let $f : [a, b] \to \mathbb{R}$

(i) Given a partition $\mathcal{P}$, definition of p, n, and t

(ii) Definition of positive, negative and total variation of f

(iii) Definition of $BV[a, b]$

2.2. [HLR, Lemma 5.4]

2.3. [HLR, Theorem 5.5]

2.4. [HLR, Corollary 5.6, Problem 5.11]

3. Differentiation of an integral

3.1. [HLR, Lemma 5.7]

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3.2. [HLR, Proposition 4.14]

3.3. [HLR, Lemma 5.8]

3.4. [HLR, Lemma 5.9]

3.5. [HLR, Theorem 5.10]

4. Absolute Continuity

4.1. Remark: When is it true that $\int_a^b f'(x)dx = f(b) - f(a)$?

4.2. Definition of absolutely continuous function and $AC[a, b]$

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4.3. Remark : $f \in L^1[a, b]$ and $F(x) = \int_a^x f(t)dt$, then $F \in AC[a, b]$ (by Prop 3.2)

4.4. [HLR, Lemm 5.11, Corollary 5.12]

4.5. [HLR, Lemma 5.13]

4.6. [HLR, Theorem 5.14] (Fundamental Theorem of Calculus)

4.7. Remark: Given $F \in AC[a,b]$ increasing ($\Leftrightarrow F' \geq 0$), then $\int_{[a,b]} g d\mu_F = \int_a^b g(t)F'(t)dt$ for any $g \geq 0$ or $g \in L^1(\mu_F)$ [Compare with V.3.6(iii)]

4.8. [Folland, Theorem 3.36]

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VIII. Integration as a Linear Functional

1. Bounded Linear Functionals

1.1. Definition :

 (i) Continuous linear map $T : V \to W$ between two normed linear spaces.

 (ii) Continuous linear function $T : V \to \mathbb{C}$. Dual space V^*

1.2. Examples :

 (i) If $V = \mathbb{C}^n$ with $\|\cdot\|_2$. For any fixed $w \in V$, define $T : V \to \mathbb{C}$ by $v \mapsto \langle v, w \rangle$

 (ii) If μ is a regular Borel measure on X compact, and $V = C(X)$, $T_\mu : V \to \mathbb{C}$ by $f \mapsto \int f d\mu$

 (iii) $V = C[0,1]$ with $\|\cdot\|_1$, then $T(f) = f(1/2)$ is discontinuous (Homework 11)

1.3. [Folland, Proposition 5.2]

1.4. Definition of $\mathcal{B}(V,W)$ and $\|T\|$ for $T \in \mathcal{B}(V,W)$.

1.5. [Folland, Exercise 5.2]

1.6. Remark :

 (i) $\forall v \in V, \|Tv\| \leq \|T\|\|v\|$

 (ii) If $C > 0$ such that $\|Tv\| \leq C\|v\|$, then $\|T\| \leq C$. Furthermore, if $\exists v_0 \in V$ such that $\|Tv_0\| = C\|v_0\|$, then $\|T\| = C$

1.7. Examples : See Examples 1.2

 (i) $\|T_w\| = \|w\|$

 (ii) $\|T_\mu\| = \mu(X)$

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2. Dual of $L^p[0,1]$

2.1. [HLR, Proposition 6.11]

2.2. [HLR, Lemma 6.12]

2.3. Lemma : If $T \in (L^p[0,1])^*$ and $g \in L^1[0,1]$ such that $T(\psi) = \int \psi g$ for all step functions ψ, then $g \in L^q[0,1]$ and $T = T_g$ on $L^p[0,1]$

2.4. [HLR, Theorem 6.13]

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2.5. Remark :

(i) Theorem 2.3 works for any σ-finite measure space. We need the Radon-Nikodym theorem for this. In particular, $(\ell^p)^* \cong \ell^q$ (HW 11)

(ii) $(L^\infty)^* \neq L^1$ because L^1 is separable and L^∞ is not. Another proof is in HW 11.

(iii) However, Theorem 2.1 holds even for $p = \infty$. ie. There is an injection $L^1[0,1] \hookrightarrow (L^\infty[0,1])^*$

(iv) $(L^2)^* \cong L^2$. This fact is true for any complete inner product space.

3. Positive linear functionals on $C(X)$

3.1. Definition: Let X be a compact Hausdorff space.

(i) Recall definition of regular Borel measure

(ii) A positive linear functional $T : C(X) \to \mathbb{C}$. Note: Every regular Borel measure μ on X defines a positive linear function on $C(X)$ by $T(f) = \int_X f d\mu$.

3.2. (Riesz Representation Theorem) If T is a positive linear functional on $C(X)$, then there exists a unique Borel measure μ on X such that $T(f) = \int_X f d\mu$

3.3. Motivation: Suppose μ is a regular Borel on a compact *metric* space X, then μ is uniquely determined by its values on open sets. So, consider an open set $O \subset X$ and define

$$f_n(x) = \begin{cases} 0 & : x \in X \setminus O \\ nd(x, X \setminus O) & : x \in O, \text{ and } d(x, X \setminus O) \leq 1/n \\ 1 & : x \in O, \text{ and } d(x, X \setminus O) > 1/n \end{cases}$$

Then, $0 \leq f_n \leq 1, \text{supp}(f) \subset O$ and $f_n \in C(X)$ and $f_n \to \chi_O$ pointwise. Hence, by the dominated convergence theorem,

$$\mu(O) = \lim_{n\to\infty} T(f_n)$$

Hence, we make the following definition :

3.4. Definition : $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is defined by

(i) If O open, $\mu^*(O) = \sup\{T(f) : f \in C(X), 0 \leq f \leq 1, \text{supp}(f) \subset O\}$

(ii) For any $A \subset X$, $\mu^*(A) = \inf\{\mu^*(O) : O \text{ open s.t. } A \subset O\}$

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3.5. Theorem (Partitions of Unity): If $\{O_i\}_{i=1}^n$ open and K compact such that $K \subset O_1 \cup O_2 \cup \ldots \cup O_n$, then there exist $g_i \in C(X)$ such that $0 \leq g_i \leq 1$, $\sum g_i = 1$ on K and $\text{supp}(g_i) \subset O_i$ (Proof omitted)

3.6. Lemma: μ^* is a well-defined outer measure on X

3.7. Lemma: Every open set is μ^*-measurable. Hence, μ^* restricts to a measure on $\mathcal{B}_X$

3.8. Lemma: For any compact $K \subset X$, $\mu(K) = \inf\{T(f) : f \in C(X), f \geq \chi_K\}$

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3.9. Lemma: μ is a regular Borel measure.

Proof of Theorem 3.2

3.10. Corollary: Let $T : C[0,1] \to \mathbb{C}$ be a positive linear functional, then there exists an increasing, right-continuous function F such that

$$T(f) = \int_0^1 f dF$$

This function F is unique upto additive constant.

3.11. Corollary: If $T : C(X) \to \mathbb{C}$ is a positive linear functional, then $\|T\| = T(1)$

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4. Dual of $C(X)$

4.1. Remark: Every bounded linear functional $T \in (C(X))^*$ is determined by its values on real-valued $f \in C(X)$.

4.2. Theorem : Let $T \in C(X, \mathbb{R})^*$, then there exist unique positive linear functionals T_+ and T_- such that $T = T_+ - T_-$, and $\|T\| = T_+(1) + T_-(1)$.

4.3. Definition: Let $M(X)$ denote the $\mathbb{C}$-span of all regular Borel measures on X.

4.4. Corollary: Let X be a compact Hausdorff space, then $C(X)^* \leftrightarrow M(X)$

4.5. Corollary: Define

$$NBV[0,1] = \{F \in BV[0,1] : F(0) = 0\}$$

Then, $C[0,1]^* \cong NBV[0,1]$

4.6. Remark: One can put a norm on $M(X)$ such that the linear bijection in Theorem 4.4 is an isomorphism of normed linear spaces. In Corollary 4.5, this norm is given by $\|F\| = T_0^1(F)$ =the total variation of F on $[0,1]$ (See HW 10).

(End of Day 34)

Review for Final Exam

(End of Day 35)

IX. Instructor Notes

0.1. Chapters I-IV, and VII were taken mostly from Royden. Chapters V, VI and VII were from other places, but mostly Folland. I believe Folland should be added to the list of suggested textbooks.

0.2. The Caratheodory definition of measurability was tough to grasp. Perhaps it should have been motivated with the case of subsets of $[0,1]$.

0.3. I did not touch upon Hausdorff measures, as it did not seem necessary.

0.4. I did not do Jensen's inequality. Although it is important, we only need it for Minkowski's inequality here, and that can be done using a simpler lemma [Folland, Lemma 6.1]

0.5. I did do Product measures and Fubini's theorem. This seems essential, and should be added to the syllabus.

0.6. I did not do the Riesz-Representation Theorem for L^p spaces in full generality (I did it only for $[0,1]$ and for the ℓ^p spaces), because I did not have time to do Radon-Nikodym.

Bibliography

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[Bartle] R. Bartle, *Introduction to Measure Theory*