MTH308/412: Combinatorics and Graph Theory

Semester 2, 2022-2023

Prahlad Vaidyanathan

Contents

I. What is Combinatorics?

1. Example: Who Moved My Cheese?

A mouse named Ashwini wants to steal some cheese and run back home as quickly as possible. She needs to run along the paths between some obstacles, but must always choose the shortest possible route (as shown below).

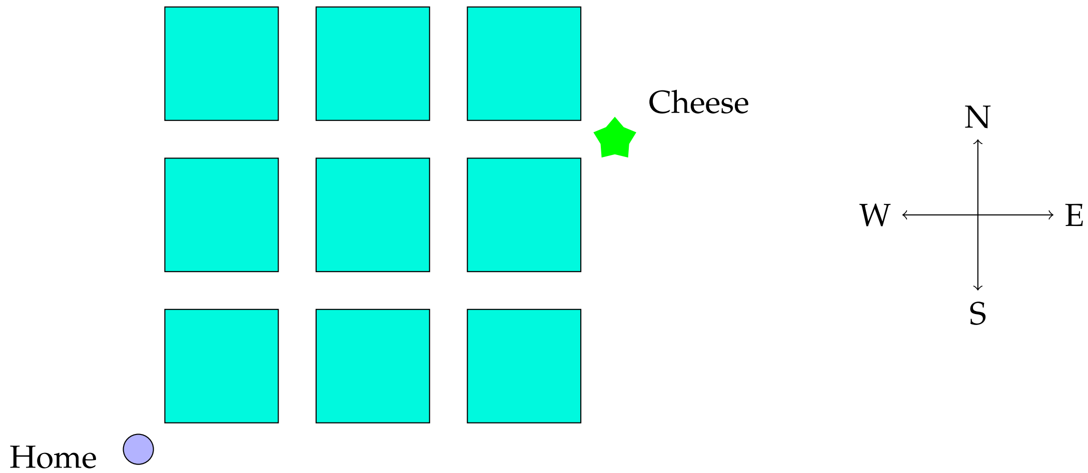


Figure I.1.: Ashwini Moved my Cheese

There are, however, many possible 'shortest' routes (we think of each block as a step of one unit in that direction).

1.1. Example: Two examples are shown below:

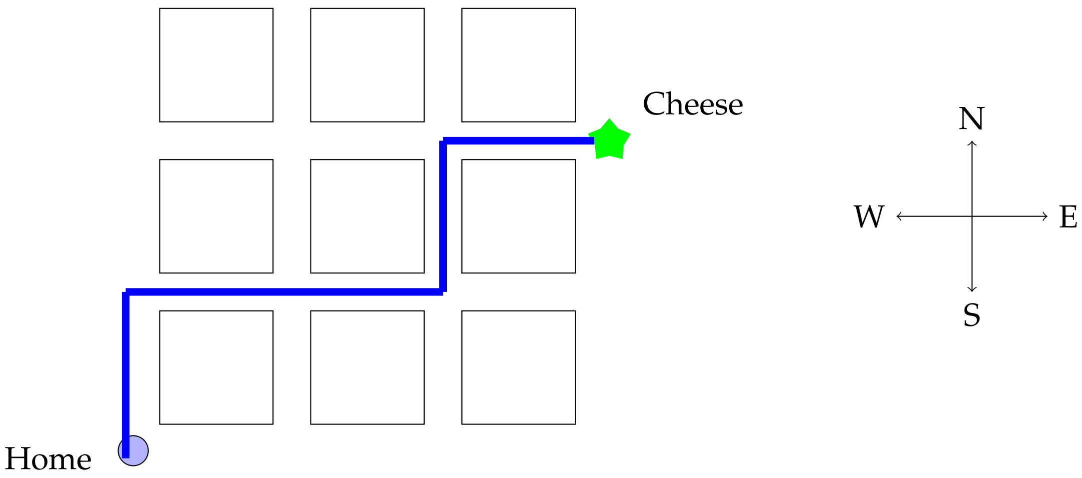


Figure I.2.: One possible route for Ashwini.

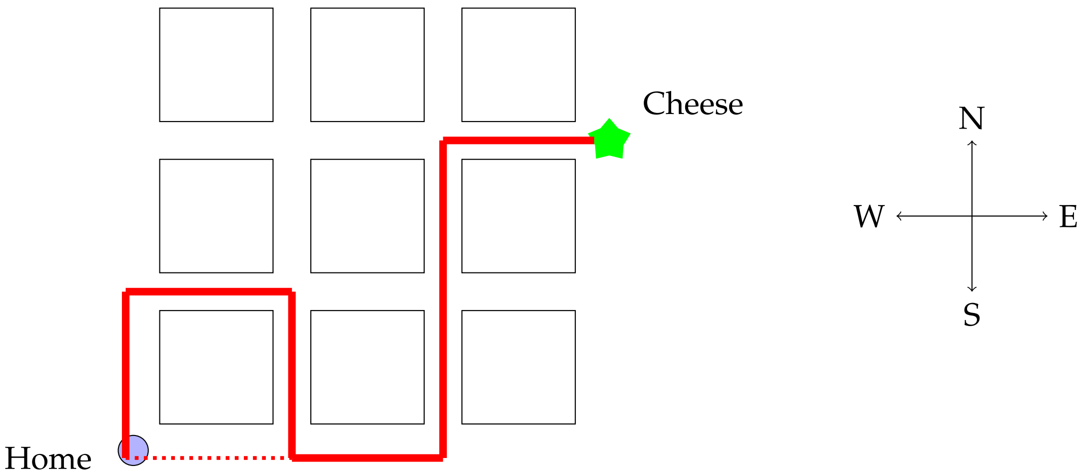


Figure I.3.: A route Ashwini would not take.

1.2. Question: Determine the number of such routes
Hint: Use the directions shown in the picture.

Solution: Note that Ashwini must walk only West (W) or South (S). Otherwise the route would be longer.

Now we may draw all such "S/W" routes. Observe that one has to go "W" 3 units, and "S" two units. Therefore, the route may be described as a word of the form *SWWSW*. Thus, there are 10 such words.

Why? Because the first W has 5 options, the second has 4 options, and the third has 3 options. With those three fixed, the remaining two places must be filled with an S. This gives us $5 \times 4 \times 3$ such routes. However, the three W's may be interchanged without any change in the route. There are 3! ways of interchanging them, so one has

$$\frac{5 \times 4 \times 3}{1 \times 2 \times 3} = 10$$

possible routes. $\square$

1.3. Ashwini's Friends: The next day, Ashwini finds more cheese in a new place. Since there is a lot of it, Ashwini decides to share it with her friends. Now, seven mice (Ashwini, Bhanu, Cyrus, Dhruva, Elango, Fatima, Gaurav) must steal pieces of the cheese and find their way back to their respective homes through a new maze. Once again, each one takes the shortest route possible.

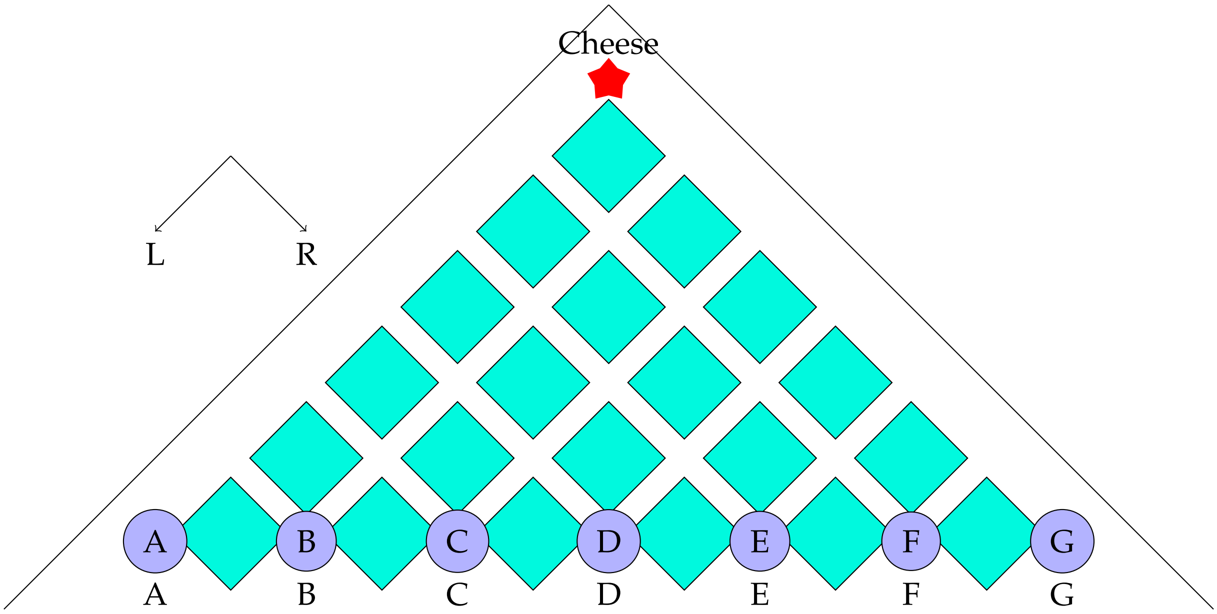


Figure I.4.: Shortest Route for each of the mice.

1.4. Example: Two examples are shown below:

1.5. Question: How many such shortest routes does Dhruva have?

Hint: Look at the directions again!

Solution: Once again, each route must take only L or R steps. Hence, each route is a *word* in the alphabet $\{L, R\}$. Moreover, the word consists of 6 letters (so we get words of the form $LLLRLR$). For Dhruva, there must be exactly three L's and three R's. Therefore, the number of such routes is

$$\frac{6 \times 5 \times 4}{1 \times 2 \times 3} = 20.$$

□

1.6. Question: For each mouse, determine the number of such routes.

Before we proceed, fill in the each circle of Figure I.7 with the number of such routes.

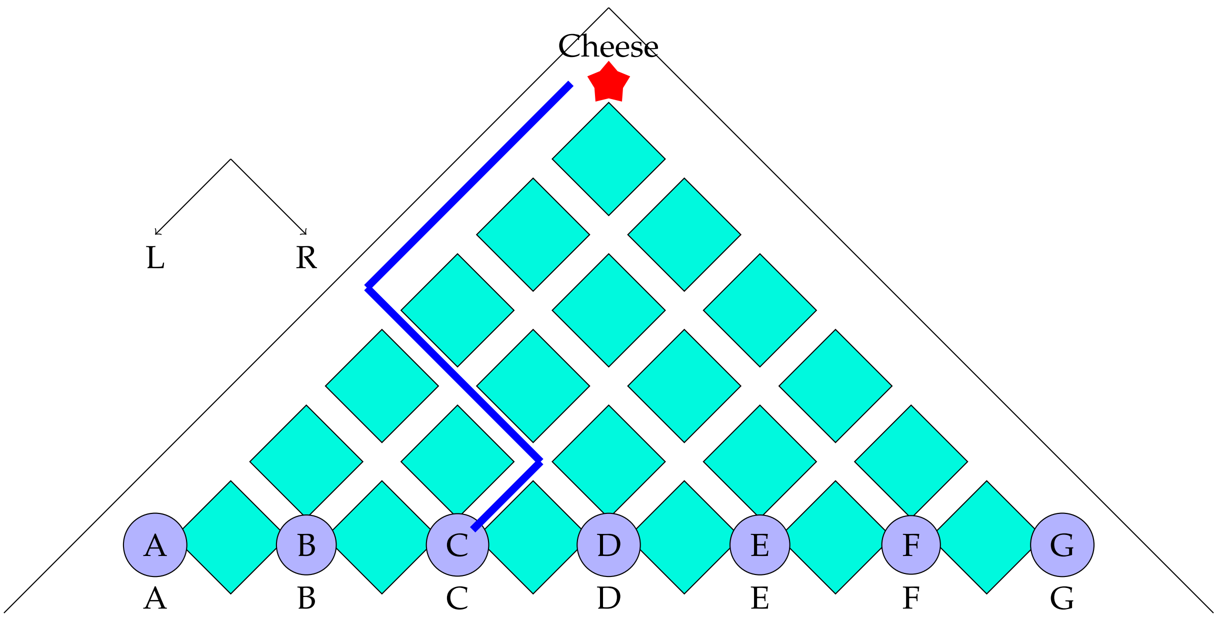


Figure I.5.: A route that Cyrus could take

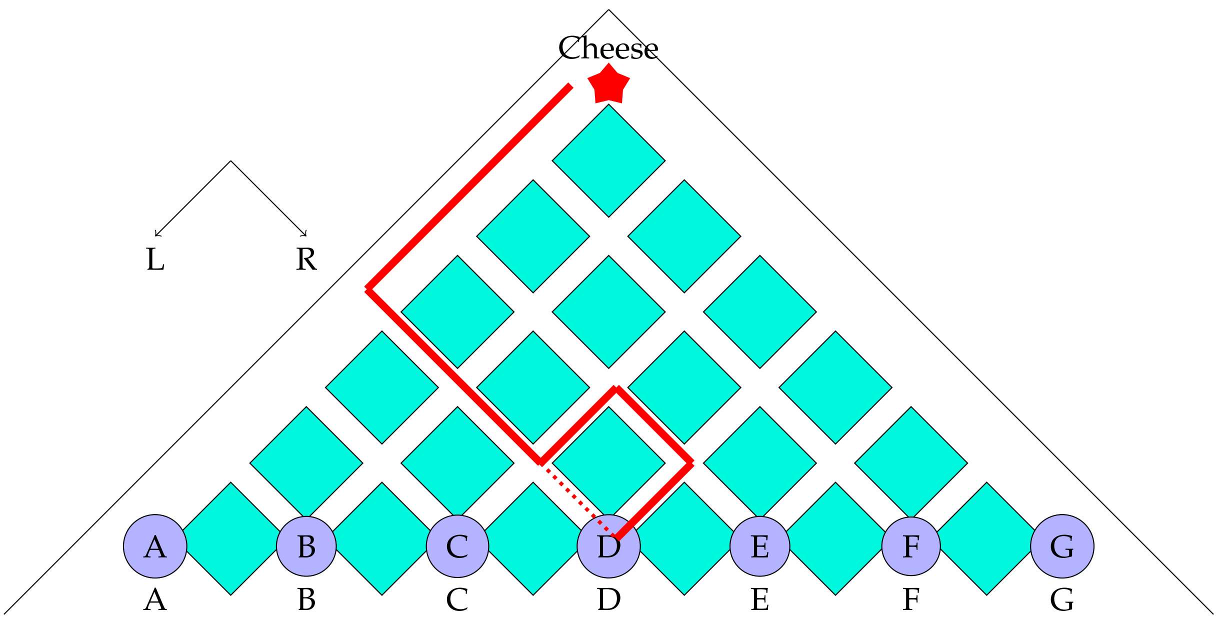


Figure I.6.: A route that Dhruva would not take

1.7. Question: Do you notice any patterns?

Solution: Notice that A and G must have only 1. The numbers appearing are also symmetric (Cyrus and Elango both have the same number). □

1.8. More Mice!: Let us add more mice to the problem (because, why not?). Suppose

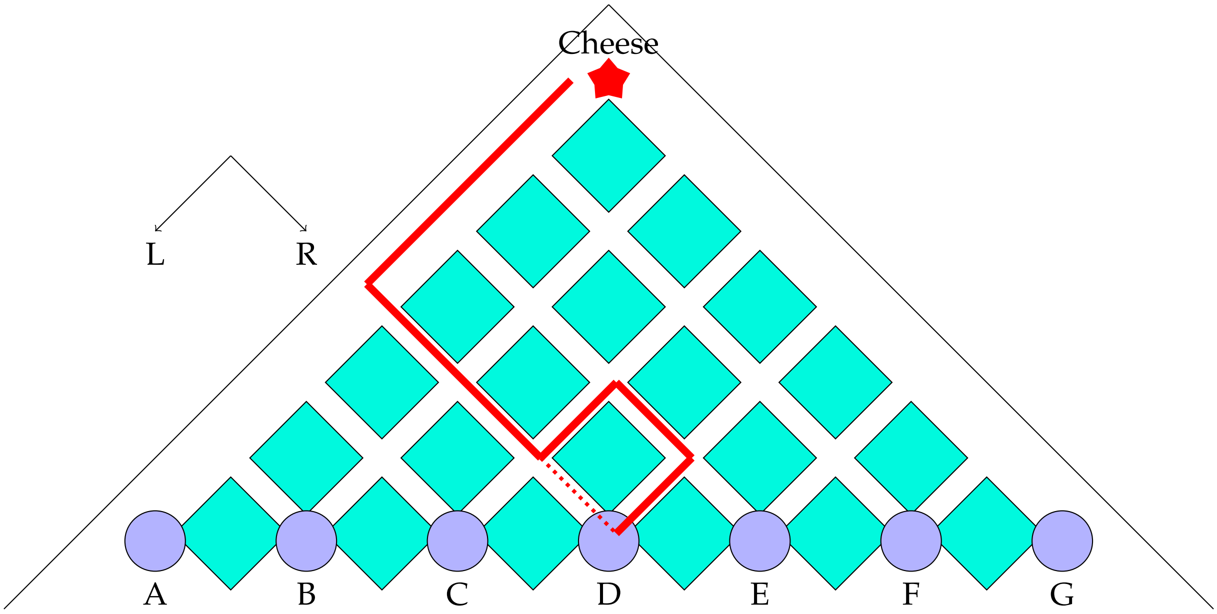


Figure I.7.: Fill in the number of shortest routes for each mouse

there are six more mice (Hari, Ibrahim, John, Kabir, Lalit and Meena), attempting the same robbery. The homes of these new mice, however, are closer to the prize! (See Figure I.8).

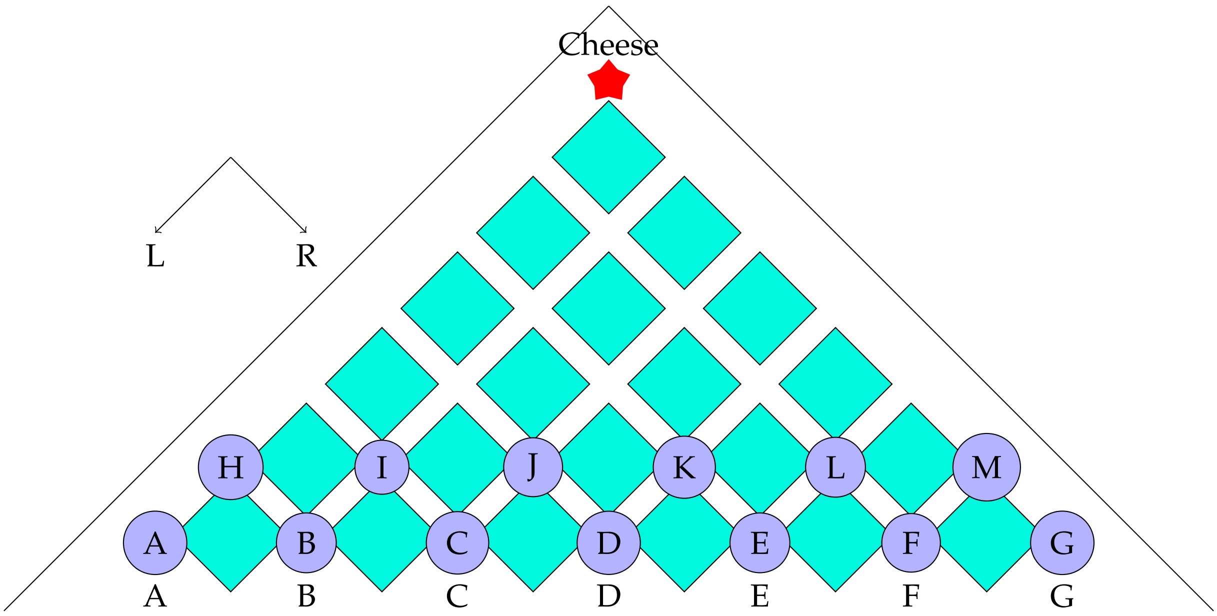


Figure I.8.: More Mice!

1.9. Question: Compute the number of shortest routes for each of the new mice as well.

Solution: For instance, the number of routes for John is computed as follows: It must be a 5-letter word with 3 L's and 2 R's. Therefore, the number is

$$\frac{5 \times 4 \times 3}{1 \times 2 \times 3} = 10.$$

□

Using the data from Figure I.7, let us now complete the triangle in Figure I.9 with these numbers placed at each corner.

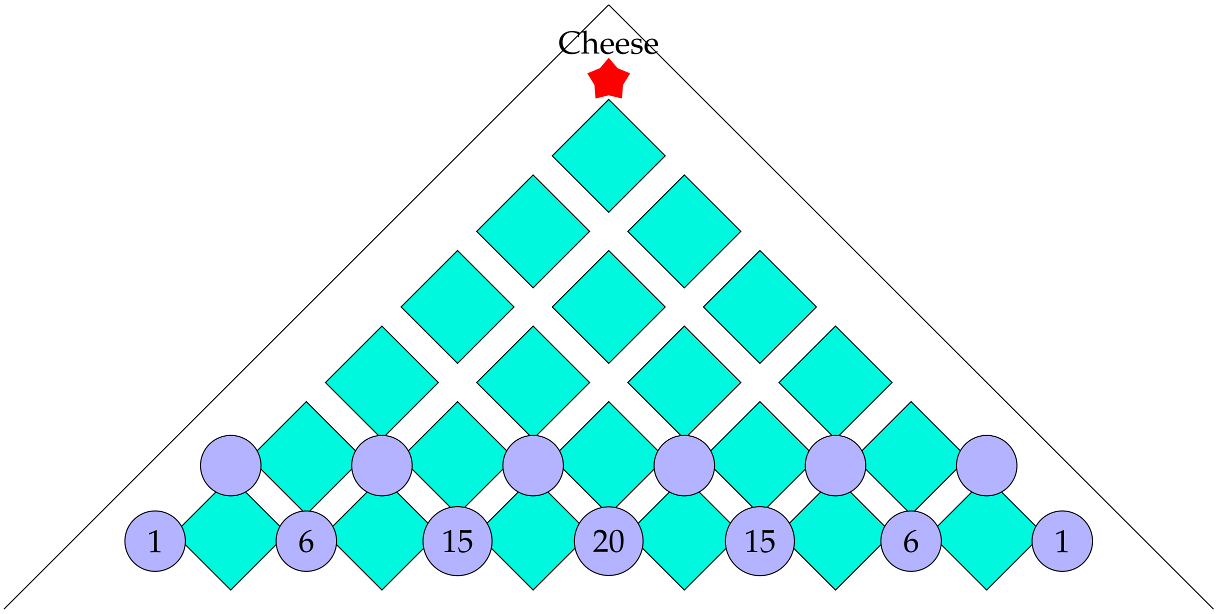


Figure I.9.: More Mice!

1.10. Question: Do you see a pattern emerging? What do you infer?

Solution: Once again, a 1 appears on the outer two entries. Also, the same symmetry as before (Ibrahim = Lalit, and John = Kabir).

A new phenomenon: Observe that the number appearing on the lowest row is the sum of the entries appearing in the previous two rows directly joining it. For instance, Cyrus (15) = Ibrahim (5) + John (10). □

1.11. Mice Everywhere!: Suppose there was a mouse at every corner. First of all, there would (sadly) not be any cheese left. On the bright side, we would have learnt something. Can you fill in all these empty circles?

Indeed, you don't need to count anymore - you may simply apply the rules from Question 1.10 blindly!

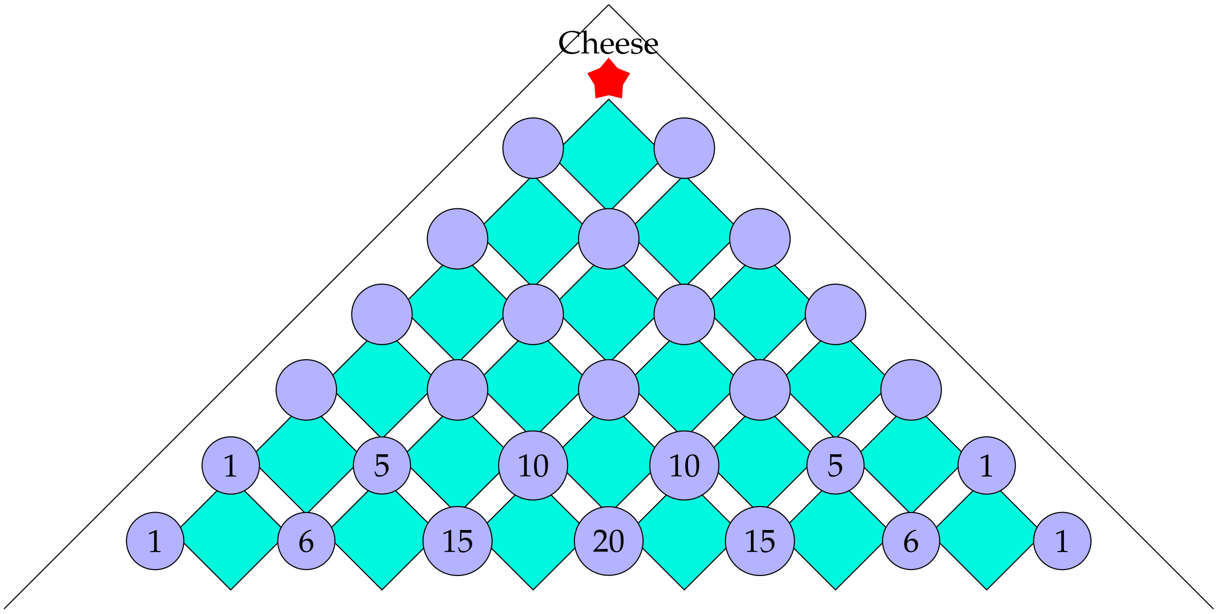


Figure I.10.: Mice Everywhere!

1.12. The Conclusion: Pascal's Triangle. The triangle you end up with should look like this (the cheese has been eaten and replaced by the house of a mouse).

Some explanation is in order. Let us introduce some terminology first.

- The *level* of a mouse is the horizontal row on which it appears (The *Cheese* is on level 0, and is assigned the value 1). For instance, Hari is on level 5, while Chitra is on level 6. This number is denoted by n.
- The number of *right* turns that a mouse needs to take is denoted by r. For instance, for John, $r = 2$ (See Figure I.8).
- Similarly, the number of *left* turns needed by a mouse is denoted by ℓ. Observe that

$$n = \ell + r.$$

- The number of shortest routes from the choose to a mouse is called a *binomial coefficient*, and is denoted by

$$\binom{n}{r}.$$

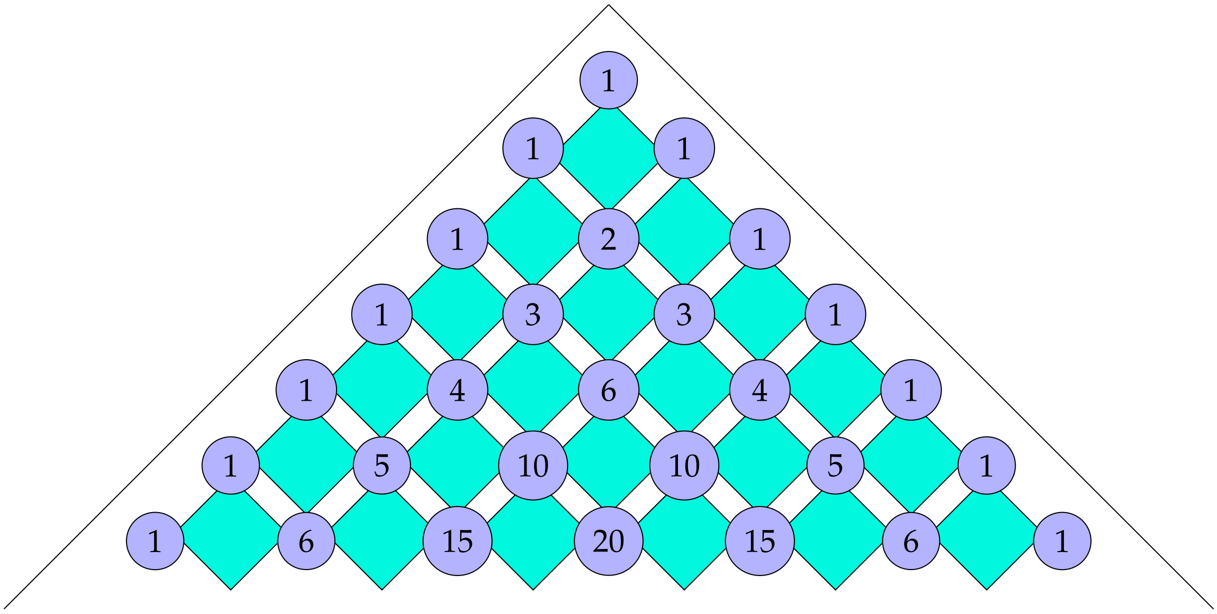


Figure I.11.: Pascal's Triangle

In Question 1.6, we had seen that

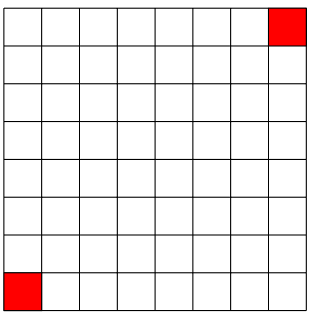
$$\binom{n}{n-r} = \binom{n}{r}.$$

- In Question 1.10, you have observed that a given value at level n is obtained by adding the two values at level $(n-1)$ 'adjacent' to it. This is captured by the formula

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

2. Example: Tiling a Chessboard

- Consider an 8×8 chessboard, and delete the tiles from two diagonally opposite corners. Can the remaining region be covered by dominoes? (a domino is a 1×2 tile - it may be placed vertically or horizontally).



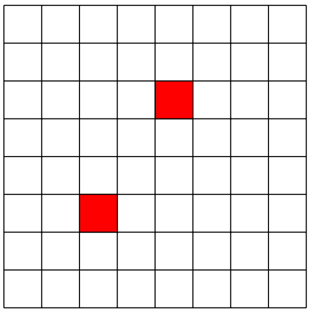
Hint: Think of an actual chessboard!

(End of Day 1)

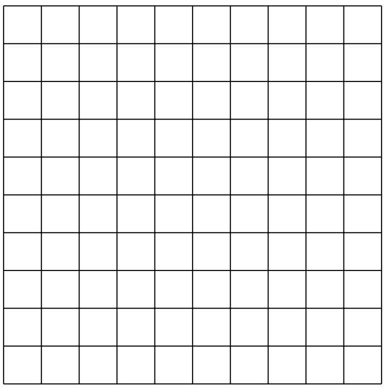
Solution: Colour the chessboard (like a normal chessboard) in white and black. Notice that the two deleted squares must have the same colour. We may assume that the the number of white squares is 2 less than the number of black squares.

Now, each domino covers each colour once. Therefore, if such a tiling exists, then the number of white squares must equal the number of black squares. Thus, such a tiling is impossible. □

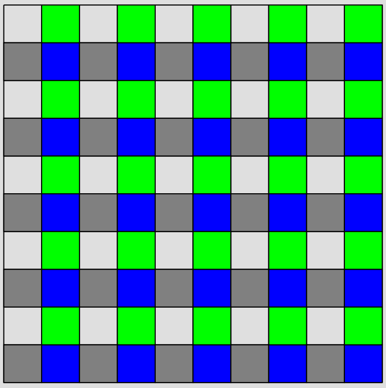
- Consider an 8×8 chessboard, and delete two tiles from it - one white and one black. Is it now possible to cover the remaining region by dominoes?



- 10×10 board: Consider a 10×10 board. Can you cover it with 1×4 rectangles?



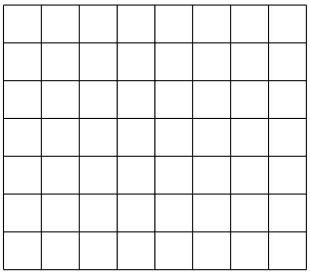
Solution: Colour the board with 4 colours, like so



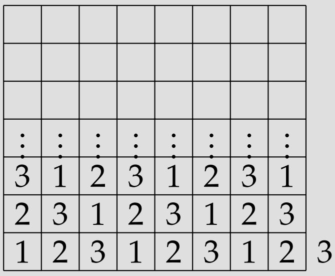
Then, no matter how we place it, each 1×4 tile covers an *even* number (possibly zero) of squares of each colour. Therefore, if the board had such a tiling, the total number of squares of each colour would have to be even. However, there are 25 squares of each colour, so a tiling is impossible. $\square$

- An $m \times n$ board: Consider an $m \times n$ board, and an integer k. When is it possible to tile the board with $1 \times k$ tiles?

For simplicity, try the case $m = 7, n = 8$ and $k = 3$.



Solution: Assign k colours to the board, as below (for $k = 3$).



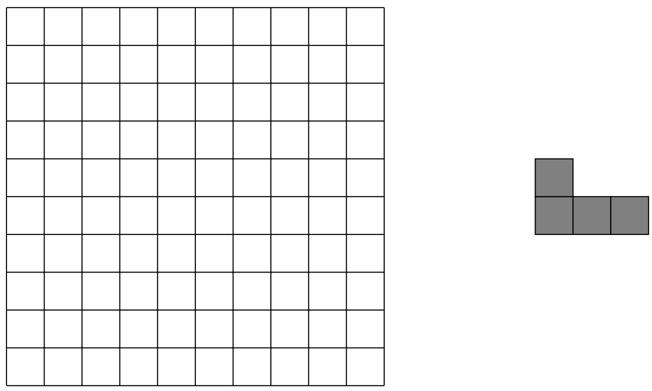
Notice that if $k \mid n$ or $k \mid m$, then one may construct a tiling quite easily.

Now suppose that $k \nmid n$ and $k \nmid m$. Each tile covers all three colours exactly once. Therefore, for such a tiling to exist, each colour must appear the same number of times. Write $m = ck + a$ with $0 < a < k$. Then, the top $ck \times n$ grid may be tiled. So we consider the bottom $a \times n$ strip. Similarly, let b denote the remainder when n is divided by k, and consider the left bottom $a \times b$ strip. Assume $a \leq b$, then it looks like

a				s
$\vdots$		$\vdots$		
2	3			
1	2	3	$\cdots$	b

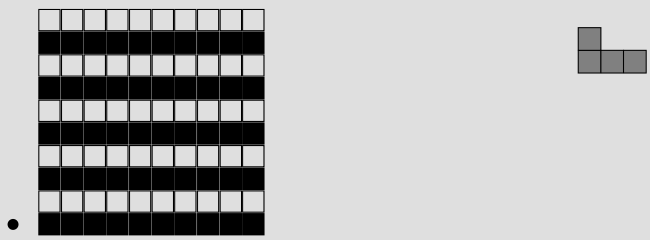
where $s = a + b - 1$. Since $a \leq b$, colour b shows up in exactly a rows, while colour 1 does not show up in row 2, and must therefore appear atmost $(a-1)$ times. Therefore, the grid cannot be tiled unless $k \mid n$ or $k \mid m$. $\square$

- L-shaped tiles: Can one tile a 10×10 grid with L-shaped tiles as shown below?



Hint: Use two colours, but not like a chessboard.

Solution: Colour alternate rows with the same colour to arrive at the following picture.



Then, each L-shape covers exactly 3 White + 1 Black or 3 Black + 1 White. Let a denote the number that cover $(3w + b)$ and b be the number that cover $(3b + w)$. The total number of white = The total number of black is 50. Therefore,

$$3w + b = 50 = 3b + w.$$

Thus, $w = b$. But in that case, $4w = 50$, which does not have an integer solution. Therefore, such a tiling is not possible. $\square$

There are many variations and generalizations of these problems. Some examples:

- Can you 'tile' a given shape A with another given shape B? Typically, both A and B are made from square tiles attached in some configuration.
- If such a tiling is possible, then how many different tilings (arrangements) are possible?
- Similar problems exist in higher dimensions - can one 'pack' a cube with 'bricks' of some prescribed length/height/width?

(End of Day 2)

3. What is Combinatorics?

Combinatorics is concerned with arrangements of the objects of a set into patterns satisfying specified rules. The general types of problems that occur repeatedly are:

- *Existence of the arrangement*. If one wants to arrange the objects of a set so that certain conditions are fulfilled, it may not be at all obvious whether such an arrangement is possible. This is the most basic of questions. If the arrangement is not always possible, it is then appropriate to ask under what conditions, both necessary and sufficient, the desired arrangement can be achieved.
- *Enumeration or classification of the arrangements*. If a specified arrangement is possible, there may be several ways of achieving it. If so, one may want to count or to classify them into types.
- *Study of a known arrangement*. After one has done the (possibly difficult) work of constructing an arrangement satisfying certain specified conditions, its properties and structure can then be investigated.
- *Construction of an optimal arrangement*. If more than one arrangement is possible, one may want to determine an arrangement that satisfies some optimality criterion-that is, to find a "best" or "optimal" arrangement in some prescribed sense.

II. Permutations and Combinations

1. Four Basic Counting Principles

Definition 1.1. Let S be a set. A partition of S is a collection $S_1, S_2, \ldots, S_m$ of subsets of S such that each element of S is in exactly one of those subsets:

$$S = S_1 \cup S_2 \cup \ldots \cup S_m, \text{ and}$$
$$S_i \cap S_j = \emptyset \text{ if } i \neq j.$$

Thus, the sets $S_1, S_2, \ldots, S_m$ are *pairwise disjoint* sets, and their union is S. The subsets $S_1, S_2, \ldots, S_m$ are called the *parts* of the partition.

Proposition 1.2 (Addition Principle). *If S is partitioned into pairwise disjoint parts $S_1, S_2, \ldots, S_m$, then*

$$|S| = |S_1| + |S_2| + \ldots + |S_m|.$$

Proposition 1.3 (Multiplication Principle). *Let S be the set of ordered pairs (a, b) of objects, where a can be chosen from a set with p elements, and for each choice of a, there are q choices for b. Then,*

$$|S| = p \times q.$$

Example 1.4. Determine the number of positive integers that are factors of the number

$$3^3 \times 5^2 \times 11^7 \times 19^6.$$

Note that every such number is of the form

$$3^i \times 5^j \times 11^k \times 19^\ell$$

where $0 \leq i \leq 3, 0 \leq j \leq 2, 0 \leq k \leq 7, 0 \leq \ell \leq 6$. Therefore there are

$$4 \times 3 \times 8 \times 7$$

such divisors (including 1 and the number itself).

Example 1.5. How many integers between 0 and $10,000$ have only one digit equal to 5?

Let S be the set of all such integers. Let S_1 be the set of such integers which have only one digit. Let S_2, S_3 and S_4 be defined analogously. Then, $\{S_1, S_2, S_3, S_4\}$ is a partition of S. Also,

$$|S_1| = 1.$$

Now, $S_2 = S_{2,1} \cup S_{2,2}$ where $S_{2,1} = \{\text{units digit is } 5\}$ and $S_{2,2} = \{\text{tens digit is } 5\}$. Then,

$$|S_{2,1}| = 8$$

since the tens digit cannot be zero. Also, $|S_{2,2}| = 9$ sincee the units digit cannot be 5. Therefore,

$$|S_2| = 8 + 9 = 17.$$

Similarly, $S_3 = S_{3,1} \sqcup S_{3,2} \sqcup S_{3,3}$ and

$$|S_{3,1}| = 9 \times 8, |S_{3,2}| = 9 \times 8, |S_{3,3}| = 9 \times 9.$$

Therefore,

$$|S_3| = 72 + 72 + 81 = 225.$$

Finally, $S_4 = S_{4,1} \sqcup S_{4,2} \sqcup S_{4,3} \sqcup S_{4,4}$ and

$$|S_{4,1}| = 8 \times 9 \times 9, |S_{4,2}| = 8 \times 9 \times 9, |S_{4,3}| = 8 \times 9 \times 9, |S_{4,4}| = 9 \times 9 \times 9.$$

Thus, $|S_4| = \ldots$, so

$$|S| = |S_1| + |S_2| + |S_3| + |S_4| = \ldots.$$

Proposition 1.6 (Subtraction Principle). *Let U be a set and $A \subset U$. If*

$$A^c := \{x \in U : x \notin A\},$$

then

$$|A| = |U| - |A^c|.$$

Example 1.7. OTPs are to consist of strings of six symbols taken from digits $0, 1, \ldots, 9$. How many OTPs have a repeated digit?

Let U be the set of all OTPs and A be the set of all OTPs with no repeated digits. Then,

$$|U| = 10^6$$

and

$$|A^c| = 10 \times 9 \times 8 \times 7 \times 6 \times 5.$$

So,

$$|A| = 10^6 - |A^c| = \ldots.$$

Proposition 1.8 (Division Principle). *Let S be a finite set that is partitioned into k parts in such a way that each part has the same number of objects. Then,*

$$k = \frac{|S|}{\textit{the number of objects in a single part}}.$$

Example 1.9. We wish to divide a class of 24 students into groups of 4. How many ways can one do this?

The number of ways to choose 4 students is $24 \times 23 \times 22 \times 21 =: n$. The number of ways to arrange four students is $4 \times 3 \times 2 \times 1 = d$. So the number of distinct groups is

$$\frac{n}{d} = \frac{24 \times 23 \times 22 \times 21}{4 \times 3 \times 2 \times 1} = \dots$$

Remark 1.10. There are four ways in which counting problems may be classified. They are:

- Ordered arrangements with repetition, such as the set U in Example 1.7.
- Ordered arrangements without repetition, such as the set A^c in Example 1.7.
- Unordered arrangements with repetition, such as in Example 1.4.
- Unordered arrangements without repetition, such as in Example 1.9.

(End of Day 3)

2. Permutations of Sets

Definition 2.1. Let S be a set with n elements and $r \in \mathbb{N}$. An *r-permutation* of S is an ordered arrangement of r elements of S.

Example 2.2. If $S = \{a, b, c\}$, then the 1-permutations of S are

$$a \quad b \quad c$$

The 2-permutations of S are

$$ab \quad ac \quad ba \quad bc \quad ca \quad cb$$

and the 3-permutations of S are

$$abc \quad acb \quad bac \quad bca \quad cab \quad cba$$

Definition 2.3. $P(n, r)$ denotes the number of r-permutations of a set of n elements.

Example 2.4.

2.1. $P(n, r) = 0$ if $r > n$ and $P(n, 1) = n$.

2.2. By the previous example, $P(3, 1) = 3$, $P(3, 2) = 6$ and $P(3, 3) = 6$.

Theorem 2.5. *If $r \leq n$, then*

$$P(n, r) = \frac{n!}{(n - r)!}$$

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Proof. There are n ways to choose the first element, $(n-1)$ ways to choose the second element, and so on. Finally, there are $(n-(r-1))$ ways to choose the r^{th} element. Therefore,

$$P(n,r) = n \times (n-1) \times \ldots \times (n-r+1) = \frac{n!}{(n-r)!}.$$

□

Remark 2.6. By convention, we write

$$P(n,0) = 1$$

for all $n \in \mathbb{N}$. This agrees with the convention that $0! = 1$.

Example 2.7. The *15-puzzle* consists of sliding squares labelled $1, 2, \ldots, 15$ on a 4×4 grid to move from an initial position to a prescribed final position.
Ignoring whether it is possible to move from one position to another given position, what is the total number of positions in the puzzle?

Solution: Assign 16 to the empty square. Then, the answer is precisely

$$P(16,16) = 16!$$

□

Example 2.8. What is the number of ways to order the 26 alphabets in such a way that no two of the vowels $\{a, e, i, o, u\}$ occur consecutively?

Solution: First arrange the consonants. There are 21! ways of doing that. Then, there are 22 places for a, 21 places for e, 20 for i, 19 for o and 18 for u. Therefore, the answer is

$$21! \times \frac{22!}{17!}$$

□

Example 2.9. How many 7-digit numbers are there such that the digits are distinct integers taken from $\{1, 2, \ldots, 9\}$ and such that the digits 5 and 6 do not appear consecutively in either order?

Solution: There are 4 different types of such numbers, which we count:

2.1. If the number has neither 5 nor 6: We can count the number of 7 permutations of the set $\{1,2,3,4,7,8,9\}$. There are

$$P(7,7) = 7!$$

such numbers.

2.2. If only the number 5 occurs, but 6 does not occur: The six remaining digits (other than 5) can be obtained as a 6-permutation of the set $\{1,2,3,4,7,8,9\}$. There are $P(7,6)$ ways of doing this. Then, there are 7 places where the 5 can go, so there are

$$7 \times P(7,6) = 7 \times 7!$$

such numbers.

2.3. If only 6 occurs and 5 does not: Again, there are

$$7 \times 7!$$

such numbers.

2.4. If both 5 and 6 occur: The 5 remaining digits can be obtained as a 5-permutation of the set $\{1,2,3,4,7,8,9\}$. There are $P(7,5)$ ways of doing this. Then, there are 6 places for the 5, and 5 places for the 6, so there are

$$6 \times 5 \times P(7,5) = 30 \times \frac{7!}{2!}$$

such numbers.

So the total number of such numbers is

$$7! + 2 \times 7 \times 7! + (30 \times \frac{7!}{2!}) = 30 \times 7!.$$

$\square$

Definition 2.10. Let S be a set with n elements and $r \in \mathbb{N}$. A *circular r-permutation* of S is an ordered arrangement of r elements of S in a circle. In other words, it is a linear arrangement as before, but we do not distinguish between arrangements if one can be obtained from the other by a cyclic permutation.

Example 2.11. If $S = \{1,2,3\}$, then it has 6 3-permutations

$$123 \quad 132 \quad 213 \quad 231 \quad 312 \quad 321$$

However, $\{123, 312, 231\}$ all represent the same circular permutation. Therefore, S has only 2 circular 3-permutations

$$123 \quad 132$$

Theorem 2.12. *If $r \leq n$, the number of circular r-permutations of n elements is*

$$\frac{P(n,r)}{r} = \frac{n!}{r \times (n-r)!}.$$

Proof. Let T denote the set of all linear r-permutations of $S = \{1, 2, \ldots, n\}$. Then, partition T into parts such that two linear permutations are in the same part if and only if one can be obtained from the other by a cyclic permutation. Then, the number of circular r-permutations is the number of parts. Each part contains r elements, so by the division principle, we get

$$\frac{P(n,r)}{r}$$

circular r-permutations. □

Example 2.13. How many necklaces can be made from 20 beads, each of a different colour?

Solution: This is a circular arrangements of the 20 beads. However, an arrangement can be turned over as well. For instance, $\begin{array}{ccc} & \mathrm{A} & \\ \mathrm{D} & & \mathrm{C} \\ \mathrm{F} & & \mathrm{B} \\ & \mathrm{E} & \end{array}$ is the same as $\begin{array}{ccc} & \mathrm{A} & \\ \mathrm{C} & & \mathrm{D} \\ \mathrm{B} & & \mathrm{F} \\ & \mathrm{E} & \end{array}$

Therefore, there are precisely

$$\frac{1}{2}\frac{20!}{20}$$

such necklaces. □

(End of Day 4)

3. Combinations (Subsets) of Sets

Definition 3.1. Let S be a set with n element and $r \in \mathbb{N}$. An *r-combination* from S is an unordered selection of r elements from S. The result is a subset $A \subset S$ (so *combination* and *subset* are interchangeable terms).

Example 3.2. If $S = \{a, b, c, d\}$, the possible 3-combinations from S are

$$\{a, b, c\} \quad \{a, c, d\} \quad \{a, b, d\} \quad \{b, c, d\}.$$

Definition 3.3. The number of r-combinations from a set of n elements is denoted by

$$\binom{n}{r}$$

Example 3.4.

3.1. If $r > n$, then
$$\binom{n}{r} = 0$$

3.2. If $r > 0$, then
$$\binom{0}{r} = 0.$$

3.3.
$$\binom{n}{0} = 1, \quad \binom{n}{1} = n, \quad \text{and} \quad \binom{n}{n} = 1.$$

3.4. In particular,
$$\binom{0}{0} = 1.$$

Theorem 3.5. *If* $0 \leq r \leq n$, *then*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Proof. We may choose a subset of r elements in two steps: First choose an ordered collection of r elements, and then re-arrange them in any order. The first step can be done in $P(n,r)$ ways, and the second step can be done in $r!$ ways. Therefore,

$$\binom{n}{r} = \frac{P(n,r)}{r!}.$$

Now apply Theorem 2.5. □

Example 3.6. Twenty-five points are chosen on the plane so that no three of them are collinear. How many straight lines do they determine? How many triangles do they determine?

Solution: Since no three are collinear, any pair of points determines a unique straight line. Therefore, there are

$$\binom{25}{2}$$

such lines. Similarly, tere are

$$\binom{25}{3}$$

such triangles. □

Example 3.7. How many 8-letter English words can be constructed if each word contains three vowels?

Solution: The three positions for the vowels can be chosen in

$$\binom{8}{3}$$

ways. Then, the vowel positions can be filled in 5^3 ways and the consonant places can be filled in 21^5 ways. So the total number of ways to form such a word is

$$\binom{8}{3} \times 5^3 \times 21^5.$$

□

Corollary 3.8. *If* $0 \leq r \leq n$*, then*

$$\binom{n}{r} = \binom{n}{n-r}.$$

Theorem 3.9 (Pascal's Formula)**.** *If* $1 \leq r \leq n-1$*, then*

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

This formula was proved in section 1. A proof is also given in the textbook.

Example 3.10. Suppose $n = 5, r = 3$ and $S = \{x, a, b, c, d\}$. If

$$\begin{aligned} A &:= \{\text{3-subsets of S not containing x}\} \\ B &:= \{\text{3-subsets of S containing x}\} \end{aligned}$$

then the 3-subsets of S in A are

$$\{a, b, c\} \quad \{a, c, d\} \quad \{a, b, d\} \quad \{b, c, d\}.$$

The 3-subsets of S in B are

$$\{x, a, b\} \quad \{x, b, c\} \quad \{x, a, c\} \quad \{x, a, d\} \quad \{x, b, d\} \quad \{x, c, d\}$$

Deleting x, the sets in B may be identified with

$$\{a, b\} \quad \{b, c\} \quad \{a, c\} \quad \{a, d\} \quad \{b, d\} \quad \{c, d\}$$

which are precisely all the 2-subsets of $S \setminus \{x\}$. Hence,

$$\begin{aligned}\binom{5}{3} &= |\{\text{3-subsets of } S\}| \\ &= |A| + |B| \\ &= |\{\text{3-subsets of } S \setminus \{x\}\}| + |\{\text{2-subsets of } S \setminus \{x\}\}| \\ &= \binom{4}{3} + \binom{4}{2}.\end{aligned}$$

Theorem 3.11. *For all $n \geq 0$,*

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$$

and the common value equals the number subsets of a set with n elements.

Proof. Note that the LHS is precisely the number of subsets of a set with n elements, denoted by S. For a subset $A \subset S$, consider the function

$$\chi_A : S \to \{0, 1\}.$$

Conversely, given a function $f : S \to \{0, 1\}$, there is a set $A \subset S$ such that $f = \chi_A$. Therefore,

$$\{\text{number of subsets of S}\} = \{\text{number of functions } f : S \to \{0, 1\}\}.$$

To define a function, we must assign either 0 or 1 to each member of S, so there are 2^n choices for such a function. □

Note: This is an example of *double-counting*, where you count the same number in two different ways. This is a useful technique in combinatorics.

Example 3.12. The number of 2-subsets of the set $\{1, 2, \ldots, n\}$ is

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

For each $i \in \{1, 2, \ldots, n\}$, the number of 2-subsets in which i is the largest integer is $(i - 1)$ [the other integer is chosen from $\{1, 2, \ldots, i - 1\}$]. Therefore,

$$0 + 1 + 2 + \ldots + (n-1) = \frac{n(n-1)}{2}.$$

(End of Day 5)

4. Permutations of Multisets

Definition 4.1. A multiset is a set of the form $\{(a, m(a)) : a \in A\}$ where A is a set and $m : A \to \mathbb{N}_0$ is a function. We will denote a multiset in the form $M = \{3 \cdot a, 1 \cdot b, 4 \cdot c, \infty \cdot d\}$

For each $a \in A$, the number $m(a)$ is called the *repetition number* of a. Note that the total number of objects in a multiset is n if $n = \sum_{a \in A} m(a)$.

Definition 4.2. Given a multiset S and $r \in \mathbb{N}$, an *r-permutation* of S is an ordered arrangement of r objects of S. If r is the total number of objects in S, then an r-permutation is simply called a *permutation*.

Theorem 4.3. *Let S be a multiset with objects of k different types, where each object has an infinite repetition number. Then, the number of r-permutations of S is k^r.*

Example 4.4. A ternary numeral is a number represented in base 3. How many ternary numerals are there with at most four digits?

Solution: We are looking for four permutations of the multiset $\{\infty \cdot 0, \infty \cdot 1, \infty \cdot 2\}$, so this number is

$$3^4 = 81.$$

□

Theorem 4.5. *Let $S = \{n_1 \cdot 1, n_2 \cdot 2, \ldots, n_k \cdot k\}$. If $n = n_1 + n_2 + \ldots + n_k$, then the number of permutations of S is*

$$\frac{n!}{n_1! n_2! \ldots n_k!}.$$

Proof. We have n boxes in which we need to place these numbers. There are $\binom{n}{n_1}$ ways to determine where the 1 goes. Then, there are $(n - n_1)$ places left, and there are $\binom{n-n_1}{n_2}$ ways to determine where 2 goes. Thus proceeding, the total number of permutations is given by

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\ldots\binom{n-n_1-n_2-\ldots-n_{k-1}}{n_k}.$$

By Theorem 3.5, this is

$$\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdots\frac{(n-n_1-n_2-\ldots-n_{k-1})!}{n_k!(n-n_1-n_2-\ldots-n_k)!}$$

Cancellation gives the result we need. □

Example 4.6. Find the number of permutations of the word MISSISSIPPI.

Solution: It is
$$\frac{11!}{1!4!4!2!}$$
$\square$

Example 4.7. Let $S = \{a, b, c, d\}$, and we wish to partition S into two sets, each of size 2. How many ways are there to do this?

Solution: Clearly, we have
$$\{\{a, b\}, \{c, d\}\}, \{\{a, c\}, \{b, d\}\}, \{\{a, d\}, \{b, c\}\}.$$
$\square$

Suppose that the partition represents putting balls into boxes. Now, if the boxes are coloured (Blue and Red), then how many ways are there to partition?

Solution: For each unlabelled partition, there are two options. For instance, for the partition $\{\{a, b\}, \{c, d\}\}$, one has
$$R(\{a, b\}), B(\{c, d\}) \text{ and } R(\{c, d\}), B(\{a, b\}).$$
$\square$

We think of the first case as *unlabelled* partitions, while the second is a *labelled* partition.

Theorem 4.8. *Let $n = n_1 + n_2 + \ldots + n_k$. The number of ways to partition a set of n objects into k labelled boxes in which Box 1 contains n_1 objects, Box 2 contains n_2 objects, ..., Box k contains n_k objects is*
$$\frac{n!}{n_1!n_2!\ldots n_k!}.$$
If the boxes are not labelled and $n_1 = n_2 = \ldots = n_k = \ell$, then the number of partitions is
$$\frac{n!}{k!(\ell!)^k}.$$

Proof. Choose n_1 objects for Box 1. This can be done in $\binom{\{n}{n_1\}}$ ways. Then, choose n_2 objects for Box 2 out of the remaining $(n - n_1)$. This can be done in $\binom{\{n-n_1}{n_2\}}$ ways. Thus proceeding, we get the total number of ways to be
$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\ldots-n_{k-1}}{n_k}$$

which gives is the desired number.

Now if the boxes are not labelled and $n_1 = n_2 = \ldots = n_k = \ell$, then we let T be the set of all labelled partitions, and identity two partitions if one is obtained from the other by permuting the boxes. Since there are k boxes, there are $k!$ elements in each part. Therefore, the number of ways to partition the set is given by

$$\frac{|T|}{k!} = \frac{n!}{k!(\ell!)^k}.$$

□

Example 4.9. Rooks on a chessboard can only attack another piece if they travel in the same row or in the same column. So a set of rooks are said to be *non-attacking* if no two rooks share the same row or the same column.

	Φ						
Φ							
			Φ				
		Φ					
					Φ		
				Φ			
						Φ	
							Φ

4.1. How many possible arrangements are there for eight non-attacking rooks on an 8×8 board?

Solution:

Each square is labelled (i, j) as usual. There are eight rooks, so they must occupy positions of the form

$$(1, j_1), (2, j_2), \ldots (8, j_8).$$

Now the set $\{j_1, j_2, \ldots, j_8\}$ is a permutation of $\{1, 2, \ldots, 8\}$ since no two j_i's can be equal. Therefore, there are precisely 8! ways of arranging the rooks. □

4.2. Now suppose each rook is coloured by a different colour. How many such arrangements are there?

Solution: In the previous step, we have chosen the positions for eight unlabelled rooks. When we label them, then there are a further 8! ways of permuting the colours, so we get

$$(8!)^2$$

such arrangements. □

4.3. What if there are four yellow, three blue and one red rook. How many arrangements are there?

Solution: This is now a multiset $\{1 \cdot R, 3 \cdot B, 4 \cdot Y\}$. The number of permutations of the multiset is

$$\frac{8!}{1!3!4!}.$$

So the number of ways to arrange the rooks is

$$\frac{(8!)^2}{1!3!4!}.$$

□

(End of Day 6)

Theorem 4.10. *There are n rooks of k colours, with n_1 rooks of the first colour, n_2 rooks of the second colour, ..., n_k rooks of the k^{th} colour. The number of ways these can be arranged on an $n \times n$ board so that no rook can attack another is*

$$\frac{(n!)^2}{n_1!n_2!\dots n_k!}.$$

Remark 4.11. If S is a multiset with n elements, we have a formula for the number of n-permutations of S. However, if $r < n$, there is no simple formula for the number of r-permutations of S.

5. Combinations of Multisets

Definition 5.1. Given a multiset and $r \in \mathbb{N}$, an *r-combination* of S is an unordered selection of r elements from S. Equivalently, it is a *submultiset* of cardinality r (an *r-submultiset*).

Example 5.2. Let $S = \{2 \cdot a, 1 \cdot b, 3 \cdot c\}$. The 3-combinations of S are

$$\begin{aligned} \{2 \cdot a, 1 \cdot b\}, \{2 \cdot a, 1 \cdot c\}, \\ \{1 \cdot a, 1 \cdot b, 1 \cdot c\}, \{1 \cdot a, 2 \cdot c\}, \\ \{1 \cdot b, 2 \cdot c\}, \{3 \cdot c\}. \end{aligned}$$

Theorem 5.3. *Let S be a multiset with objects of k types, each with an infinite repetition number. Then, the number of r-permutations of S is*

$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1}.$$

Proof. Let $S = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\}$. An r-combination is a multiset of the form $\{x_1 \cdot a_1, x_2 \cdot a_2, \ldots, x_k \cdot a_k\}$ where $x_i \in \mathbb{N}_0$ are such that $x_1 + x_2 + \ldots + x_k = r$. Conversely, any such sequence $(x_1, x_2, \ldots, x_k)$ gives such a r-combination. Therefore, we wish to count the number of solutions to the equation

$$x_1 + x_2 + \ldots + x_k = r \text{ with } x_i \in \mathbb{N}_0. \tag{II.1}$$

Given such a solution, we may associate an arrangement of two objects $\{A, B\}$ as follows: We place x_1 A's down, then a B, then x_2 A's, then a B and so on. When we finish, we would have placed $x_1 + x_2 + \ldots + x_k = r$ A's, and $(k-1)$ B's. This gives us a *permutation* (an ordered arrangement) of a multiset

$$T = \{r \cdot A, (k-1) \cdot B\}.$$

Hence, to each solution to Equation II.1, we get a permutation of T and vice-versa (check!). Therefore, the number of solutions to Equation II.1 is the number of permutations of T, given by Theorem 4.8, and is

$$\frac{(r+k-1)!}{r!(k-1)!}.$$

□

Example 5.4. What is the number of nondecreasing sequences of length r whose terms are taken from $\{1, 2, \ldots, k\}$?

Solution: Here, $S = \{\infty \cdot 1, \infty \cdot 2, \infty \cdot 3, \ldots, \infty \cdot k\}$, so by the previous theorem, it is

$$\binom{r+k-1}{r}$$

□

Example 5.5. Let $S = \{10 \cdot a, 10 \cdot b, 10 \cdot c, 10 \cdot d\}$ with objects of four types $\{a, b, c, d\}$. What is the number of 10-combinations of S that have the property that each of the four types of objects occurs at least once?

Solution: Since the repetition number of all types is 10 and we are looking for 10-combinations, we may as well assume that

$$S = \{\infty \cdot a, \infty \cdot b, \infty \cdot c, \infty \cdot d\}.$$

As before, we are looking to solve the equation

$$x_1 + x_2 + x_3 + x_4 = 10$$

with $x_i \geq 1$ for all $1 \leq i \leq 4$. Let $y_i := x_i - 1$, so we are looking to solve the equation

$$y_1 + y_2 + y_3 + y_4 = 6, \text{ with } y_i \in \mathbb{N}_0.$$

The number of such solutions is given by

$$\binom{6+4-1}{6} = \binom{9}{6} = 84.$$

□

Example 5.6. What is the number of integral solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 20$$

with the conditions $x_1 \geq 3, x_2 \geq 1, x_3 \geq 0$ and $x_4 \geq 5$?

Solution: Let $y_1 := x_1 - 3, y_2 := x_2 - 1, y_3 = x_3$ and $y_4 = x_4 - 5$. Then, we are looking to solve the equation

$$y_1 + y_2 + y_3 + y_4 = 11, \text{ with } y_i \in \mathbb{N}_0.$$

So the number of solutions is

$$\binom{10+4-1}{4}.$$

□

Remark 5.7. Given a multiset $S = \{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$ with k types of objects with repetition numbers $n_1, n_2, \ldots, n_k$ respectively. The number of r-combinations of S is the number of integral solutions to the equation

$$x_1 + x_2 + \ldots + x_k = r, \text{ with } 0 \leq x_i \leq n_i \text{ for all } 1 \leq i \leq k.$$

Here, we have upper bounds for the x_i. This situation cannot be handled yet - we will tackle it later.

(End of Day 7)

6. Finite Probability

An experiment $\mathcal{E}$ is conducted, which has finitely many possible outcomes and each outcome is equally likely. We say that the experiment is conducted randomly. The set of all possibly outcomes is called the sample space, denoted by S. So,

$$S = \{s_1, s_2, \ldots, s_n\}$$

and

$$P(s_i) = \frac{1}{n} \text{ for all } 1 \leq i \leq n.$$

An event is a subset $E \subset S$.

Example 6.1. Consider the experiment $\mathcal{E}$ of tossing three coins, each coin shows either H or T. Then,

$$S = \{(H,H,H),(H,H,T),(H,T,H),(T,H,H),(H,T,T),(T,H,T),(T,T,H),(T,T,T)\}.$$

If E is the event that at least two coins come up H, then

$$E = \{(H,H,H),(H,H,T),(H,T,H),(T,H,H)\}$$

Therefore, $P(E) = |E|/|S| = 1/2$.

Example 6.2. Consider a deck of cards (52 cards with 13 cards per suit (C,D,H,S)). Let $\mathcal{E}$ be the experiment of drawing a card at random. Then,

$$S = \{\{all52cards\}$$

If E is the event that the card drawn is a five, then $E = \{(C,5),(D,5),(H,5),(S,5)\}$, so that

$$P(E) = \frac{4}{52}.$$

Example 6.3. Let $n \in \mathbb{N}$ fixed. We choose n integers $(i_1, i_2, \dots, i_n)$ with each integer from the set $\{1, 2, \dots, n\}$.

6.1. What is the probability that the chosen sequence is a permutation of $(1, 2, \dots, n)$?

Solution: Note that S is the set of all possible such tuples, so

$$|S| = n^n.$$

Also, E is the set of all tuples that are permutations of the tuple $(1, 2, \dots, n)$. So,

$$|E| = n!.$$

Hence, $P(E) = n!/n^n$. □

6.2. What is the probability that the sequence contains exactly $(n-1)$ different integers?

Solution: Again, $|S| = n^n$. Now, F is the event that the tuple contains exactly one repeated integer. There are n ways to choose the repeated integer, and $(n-1)$ ways to choose the missing integer. Then, there are $\binom{n}{2}$ ways to place the repeated integer and $(n-2)!$ ways to choose the other integers. Therefore,

$$|F| = n(n-1)\binom{n}{2}(n-2)! = \frac{(n!)^2}{2(n-2)!}.$$

Hence, $P(E) = |F|/|S|$. $\square$

Example 6.4. Five identical rooks are placed at random in non-attacking positions on an 8×8 board. What is the probability that the rooks are places in rows $\{1, 2, \ldots, 5\}$ and columns $\{4, 5, \ldots, 8\}$?

Solution: Note that S consists of all placements of five non-attacking rooks on the board, so

$$|S| = \binom{8}{5}^2 \times 5!.$$

Also, E is the event that the five rooks are placed in the prescibed rows and columns. Hence, $|E| = 5!$, so

$$P(E) = \frac{1}{\binom{8}{5}^2}.$$

$\square$

Example 6.5. Poker is a game where each player gets a hand with 5 cards out of the usual 52-card deck. Our experiment is to select a hand at random and examine what kind of cards it has. Hence, the sample space S is the set of all poker cards, so

$$|S| = \binom{52}{5}.$$

6.1. What is the probability that a poker hand contains at least one ace.

Solution: If E is the event that a hand contains at least one ace, then E^c is the event that it contains no aces. Hence,

$$|E^c| = \binom{48}{5}.$$

By the subtraction principle,

$$|E| = \binom{52}{5} - \binom{48}{5}$$

and $P(E) = |E|/|S|$. $\square$

Note that by the subtraction principle, for any event E,

$$P(E^c) = 1 - P(E).$$

6.2. What is the probability that a poker hand contains exactly two pairs (two cards of one rank, and two cards of a different rank, and one additional card of a different rank)?

Solution: If E is the given event, then we can compute $|E|$ by the multiplication principle. We have three tasks:

- Choose the two ranks occurring in the two pairs. This can be done in $\binom{13}{2}$ ways.
- Choose the two suits for each of the two ranks. This can be done in $\binom{4}{2} \times \binom{4}{2}$ ways.
- Choose the remaining card. This can be done in 44 ways.

Hence,

$$|E| = \binom{13}{2} \times \binom{4}{2}^2 \times 44$$

and $P(E) = |E|/|S|$. $\square$

(End of Day 8)

III. The Pigeonhole Principle

1. Simple Form

Theorem 1.1. *If $n+1$ objects are distributed into n boxes, then at least one box contains two or more of the objects.*

Proof. By contradiction. If each box contains ≤ 1 object, then there would be $\leq n$ elements in total. $\square$

Remark 1.2. The pigeonhole principle does not tell us which box has more than one element, just that there is one such box. Hence, it can be used to prove *existence* of a certain arrangement, but may not be able to help construct such an arrangement.

Remark 1.3. Equivalent statements:

1.1. If $n+1$ colours are used to colour n objects, then at least two objects must share a colour.

1.2. If n objects are put into n boxes and no box is empty, then each box must contain exactly one object.

1.3. If n objects are put into n boxes and no box has more than object, then each box must contain exactly one object.

1.4. Let $f : X \to Y$ be a function between two sets.

   (i) If X has more elements than Y, then f is not injective.

   (ii) If $|X| = |Y|$ and f is onto, then f is one-to-one.

   (iii) If $|X| = |Y|$ and f is injective, then f is surjective.

Example 1.4. If there are 13 people in a room, at least two must have their birthday in the same month.

Proposition 1.5. *Given m integers $\{a_1, a_2, \ldots, a_m\}$, there exist integers $0 \leq k < l \leq m$ such that the sum $(a_{k+1} + a_{k+2} + \ldots + a_l)$ is divisible by m.*

Proof. Consider the m sums

$$s_1 := a_1, s_2 := a_1 + a_2, s_3 := a_1 + a_2 + a_3, \ldots, s_m := a_1 + a_2 + \ldots + a_m.$$

If any of the s_i are divisible by m, then we are done. If not, then each s_i leaves a non-zero remainder r_i when divided by m. Since $r_i \in \{1, 2, \ldots, m-1\}$ for all $1 \leq i \leq m$, it follows by the pigeonhole principle that there exist integers $k < \ell$ such that

$$r = r_k = r_\ell.$$

In other words, there exist $b, c \in \mathbb{N}_0$ such that

$$s_k = bm + r \text{ and } s_\ell = cm + r.$$

Subtracting, we get $(a_{k+1} + a_{k+2} + \ldots + a_\ell) = s_\ell - s_k$ is divisible by m. □

Example 1.6. Let $m = 5$ and consider the integers $2, 6, 8, 11, 12$. Then,

$$s_1 = 2, s_2 = 8, s_3 = 16, s_4 = 27, s_5 = 39.$$

None of the s_i are divisible by 5, so we divide and check the remainders

$$r_1 = 2, r_2 = 3, r_3 = 1, r_4 = 2, r_5 = 4.$$

Hence, $r_1 = r_4 = 2$, so that $a_2 + a_3 + a_4 = 6 + 8 + 11 = 25$ is divisible by 5.

Example 1.7. From the integers $1, 2, \ldots, 200$, we choose 101 integers. Show that among the chosen integers, there are two such that one of them is divisible by the other.

Solution: Write each number in the form $2^k \times a$ where $2 \nmid a$. Then, $a \in \{1, 3, 5, 7, \ldots, 199\}$. Hence, among the 101 numbers, there exist two such whose 'remainder' a value is the same, say $x = 2^k \times a$ and $y = 2^\ell \times a$. If $k < \ell$, then $x \mid y$ and if $\ell < k$, then $y \mid x$. □

Can you do this if you choose 100 numbers?

Solution: No. You can choose $\{101, 102, \ldots, 200\}$ none of which divide each other. □

Definition 1.8. Two positive integers m and n are said to be *relatively prime* if $\gcd(m, n) = 1$.

Lemma 1.9 (Euclid's Lemma)**.** *If m and n are relatively prime and $n \mid mk$, then $n \mid k$.*

Theorem 1.10 (Chinese Remainder Theorem)**.** *Let $m, n \in \mathbb{N}$ be two relative prime numbers, and choose $0 \leq a \leq m - 1$ and $0 \leq b \leq n - 1$. Then, there is a positive integer x such that*

$$x = pm + a \text{ and } x = qn + b.$$

In other words, the remainder when x is divided by m is a and when divided by n it is b.

Proof. Consider the n integers

$$a, m + a, 2m + a, \ldots, (n - 1)m + a.$$

Note that each of these numbers has remainder a when divided by m. Suppose that two of them had the same remainder r when divided by n. Then, there exist $0 \leq i < j \leq n-1$ such that

$$im + a \text{ and } jm + a$$

have the same remainder when divided by n. Then,

$$(jm + a) - (im + a) = nq = (j - i)m.$$

By Euclid's lemma, $n \mid (i - j)$. But $0 \leq i < j \leq n-1$, so this is impossible.

Therefore, our assumption is false, and the n numbers

$$a, m + a, \ldots, (n-1)m + a$$

have different remainders when divided by n. There are n numbers and the remainders are in $\{0, 1, \ldots, n-1\}$, so by the pigeonhole principle, each remainder must occur once. In particular, b must occur once. Hence, there exists $0 \leq i \leq n-1$ and $q \in \mathbb{N}$ such that

$$im + a = qm + b.$$

So $x := qm + b$ works. □

(End of Day 9)

2. Strong Form

Theorem 2.1. *Let $q_1, q_2, \ldots, q_n$ be positive integers. If*

$$q_1 + q_2 + \ldots + q_n - n + 1$$

objects are distributed into n boxes, then there exists $1 \leq i \leq n$ such that the i^{th} box contains at least q_i objects.

Proof. Suppose not, then for each $1 \leq i \leq n$, the i^{th} box contains atmost $(q_i - 1)$ objects. Then, the total number of objects would be atmost

$$(q_1 - 1) + (q_2 - 1) + \ldots + (q_n - 1) = q_1 + q_2 + \ldots + q_n - n.$$

This is a contradiction. □

Remark 2.2. Notice that it is possible to distribute $(q_1 + q_2 + \ldots + q_n - n)$ objects into n boxes in such a way that each box contains exactly $(q_i - 1)$ elements.

Remark 2.3. The simple form of the pigeonhole principle follows from Theorem 2.1: Take $q_1 = q_2 = \ldots = q_n = 2$ in the previous theorem, so that

$$q_1 + q_2 + \ldots + q_n - n + 1 = n + 1.$$

Corollary 2.4. *Let n, r be positive integers. If $n(r-1)+1$ objects are distributed into n boxes, then at least one box contains r or more objects.*

Proof. Take $q_1 = q_2 = \ldots = q_n = r$ in Theorem 2.1. $\square$

Proposition 2.5 (Averaging Principles)**.**

2.1. *If the average of n positive integers is greater than $(r-1)$, then at least one integer is greater than or equal to r.*

2.2. *If the average of n positive integers is less than $(r+1)$, then at least one integer is less than or equal to r.*

Proof. We prove only part (1) as part (2) is similar.

Suppose $m_1, m_2, \ldots, m_n$ are positive integers such that

$$\frac{m_1 + m_2 + \ldots + m_n}{n} > (r-1).$$

If $m_i \leq (r-1)$ for all $1 \leq i \leq n$, then

$$\frac{m_1 + m_2 + \ldots + m_n}{n} \leq \frac{n(r-1)}{n} = (r-1).$$

This is a contradiction. $\square$

Example 2.6. A basket of fruit is being arranged out of apples, bananas and oranges. What is the smallest number of pieces of fruit that should be put in the basket to ensure that there are at least 8 apples, or 6 bananas, or 9 oranges?

Solution: By Theorem 2.1, the number of pieces of fruit is

$$8 + 6 + 9 - 3 + 1 = 21.$$

$\square$

Definition 2.7. Given a sequence $a_1, a_2, \ldots, a_k$ of integers, a <u>subsequence</u> is a sequence of the form $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$ where $1 \leq i_1 < i_2 < \ldots < i_m \leq k$. Moreover, m is called the <u>length</u> of the subsequence.

Proposition 2.8 (Erdös, Szekeres)**.** *Show that every sequence of (n^2+1) real numbers contains an increasing subsequence of length $(n+1)$ or a decreasing subsequence of length $(n+1)$.*

Proof. Label the sequence $a_1, a_2, \ldots, a_{n^2+1}$. Suppose there is no increasing subsequence of length $(n+1)$. For each $k \in \mathbb{N}$, let m_k denote the length of the longest increasing subsequence beginning at a_k. Then, by hypothesis,

$$1 \leq m_k \leq n$$

for all $1 \leq k \leq n^2+1$. Hence, we have (n^2+1) integers $S := \{m_1, m_2, \ldots, m_{n^2+1}\}$ which all lie in the set $\{1, 2, \ldots, n\}$. Taking $r = n+1$ in Corollary 2.4, we see that there must exist $i \in \{1, 2, \ldots, n\}$ such that $(n+1)$ integers of S are equal to i. Hence, there exists $1 \leq k_1 < k_2 < \ldots < k_{n+1} \leq n^2+1$ such that

$$m_{k_1} = m_{k_2} = \ldots = m_{k_{n+1}}.$$

We claim that $a_{k_1} \geq a_{k_2} \geq \ldots \geq a_{k_{n+1}}$ (which would be the decreasing sequence we are looking for). Suppose not, then there exists $1 \leq j \leq n+1$ such that

$$a_{k_j} < a_{k_{j+1}}.$$

By hypothesis, there is an increasing subsequence of length $m_{k_{j+1}}$ starting at $a_{k_{j+1}}$. Putting a_{k_j} in front, we get an increasing subsequence of length $(m_{k_{j+1}}+1)$ starting at a_{k_j}. By construction,

$$m_{k_j} \geq m_{k_{j+1}} + 1.$$

This contradiction proves that $a_{k_j} \geq a_{k_{j+1}}$ for all $1 \leq i \leq n+1$. $\square$

Example 2.9. Suppose (n^2+1) people are lined up shoulder-to-shoulder, then it is possible to choose $(n+1)$ people from them to step forward so that, going from left to right, either their heights are increasing or they are decreasing.

3. A Theorem of Ramsey

Proposition 3.1. *Given 6 people in a room, either there are three who are mutually acquainted, or there are three who are not.*

We state this symbolically by

$$K_6 \to K_3, K_3.$$

Definition 3.2. For $n \in \mathbb{N}$, K_n is a set of n objects and all pairs of these n objects.

It can be represented pictorially by points on a page (the objects) and edges connecting those points. Note that K_3 is a triangle.

Now, given the graph K_6 representing the six people, we colour an edge blue if the two people are not acquainted, and red if they are. Hence, the assertion of Proposition 3.1 is that K_6 contains either a red K_3 or a blue K_3.

(End of Day 10)

Proof of Proposition 3.1. Suppose the edges of K_6 have been coloured red and blue in some way. Fix a vertex p in K_6. There are 5 edges starting at p. By Theorem 2.1, there are at least three edges that share a colour. Suppose that three of them are red, and suppose the edges end at a, b and c. Consider the edges connecting a, b and c. If all of them are blue, then we have found a blue K_3. If not, then one edge is red. Suppose the edge connecting a and b is red. Then, the triangle (p, a, b) is red and we are done. □

Remark 3.3. The statement $K_5 \to K_3, K_3$ is false: One can colour K_5 red and blue in such a way that no K_3 is either red or blue.

Theorem 3.4 (Ramsey's Theorem)**.** *Given positive integers $m \geq 2$ and $n \geq 2$, there is $p \in \mathbb{N}$ such that*

$$K_p \to K_m, K_n.$$

In other words, if the edges of K_p are coloured red and blue, then K_p contains either a red K_m or a blue K_n. Notice that if $K_p \to K_m, K_n$, then $K_q \to K_m, K_n$ for all $q \geq p$.

Definition 3.5. Given $m, n \geq 2$, the Ramsey number $r(m, n)$ is the least integer p such that $K_p \to K_m, K_n$.

Example 3.6.

3.1. By symmetry, $r(m, n) = r(n, m)$.

3.2. By our earlier arguments, $K_6 \to K_3, K_3$ but $K_5 \not\to K_3, K_3$. Hence,

$$r(3, 3) = 6.$$

3.3. We claim that $r(2, n) = n$ (These are called the trivial Ramsey numbers).

Proof.

(i) We check that $r(2, n) \leq n$. In other words, we need to show that $K_n \to K_2, K_n$: Notice that if K_n is coloured red and blue, then either some edge is red, or some edge is blue. Hence, $r(2, n) \leq n$. Also, if we colour all the edges

(ii) We check that $r(2, n) > n$. In other words, we need to show that $K_{n-1} \not\to K_2, K_n$: Suppose that K_{n-1} is coloured entirely blue. Then, we do not have a red K_2 nor a blue K_n.

□

Proof of Theorem 3.4. We prove this by double induction on (m, n) with $m \geq 2, n \geq 2$. If $m = 2$, then

$$r(m, n) = r(2, n) = n.$$

If $n = 2$, then

$$r(m, 2) = r(2, m) = m.$$

Now suppose $m \geq 3$ and $n \geq 3$, then by the induction hypothesis, both $r(m-1, n)$ and $r(m, n-1)$ exist. Let

$$p := r(m-1, n) + r(m, n-1).$$

We claim that $K_p \mapsto K_m, K_n$: Suppose the edges of K_p are coloured red and blue. Fix one vertex x of K_p. Let R_x be the edges of K_p joined to x that are red, and let B_x be the edges that are joined to x and are blue. Then,

$$|R_x| + |B_x| = p - 1 = r(m-1, n) + r(m, n-1) - 1.$$

Hence, either $|R_x| \geq r(m-1, n)$ or $|B_x| \geq r(m, n-1)$.

3.1. Suppose that $q := |R_x| \geq r(m-1, n)$. Then, consider K_q with the points on R_x. Then,

$$K_q \mapsto K_{m-1}, K_n.$$

In other words, inside K_q there is either a red K_{m-1} or a blue K_n.

- If there is a blue K_n, then we are done.
- If there is a red K_{m-1}. In that case, since R_x consists of red edges, we may add x to the red K_{m-1} and get a red K_m in K_p.

In either case, we have $K_p \to K_m, K_n$.

3.2. Suppose that $q := |B_x| \geq r(m, n-1)$, then the same argument holds.

$\square$

Remark 3.7. Notice that in our proof of Theorem 3.4, we have also proved that whenever $m, n \geq 3$, then

$$r(m, n) \leq r(m-1, n) + r(m, n-1).$$

So if we let

$$f(m, n) := \binom{m+n-2}{m-1}$$

with $m, n \geq 2$, then by Pascal's formula (Theorem 3.9)

$$f(m, n) = f(m, n-1) + f(m-1, n)$$

whenever $m, n \geq 3$. Since

$$r(2, n) = n = f(2, n) \text{ and } r(m, 2) = m = f(m, 2).$$

Therefore, it follows by induction that

$$r(m, n) \leq f(m, n) = \binom{m+n-2}{m-1} = \binom{m+n-2}{n-1}$$

for all $m, n \geq 2$.

(End of Day 11)

Example 3.8. Some known facts about Ramsey numbers:

3.1. $r(3,3) = 6$.

3.2. $r(3,4) = r(4,3) = 9$. In other words, $K_9 \to K_3, K_4$ but $K_8 \nrightarrow K_3, K_4$. In other words, there is a colouring of K_8 in red and blue such that there is no red K_3 nor a blue K_4.

3.3. $40 \leq r(3,10) \leq 43$. In other words, $K_43 \to K_3, K_{10}$. However, there is a colouring of K_{39} in red and blue such that there is no red K_3 or a blue K_{10}. However, neither of these conclusions is known for K_{40}, K_{41} or K_{42}.

Remark 3.9.

3.1. Ramsey's theorem generalizes to any number of colours. If $n_1, n_2, n_3 \geq 2$, then there exists $p \in \mathbb{N}$ such that

$$K_p \to K_{n_1}, K_{n_2}, K_{n_3}.$$

In other words, if the edges of K_p are coloured red, blue and green in any way, then there is a red K_{n_1}, a blue K_{n_2} or a green K_{n_3}. The least p for which this holds is denoted $r(n_1, n_2, n_3)$. It is known that

$$r(3,3,3) = 17.$$

Similarly, one may define $r(n_1, n_2, \ldots, n_k)$ as well, and a Ramsey's theorem holds in a full generality: Given integers $n_1, n_2, \ldots, n_k \geq 2$, there exists $p \in \mathbb{N}$ such that

$$K_p \to K_{n_1}, K_{n_2}, \ldots, K_{n_k}.$$

3.2. Given $t \in \mathbb{N}$ and $S = \{1, 2, \ldots, p\}$, let K_p^t denote the set of all subsets S with exactly t elements. Note that if $t = 2$, then we recover the old definition of K_p. We now have a general form of Ramsey's theorem: Given integers $t \geq 2$ and $q_1, q_2, \ldots, q_k \geq 2$, there is an integer $p \in \mathbb{N}$ such that

$$K_p^t \to K_{q_1}^t, K_{q_2}^t, \ldots, K_{q_k}^t.$$

In other words, if each t element subset of K_p is assigned one of k colours $C_1, C_2, \ldots, C_k$, then there are q_1 elements all of whose t-subsets are of colour C_1, or q_2 elements all of whose t subsets are of colour C_2, ... or there are q_k elements, all of whose t-subsets are of colour C_k.

3.3. The smallest such integer is called the Ramsey number

$$r_t(q_1, q_2, \ldots, q_k).$$

3.4. If $t = 1$, then $r_1(q_1, q_2, \dots, q_k)$ is the smallest number p such that if there are p elements colours in one of k colours $C_1, C_2, \dots, C_k$, then there are at least q_1 elements of colours C_1 or q_2 elements of colour $C_2, \dots$ or q_k elements of colour C_k. By Theorem 2.1,

$$r_1(q_1, q_2, \dots, q_k) = q_1 + q_2 + \dots + q_k - k + 1.$$

Hence, Ramsey's theorem is a generalization of Theorem 2.1.

3.5. It is difficult to compute $r_t(\cdot)$ in general. However, it is clear that

$$r_t(\cdot)$$

does not depend on the order of the numbers $(q_1, q_2, \dots, q_k)$.

IV. The Binomial Coefficients

1. Pascal's Triangle

Recall the following from Definition 3.3.

Definition 1.1. For all $n, k \geq 0$,

$$\binom{n}{k} = \text{the number of } k\text{-subsets of an } n\text{-element set.}$$

Remark 1.2.

1.1. $\binom{n}{k} = 0$ if $k > n$.

1.2. $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ for all $n \geq 0$.

1.3. If $n \geq 1$ and $1 \leq k \leq n$, then

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

1.4. If $0 \leq k \leq n$, then

$$\binom{n}{k} = \binom{n}{n-k}.$$

1.5. Pascal's formula:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

1.6.

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$$

1.7. Using properties (2) and (5), one can compute $\binom{n}{k}$ for all values of $n, k \geq 0$.

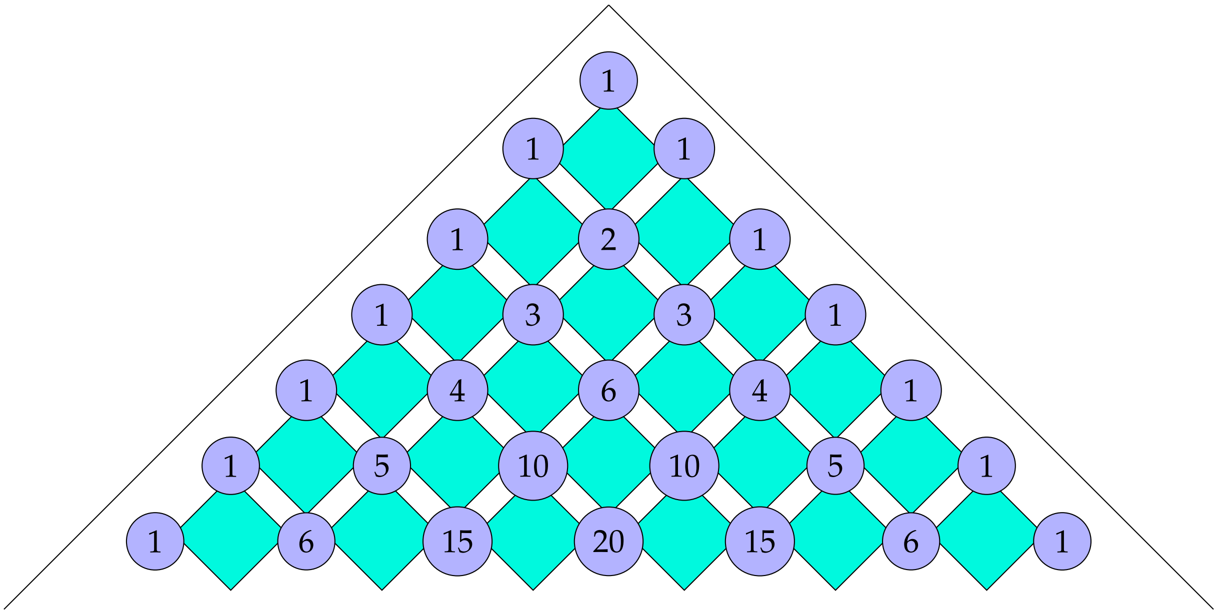
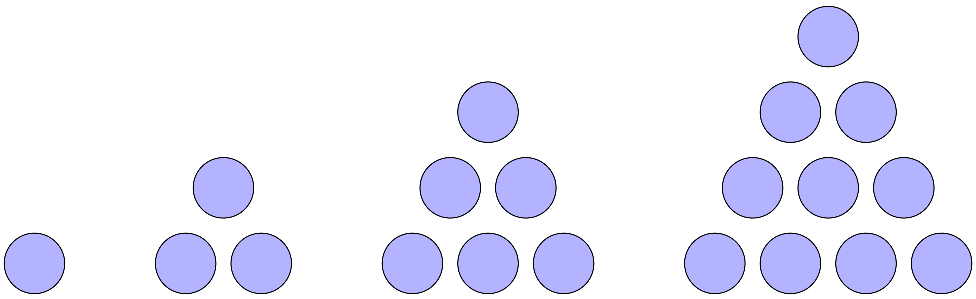


Figure IV.1.: Pascal's Triangle

1.8. Notice that

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

is equal to the number of dots in the triangles shown below: Therefore, these



numbers are called triangular numbers.

1.9. Similarly, the numbers $\binom{n}{3}$ for $n \geq 0$ are called tetrahedral numbers, as they equal the number of dots in a tetrahedral array (think of stacked balls).

1.10. By the discussion in section 1, $\binom{n}{k}$ represents the number of paths from the top of the triangle to that entry.

2. Binomial Theorem

Theorem 2.1 (Binomial Theorem)**.** *Let $n \in \mathbb{N}$. For all $x, y \in \mathbb{R}$,*

$$\begin{aligned}(x+y)^n &= x^n + \binom{n}{1}x^{n-1}y^1 + \ldots + \binom{n}{k}x^{n-k}y^k + \ldots + y^n \\ &= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^y.\end{aligned}$$

Proof. Write

$$(x+y)^n = \underbrace{(x+y)\cdot(x+y)\cdot\ldots\cdot(x+y)}_{n\text{ times}}$$

Expand this product using the distributive law and combine like terms. There are 2^n terms in the sum, each of the form $x^k y^{n-k}$ for some $0 \le k \le n$. We obtain the term $x^k y^{n-k}$ by choosing y in k of the n factors and x in $(n-k)$ factors. Thus, the number of times $x^{n-k}y^k$ appears is $\binom{n}{k}$. Therefore,

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.$$

$\square$

Proof. Alternate proof by induction: If $n = 1$, the formula becomes

$$(x+y)^1 = \binom{1}{0}x^1y^0 + \binom{1}{1}x^0y^1 = x + y.$$

Now assume that the formula is true for a positive integer n, and we prove it for $(n+1)$. Write

$$\begin{aligned}(x+y)^{n+1} &= (x+y)\cdot(x+y)^n \\ &= (x+y)\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\right) \\ &= x\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\right) + y\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\right) \\ &= \left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k+1}y^k\right) + \left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k+1}\right) \\ &= \binom{n}{0}x^{n+1} + \sum_{k=1}^{n}\binom{n}{k}x^{n+1-k}y^k + \sum_{k=0}^{n-1}x^{n-k}y^{k+1} + \binom{n}{n}y^{n+1}.\end{aligned}$$

Now note that

$$\sum_{k=0}^{n-1}\binom{n}{k}x^{n-k}y^{k+1} = \sum_{r=1}^{n}\binom{n}{r-1}x^{n+1-r}y^r.$$

Hence, by Pascal's formula,

$$\begin{aligned}(x+y)^{n+1} &= x^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] x^{n+1-k} y^k + y^{n+1} \\ &= x^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^{n+1-k} y^k + y^{n+1} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k\end{aligned}$$

□

Example 2.2. If $n = 2$, then

$$(x+y)^2 = x^2 + 2xy + y^2.$$

If $n = 3$, then

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

And thus, the coefficients appearing in the expansion of $(x+y)^n$ are the numbers in the n^{th} of Pascal's triangle.

Corollary 2.3. *For any $n \in \mathbb{N}$ and $x \in \mathbb{R}$,*

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Proposition 2.4. *For any $n, k \geq 1$,*

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Proof. Write

$$\begin{aligned}k\binom{n}{k} &= k\frac{n!}{k!(n-k)!} \\ &= k\frac{n(n-1)\dots(n-k+1)}{k(k-1)\dots 1} \\ &= n\frac{(n-1)(n-2)\dots(n-1-(k-1)+1)}{(k-1)(k-2)\dots 1} \\ &= n\binom{n-1}{k-1}.\end{aligned}$$

□

Proposition 2.5. *For any $n \geq 0$,*

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

Moreover, if $n \geq 1$, then

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = 2^{n-1} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots.$$

Proof. The first statement follows from Corollary 2.3 taking $x = 1$. For the second statement, take $x = 1$ and $y = -1$ in the Binomial Theorem and see that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^n \binom{n}{n} = 0.$$

Hence,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots.$$

and both sides must then by equal to $1/2 \times 2^n = 2^{n-1}$. □

Remark 2.6.

2.1. Let $S = \{x_1, x_2, \ldots, x_n\}$ be set of n elements. A subset of S is obtained by the following process:

- Decide whether x_1 belongs to the subset or not (two choices)
- Decide whether x_2 belongs to the subset or not (two choices)
- ...
- Decide whether x_n belongs to the subset or not (two choices).

Since n decisions are to be made, there are 2^n subsets of S.

2.2. Now suppose we want a subset with an even number of elements. Then, as before, we have two choices for each of $x_1, x_2, \ldots, x_{n-1}$. However, for x_n there is only one choice depending on whether the subset has (up until then) an even or an odd number of elements. Therefore, there are $(n-1)$ decisions to be made, so

The number of subsets of S with an even number of elements $= 2^{n-1}$.

This agrees with

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = 2^{n-1}.$$

The other case (of sets with odd cardinality) is similar.

(End of Day 12)

Proposition 2.7. *For $n \geq 1$,*

$$1\binom{n}{1} + 2\binom{n}{2} + \ldots + n\binom{n}{n} = n2^{n-1}.$$

Proof. By Proposition 2.4,

$$\begin{aligned}
1\binom{n}{1} + 2\binom{n}{2} + \ldots + n\binom{n}{n} &= n\binom{n-1}{0} + n\binom{n-1}{1} + \ldots + n\binom{n-1}{n-1} \\
&= n\left(\sum_{k=0}^{n-1}\binom{n-1}{k}\right) \\
&= n2^{n-1}
\end{aligned}$$

by Proposition 2.5. □

Alternate Proof. Consider the polynomial

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \ldots + \binom{n}{n}x^n.$$

Differentiating both sides with respect to x gives

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \ldots + n\binom{n}{n}x^{n-1}.$$

Now set $x = 1$ to get the result. □

Remark 2.8.

2.1. The alternate proof gives us a way to produce more such identities. Multiplying $n(1+x)^{n-1}$ by x, we get

$$nx(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}x^k.$$

Differentiating with respect to x gives

$$n(1+x)^{n-1} + n(n-1)x(1+x)^{n-2} = \sum_{k=1}^{n} k^2\binom{n}{k}x^{k-1}.$$

Substituting $x = 1$ gives

$$n2^{n-1} + n(n-1)2^{n-2} = \sum_{k=1}^{n} k^2\binom{n}{k}$$

which holds for all $n \geq 1$.

2.2. Similarly, one can get identities for

$$\sum_{k=1}^{n} k^p \binom{n}{k}$$

for each $p \geq 1$. For large values of p, this gets quite complicated though.

Proposition 2.9. *For $n \geq 0$,*

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Proof. Let $S = \{x_1, x_2, \ldots, x_{2n}\}$ and set

$$A := \{x_1, x_2, \ldots, x_n\}, B := \{x_{n+1}, x_{n+2}, \ldots, x_{2n}\}.$$

Let T be the set of all n-subsets of S, then

$$|T| = \binom{2n}{n}.$$

For each $0 \leq k \leq n$, let C_k denote the set of all n-subsets of S obtained by taking k elements from A and $(n-k)$ elements from B. Then,

$$|T| = |C_0| + |C_1| + \ldots + |C_n|$$

by the addition principle. Moreover,

$$|C_k| = \binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2.$$

This proves the identity. □

Definition 2.10. If $r \in \mathbb{R}$ and $k \in \mathbb{Z}$, define

$$\binom{r}{k} = \begin{cases} \frac{r(r-1)\ldots(r-k+1)}{k!} & : k \geq 1 \\ 1 & : k = 0 \\ 0 & : k \leq -1. \end{cases}$$

Remark 2.11.

2.1.

$$\binom{5/2}{4} = \frac{(5/2)(3/2)(1/2)(-1/2)}{4!} = \frac{-5}{128}.$$

$$\binom{-8}{2} = \frac{(-8)(-9)}{2} = 36.$$

$$\binom{3.2}{0} = 1 \text{ and } \binom{3}{-2} = 0.$$

2.2. Note that Pascal's formula holds

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}.$$

and so does Proposition 2.4

$$k\binom{r}{k} = r\binom{r-1}{k-1}.$$

2.3. Repeatedly applying Pascal's formula gives

$$\begin{aligned}\binom{r}{k} &= \binom{r-1}{k} + \binom{r-1}{k-1} \\ &= \binom{r-1}{k} + \binom{r-2}{k-1} + \binom{r-2}{k-2} \\ &= \binom{r-1}{k} + \binom{r-2}{k-1} + \binom{r-3}{k-2} + \binom{r-3}{k-3} \\ &= \ldots \\ &= \binom{r-1}{k} + \binom{r-2}{k-1} + \ldots + \binom{r-k-1}{0} + \binom{r-k-1}{-1}.\end{aligned}$$

The last term is 0. Replace r by $r+k+1$ and we get

$$\binom{r+k+1}{k} = \binom{r}{0} + \binom{r+1}{1} + \ldots + \binom{r+k}{k}.$$

This holds for all $r \in \mathbb{R}$ and $k \in \mathbb{Z}$.

2.4. Now let us apply Pascal's formula to the first term. Suppose $n, k \in \mathbb{N}$, and note

$$\begin{aligned}\binom{n}{k} &= \binom{n-1}{k} + \binom{n-1}{k-1} \\ &= \binom{n-2}{k} + \binom{n-2}{k-1} + \binom{n-1}{k-1} \\ &= \ldots \\ &= \binom{0}{k} + \binom{0}{k-1} + \binom{1}{k-1} + \ldots + \binom{n-2}{k-1} + \binom{n-1}{k-1}.\end{aligned}$$

Now the first term is zero. Replace n by $n+1$ and k by $k+1$ and we obtain

$$\binom{n+1}{k+1} = \binom{0}{k} + \binom{1}{k} + \ldots + \binom{n-1}{k} + \binom{n}{k}.$$

Note that the first non-zero term is $\binom{k}{k} = 1$.

2.5. In particular, if $k = 1$, then we get

$$\binom{n+1}{2} = 1 + 2 + \ldots + (n-1) + n = \frac{(n+1)n}{2}.$$

3. Unimodality of Binomial Coefficients

Definition 3.1. A sequence of numbers $\{s_0, s_1, s_2, \ldots, s_n\}$ is <u>unimodal</u> if there exists $0 \le t \le n$ such that

$$s_0 \le s_1 \le \ldots \le s_t, \text{ and } s_t \ge s_{t+1} \ge \ldots \ge s_n.$$

Theorem 3.2. *For $n \in \mathbb{N}$, the sequence*

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$$

is unimodal. Specifically, if n is even,

$$\binom{n}{0} < \binom{n}{1} < \ldots < \binom{n}{n/2},$$

$$\binom{n}{n/2} > \ldots > \binom{n}{n-1} > \binom{n}{n}.$$

If n is odd, then

$$\binom{n}{0} < \binom{n}{1} < \ldots < \binom{n}{(n+1)/2}$$

$$\binom{n}{(n+1)/2} > \ldots > \binom{n}{n-1} > \binom{n}{n}.$$

Proof. If $1 \le k \le n$,

$$\begin{aligned}\frac{\binom{n}{k}}{\binom{n}{k-1}} &= \frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k-1)!(n-k+1)!}} \\ &= \frac{n-k+1}{k}.\end{aligned}$$

Hence,

$$\binom{n}{k} > \binom{n}{k-1}$$

if and only if

$$n - k + 1 > k \Leftrightarrow n + 1 > 2k \Leftrightarrow \frac{n+1}{2} > k.$$

If n is even, this is equivalent to $\frac{n}{2} \ge k$. If n is odd, then it is equivalent to $\frac{n-1}{2} \ge k$. Hence the result.

Notice that $\binom{n}{k} = \binom{n}{k-1}$ if and only if

$$\frac{n+1}{2} = k.$$

This cannot happen if n is even, and happens at precisely one value of k when n is odd. Thus, when n is even, no two binomial coefficients are equal, while two are equal if n is odd, namely

$$\binom{n}{\frac{n+1}{2}} = \binom{n}{\frac{n-1}{2}}.$$

□

Definition 3.3. For $x \in \mathbb{R}$, the floor of x is defined as

$$\lfloor x \rfloor = \max\{n \in \mathbb{N} : n \leq x\}$$

and the ceiling of x is defined as

$$\lceil x \rceil = \min\{n \in \mathbb{N} : n \geq x\}.$$

Example 3.4. If $x \in \mathbb{N}$, then $\lceil x \rceil = \lfloor x \rfloor = x$. Moreover,

$$\lfloor 2.5 \rfloor = 2 \text{ and } \lceil 2.5 \rceil = 3.$$

Corollary 3.5. *If $n \in \mathbb{N}$, the largest of the binomial coefficients*

$$\binom{n}{0}, \binom{n}{1}, \dots \binom{n}{n}$$

is

$$\binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}.$$

a. Sperner's Theorem

Definition 3.6. Let S be a set. An antichain of S is a collection $\mathcal{A}$ of subsets of S such that no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$.

Example 3.7.

3.1. If $S = \{a, b, c, d\}$, then

$$\mathcal{A} = \{\{a, b\}, \{c, d\}, \{b, c\}, \{b, d\}\}$$

is an antichain, while

$$\mathcal{A}' = \{\{a\}, \{a, b\}, \{b, d\}\}$$

is not an antichain.

3.2. If S has n elements, and $0 \leq k \leq n$, let $\mathcal{A}_k$ denote the set of all k-subsets of S. Then, $\mathcal{A}_k$ is an antichain. By Corollary 3.5,

$$|\mathcal{A}_k| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Sperner's Theorem states that the above bound holds for *any* antichain of S.

(End of Day 13)

Definition 3.8. Let S be a set. A collection $\mathcal{C}$ of subsets of S is called a chain if, whenever $A_1, A_2 \in \mathcal{C}$, then either $A_1 \subset A_2$ or $A_2 \subset A_1$.

Example 3.9. If $S = \{1, 2, \ldots, 5\}$, then

$$\mathcal{C} = \{\{2\}, \{2, 3, 5\}, S\}$$

is a chain, while

$$\mathcal{C}' = \{\{2\}, \{2, 3\}, \{3, 5\}, S\}$$

is not a chain.

Definition 3.10. Let S be a set. A chain $\mathcal{C}$ of S is said to be maximal if, for any $A \subset S$ which is not in $\mathcal{C}$, $\mathcal{C} \cup \{A\}$ is not a chain.

Example 3.11.

3.1. If $S = \{1, 2, 3, 4\}$, then

$$\mathcal{C} = \emptyset \subset \{1\} \subset \{1, 2\} \subset \{1, 2, 4\} \subset \{1, 2, 3, 4\}$$

is a maximal chain, but

$$\mathcal{C}' = \emptyset \subset \{1\} \subset \{1, 2, 3, 4\}$$

is not a maximal chain.

3.2. If S has n elements, then a chain of the form

$$A_0 = \emptyset \subset A_1 \subset A_2 \subset \ldots \subset A_n$$

is a maximal chain if and only if $|A_i| = i$ for all $1 \leq i \leq n$.

3.3. Hence if $S = \{1, 2, \ldots, n\}$, we may construct a maximal chain by the following steps:
- Start with $A_0 := \emptyset$.
- Choose an $i_1 \in \{1, 2, \ldots, n\}$ and take $A_1 = \{i_1\}$.
- Choose an $i_2 \in \{1, 2, \ldots, n\} \setminus A_1$ and take $A_2 = \{i_1, i_2\}$.
- $\ldots$
- Choose an $i_{n-1} \in \{1, 2, \ldots, n\} \setminus A_{n-2}$ and take $A_{n-1} = A_{n-2} \cup \{i_{n-1}\}$.
- Set $A_n := S$.

Following these steps is equivalent to choosing a permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ i_1 & i_2 & i_3 & \dots & i_n \end{pmatrix}$$

of S. Hence,

$$\text{the number of maximal chains in S} = n!.$$

3.4. More generally, if $A \subset S$ with $|A| = k$, then we construct a maximal chain containing A as follows:

- Choose a maximal chain of A, denote it by $A_0 \subset A_1 \subset \dots \subset A_k$.
- Choose an element $i_{k+1} \in S \setminus A$ and take $A_{k+1} := A \cup \{i_{k+1}\}$.
- Choose an element $i_{k+2} \in S \setminus A_{k+1}$ and set $A_{k+2} := A_1 \cup \{i_{k+2}\}$.
- $\dots$
- Choose an $i_{n-1} \in S \setminus A_{n-2}$ and take $A_{n-1} = A_{n-2} \cup \{i_{n-1}\}$.
- Set $A_n := S$.

In the process, step (1) requires $k!$ choices, and the remaining steps require $(n-k)!$ choices. Therefore,

$$\text{the number of maximal chains containing A} = k!(n-k)!.$$

Remark 3.12. If S is a set, $\mathcal{A}$ is an antichain and $\mathcal{C}$ is a chain of S, then

$$|\mathcal{A} \cap \mathcal{C}| \leq 1.$$

Theorem 3.13 (Sperner's Theorem)**.** *Let S be a set with n elements. Then an antichain on S contains atmost*

$$\binom{n}{\lfloor n/2 \rfloor}$$

elements.

Proof. Let $\mathcal{A}$ be an antichain. Consider

$$T := \{(A, \mathcal{C}) : A \in \mathcal{A}, \mathcal{C} \text{ is a maximal chain}, A \in \mathcal{C}\}$$

We count the cardinality of T in two ways.

3.1. For each $A \in \mathcal{A}$, set

$$V_A := \{\mathcal{C} : (A, \mathcal{C}) \in T\}.$$

(i) If $A_1, A_2 \in \mathcal{A}$ and $\mathcal{C} \in V_{A_1} \cap V_{A_2}$ then $\{A_1, A_2\} \subset \mathcal{C}$. By Remark 3.12, this is impossible if $A_1 \neq A_2$. Hence, the collections

$$\{V_A : A \in \mathcal{A}\}$$

are all mutually disjoint. Thus,

$$|T| = \sum_{A \in \mathcal{A}} |V_A|.$$

(ii) Suppose $|A| = k$, then by Example 3.11, the number of maximal chains containing A is $k!(n-k)!$. Therefore,

$$|V_A| = k!(n-k)!$$

(iii) For each $0 \leq k \leq n$, let α_k denote the number of subsets of $\mathcal{A}$ with cardinality k. Then,

$$|T| = \sum_{k=0}^{n} \alpha_k k!(n-k)!.$$

3.2. For each maximal chain $\mathcal{C}$, let

$$W_{\mathcal{C}} := \{A \in \mathcal{A} : (A, \mathcal{C}) \in T\}$$

For each maximal chain $\mathcal{C}$, Remark 3.12 implies that

$$|W_{\mathcal{C}}| \leq 1.$$

Hence, by Example 3.11,

$$|T| = \sum |W_{\mathcal{C}}| \leq n!$$

3.3. We conclude that

$$\sum_{k=0}^{n} \alpha_k k!(n-k)! \leq n!$$

$$\Leftrightarrow \sum_{k=0}^{n} \frac{\alpha_k}{\binom{n}{k}} \leq 1.$$

By Corollary 3.5, the largest value of $\binom{n}{k}$ is $\binom{n}{\lfloor n/2 \rfloor}$. Hence,

$$|\mathcal{A}| = \sum_{k=0}^{n} \alpha_k \leq \binom{n}{\lfloor n/2 \rfloor}.$$

$\square$

Remark 3.14. If n is even, then it can be shown that there is a unique antichain of size $\binom{n}{\lfloor n/2 \rfloor}$, and this is the antichain of all $n/2$-subsets of S.

Moreover, if n is odd, then there are precisely two antichains of size $\binom{n}{\lfloor n/2 \rfloor}$, and these are the antichains of all $(n-1)/2$-subsets of S and of all $(n+1)/2$-subsets of S.

4. The Multinomial Theorem

Definition 4.1. Let $n \in \mathbb{N}$ and $n_1, n_2, \ldots, n_t \geq 0$ such that

$$n_1 + n_2 + \ldots + n_t = n.$$

Then, the corresponding <u>multinomial coefficients</u> are defined by

$$\binom{n}{n_1\ n_2\ \ldots\ n_t} = \frac{n!}{n_1!n_2!\ldots n_t!}.$$

Remark 4.2.

4.1. In this notation, the binomial coefficient $\binom{n}{k}$ is written as

$$\binom{n}{k\ (n-k)}.$$

4.2. The number $\binom{n}{n_1\ n_2\ \ldots\ n_t}$ represents the number of permutations of the multiset

$$S = \{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_t \cdot a_t\}$$

which has t types of objects with repetition numbers $n_1, n_2, \ldots, n_t$ respectively.

4.3. Pascal's formula for multinomial coefficients takes the form

$$\binom{n}{n_1\ n_2\ \ldots\ n_t} = \binom{n-1}{(n_1-1)\ n_2\ \ldots\ n_t} + \binom{n-1}{n_1\ (n_2-1)\ \ldots\ n_t} + \ldots + \binom{n-1}{n_1\ n_2\ \ldots\ (n_t-1)}$$

This can be verified by direct substition in the formula.

Theorem 4.3. *Let $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_t \in \mathbb{R}$. Then,*

$$(x_1 + x_2 + \ldots + x_t)^n = \sum \binom{n}{n_1\ n_2\ \ldots\ n_t} x_1^{n_1} x_2^{n_2} \ldots x_t^{n_t}$$

where the summation is taken over all non-negative solutions $(n_1, n_2, \ldots, n_t)$ to the equation

$$n_1 + n_2 + \ldots + n_t = n.$$

Proof. This is a generalization of the first proof of the Binomial Theorem (Theorem 2.1). Write $(x_1 + x_2 + \ldots + x_t)^n$ as a product

$$(x_1 + x_2 + \ldots + x_t) \cdot (x_1 + x_2 + \ldots + x_t) \cdot \ldots \cdot (x_1 + x_2 + \ldots + x_t).$$

Multiplying out using the distributive law and collecting like terms, we obtain an expression of the form

$$\sum \alpha_{(n_1, n_2, \ldots, n_t)} x_1^{n_1} x_2^{n_2} \ldots x_t^{n_t}$$

where $(n_1, n_2, \ldots, n_t)$ is a non-negative solution to the equation $n_1 + n_2 + \ldots + n_t = n$. Moreover, the term $x_1^{n_1} x_2^{n_2} \ldots x_t^{n_t}$ occurs when n_1 factors out of n are chosen to be x_1, n_2 factors from $(n - n_1)$ are chosen to x_2, and so on. The number of times the term $x_1^{n_1} x_2^{n_2} \ldots x_t^{n_t}$ occurs is thus given by

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\ldots-n_{t-1}}{n_t} = \frac{n!}{n_1!n_2!\ldots n_t!}.$$

□

Example 4.4.

4.1. When $(x_1 + x_2 + \ldots + x_5)^7$ is expanded, the coefficient of $x_1^2 x_2 x_3^2 x_4 x_5$ is

$$\binom{7}{2\,1\,2\,1\,1}$$

4.2. The total number of (different) terms occurring in the expansion of $(x_1 + x_2 + \ldots + x_t)^n$ is equal to the total number of non-negative solutions $(n_1, n_2, \ldots, n_t)$ to the equation

$$n_1 + n_2 + \ldots + n_t = n.$$

By section 5, this is equal to

$$\binom{n+t-1}{n}$$

(End of Day 14)

5. Newton's Binomial Theorem

We only state the theorem here but do not prove it. Note that if $y, \beta \in \mathbb{R}$, then $y^\beta := e^{\beta \log(y)}$ and if $\alpha \in \mathbb{R}, k \in \mathbb{Z}$, then

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}$$

Theorem 5.1. *Let $\alpha \in \mathbb{R}$ and $x, y \in \mathbb{R}$ with $0 \leq |x| < |y|$. Then,*

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k},$$

Example 5.2.

5.1. If $\alpha = n \in \mathbb{N}$, then for $k > n$, $\binom{\alpha}{k} = 0$ so the expansion given above reduces to

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

This agrees with the Binomial theorem.

5.2. If $z = x/y$ in the above expansion, then $(x+y)^\alpha = y^\alpha(1+z)^\alpha$. Thus, an equivalent statement to Theorem 5.1 is that for any $z \in \mathbb{R}$ with $|z| < 1$,

$$(1+z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k.$$

5.3. If $n \in \mathbb{N}$ and $\alpha := -n$, then

$$\begin{aligned}\binom{\alpha}{k} = \binom{-n}{k} &= \frac{(-n)(-n-1)\dots(-n-k+1)}{k!} \\ &= (-1)^k \frac{n(n+1)\dots(n+k-1)}{k!} \\ &= (-1)^k \binom{n+k-1}{k}\end{aligned}$$

Hence if $|z| < 1$, then

$$(1+z)^{-n} = \frac{1}{(1+z)^n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} z^k.$$

Replacing z by $-z$ and taking $n = 1$, we get

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \tag{IV.1}$$

since $\binom{k}{k} = 1$ for all $k \geq 0$.

5.4. Now suppose we start with Equation IV.1 and try to compute

$$\frac{1}{(1-z)^n} = \underbrace{(1+z+z^2+\dots)\cdot(1+z+z^2+\dots)\dots(1+z+z^2+\dots)}_{n \text{ times}}.$$

This expansion is of the form

$$\sum_{k=0}^{\infty} \beta_k z^k.$$

To compute β_k, note that we obtain z^k by choose z^{k_1} from the first factor, z^{k_2} from the second factor, ..., z^{k_n} from the n^{th} factor in such a way that $(k_1, k_2, \dots, k_n)$ is a tuple of non-negative integers satisfying

$$k_1 + k_2 + \dots + k_n = k.$$

The number of such solutions (which is β_k) is known to be

$$\binom{n+k-1}{k}$$

from section 5. Therefore,

$$\frac{1}{(1-z)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k.$$

5.5. Now take $\alpha = \frac{1}{2}$ in Equation IV.1. Then, $\binom{\alpha}{0} = 1$ and for $k > 0$,

$$\begin{aligned}
\binom{\alpha}{k} &= \frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-k+1)}{k!} \\
&= \frac{(-1)^{k-1}}{2^k} \cdot \frac{1 \times 2 \times 3 \times \dots \times (2k-2)}{2 \times 4 \times \dots \times (2k-2) \times k!} \\
&= \frac{(-1)^{k-1}}{k \times 2^{2k-1}} \frac{(2k-2)!}{(k-1)!^2} \\
&= \frac{(-1)^{k-1}}{k \times 2^{2k-1}} \binom{2k-2}{k-1}.
\end{aligned}$$

Therefore, if $|z| < 1$, then

$$\sqrt{1+z} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k \times 2^{2k-1}} \binom{2k-2}{k-1} z^k.$$

This can be used to compute square roots upto any desired accuracy.

V. The Inclusion-Exclusion Principle and Applications

1. The Inclusion-Exclusion Principle

Example 1.1. Count the permutations $(i_1, i_2 \ldots, i_n)$ of $\{1, 2, \ldots, n\}$ in which 1 is not in the first position (i.e. $i_1 \neq 1$).

Solution: For each $k \in \{2, 3, \ldots, n\}$, we count the number of permutations with k in the first position. This gives $(n-1)!$ such permutations. Adding them up, we get $(n-1)(n-1)!$ permutations that do not have 1 in the first position. $\square$

Example 1.2. Count the number of integers in $\{1, 2, \ldots, 600\}$ which are not divisible by 6.

Solution: The number of integers divisible by 6 is precisely $600/6 = 100$. Therefore, the number we are looking for is $600 - 100 = 500$. $\square$

Remark 1.3. A generalization of the subtraction principle: Suppose S is a set and P_1 and P_2 are two properties that elements of S may or may not possess. We wish to calculate the cardinality of the set

$$X := \{x \in S : x \text{ does not satisfy } P_1 \text{ and does not satisfy } P_2\}.$$

We may do this by setting

$$A_1 := \{x \in S : x \text{ satisfies } P_1\} \text{ and } A_2 := \{x \in S : x \text{ satisfies } P_2\}.$$

Then, the set X is described as $X = A_1^c \cap A_2^c$ and

$$|A_1^c \cap A_2^c| = |S| - |A_1| - |A_2| + |A_1 \cap A_2|.$$

The last term is there to avoid undercounting those elements that satisfy *both* P_1 and P_2. We examine why this formula holds: Chooose a point $x \in S$ and count the contribution of x to either side of the equation.

1.1. If x satisfies neither P_1 nor P_2, then we get

$$1 = 1 - 0 - 0 + 0.$$

1.2. If x satisfies only P_1, but not P_2, then we get

$$0 = 1 - 1 - 0 + 0.$$

1.3. If x satisfies P_2 but not P_1, the equation holds by symmetry.

1.4. If x satisfies both P_2 and P_1, then we get

$$0 = 1 - 1 - 1 + 1$$

Thus, the contribution of each point to either side of the equation is the same, so the equation holds.

Theorem 1.4. *Let S be a set and $P_1, P_2, \ldots, P_m$ be m properties that elements of S may possess. The number of elements that do not satisfy any of these properties is given by*

$$\begin{aligned} |A_1^c \cap A_2^c \cap \ldots \cap A_m^c| = |S| - \sum_{i=1}^{n} |A_i| + \sum_{i,j} |A_i \cap A_j| \\ - \sum_{i,j,k} |A_i \cap A_j \cap A_k| + \ldots + (-1)^m |A_1 \cap A_2 \cap \ldots \cap A_m|. \end{aligned} \tag{V.1}$$

where the first sum is over the 1*-subsets of* $\{1, 2, \ldots, m\}$*, the second sum is over all* 2*-subsets of* $\{1, 2, \ldots, m\}$*, and so on.*

Remark 1.5.

1.1. If $m = 3$, then we get

$$\begin{aligned} |A_1^c \cap A_2^c \cap A_3^c| = |S| &- (|A_1| + |A_2| + |A_3| + |A_4|) \\ &+ (|A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|) \\ &- (|A_1 \cap A_2 \cap A_3| + |A_2 \cap A_3 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_1 \cap A_2 \cap A_4|) \\ &+ |A_1 \cap A_2 \cap A_3 \cap A_4|. \end{aligned}$$

There are a total of $1 + 4 + 6 + 4 + 1 = 16$ terms.

1.2. In general, the number of terms on the right of Equation V.1 is

$$\binom{m}{0} + \binom{m}{1} + \ldots + \binom{m}{m} = 2^m.$$

Proof of Theorem 1.4. Choose a point $x \in S$ and calculate its contribution to each side of Equation V.1.

1.1. If x does not satisfy any of the properties, then we get

$$1 = 1 - 0 + 0 + \ldots + (-1)^m 0.$$

1.2. Now suppose x satisfies $n \geq 1$ properties. Then, the LHS is 0. For the RHS, consider the contribution of x to each term:

- In $|S|$, it is 1.
- In $\sum_{i=1}^{m} |A_i|$, it is $n = \binom{n}{1}$.
- In $\sum_{i,j} |A_i \cap A_j|$, it is $\binom{n}{2}$.
- In $\sum_{i,j,k} |A_i \cap A_j \cap A_k|$, it is $\binom{n}{3}$.

Thus proceeding, we see that the contribution of x to the RHS is

$$\begin{aligned}
&\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^m \binom{n}{m} \\
&= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^n \binom{n}{n} \\
&= 0.
\end{aligned}$$

by Proposition 2.5.

□

(End of Day 15)

Corollary 1.6. *Let S be a set and $P_1, P_2, \ldots, P_m$ be m properties that elements of S may possess. Then, the number of elements of S that have at least one of the properties is given by*

$$\begin{aligned}
|A_1 \cup A_2 \cup \ldots \cup A_m| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| \\
+ \ldots + (-1)^{m+1} |A_1 \cap A_2 \cap \ldots \cap A_m|.
\end{aligned}$$

Proof. Observe that if $X := A_1 \cup A_2 \cup \ldots \cup A_m$, then

$$X^c = A_1^c \cap A_2^c \cap \ldots \cap A_m^c.$$

By the subtraction principle, $|X| = |S| - |A_1^c \cap A_2^c \cap \ldots \cap A_m^c|$, so the result follows from Theorem 1.4. □

Example 1.7. Find the number of integers in $\{1, 2, \ldots, 1000\}$ that are not divisible by 5, by 6 or by 8.

Solution: Let P_1 be the property that a number is divisible by 5, P_2 that it is divisible by 6, and P_3 that it is divisible by 8. Let A_i be the corresponding sets, then we wish to calculate

$$|A_1^c \cap A_2^c \cap A_3^c|.$$

Note that

$$
\begin{aligned}
|A_1| &= \lfloor \frac{1000}{5} \rfloor = 200, \\
|A_2| &= \lfloor \frac{1000}{6} \rfloor = 166, \\
|A_3| &= \lfloor \frac{1000}{8} \rfloor = 125.
\end{aligned}
$$

$A_1 \cap A_2$ is the set of integers divisible by $lcm\{5,6\} = 30$. Therefore,

$$
\begin{aligned}
|A_1 \cap A_2| &= \lfloor \frac{1000}{30} \rfloor = 33 \\
|A_2 \cap A_3| &= \lfloor \frac{1000}{24} \rfloor = 41 \\
|A_1 \cap A_3| &= \lfloor \frac{1000}{40} \rfloor = 25.
\end{aligned}
$$

Similarly,

$$|A_1 \cap A_2 \cap A_3| = \lfloor \frac{1000}{120} \rfloor = 8.$$

Therefore,

$$|A_1^c \cap A_2^c \cap A_3^c| = 1000 - (200 + 166 + 125) + (33 + 41 + 25) - 8 = 600.$$

$\square$

Remark 1.8. Suppose that the cardinality $|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|$ only depends on k and not on the sets $\{A_1, \ldots, A_m\}$, then we denote it by α_k. In other words,

$$
\begin{aligned}
&\alpha_0 = |S| \\
&\alpha_1 = |A_1| = |A_2| = \ldots = |A_m| \\
&\alpha_2 = |A_1 \cap A_2| = |A_1 \cap A_3| = \ldots = |A_{m-1} \cap A_m|
\end{aligned}
$$

and so on. Then, the Inclusion-Exclusion principle takes the form

$$|A_1^c \cap A_2^c \cap \ldots \cap A_m^c| = \alpha_0 - \binom{m}{1}\alpha_1 + \ldots + (-1)^k \binom{m}{k}\alpha_k + \ldots + (-1)^m \alpha_m$$

Example 1.9. How many integers in $\{0, 1, \ldots, 99999\}$ have among their digits each of $2, 5$ and 8?

Solution: Let P_1 be the property that a number has a 2, P_2 that it has a 5 and P_3 that it has an 8. We think of S as the set $\{1, 2, \ldots, 99999\}$, except each number is a 5-digit number with possible leading zeroes. Hence,

$$\alpha_0 = |S| = 10^5.$$

Now, A_1 is the set of numbers with a 2. Therefore,

$$\alpha_1 = |A_1| = 9^5.$$

Now, $A_1 \cap A_2$ is the set of numbers with a 2 and a 5. Such a number is a permutation of the multiset

$$\{5 \cdot 0, 5 \cdot 1, 5 \cdot 3, 5 \cdot 4, 5 \cdot 6, 5 \cdot 7, 5 \cdot 8, 5 \cdot 9\}.$$

This has 8 symbols, so

$$\alpha_2 = |A_1 \cap A_2| = 8^5.$$

Similarly, $\alpha_3 = 7^5$ so that

$$|A_1^c \cap A_2^c \cap A_3^c| = 10^5 - (3 \times 9^5) + (3 \times 8^5) - 7^5.$$

□

2. Combinations with Repetition

Remark 2.1.

2.1. The number of r-subsets of a set with n elements is given by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

2.2. The number of r-combinations of a multiset with k distinct objects, each with infinite repetition number, is given by

$$\binom{r+k-1}{r}.$$

2.3. Suppose $T = \{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$ is a multiset, and we are looking for an r-combination of T. If $n_1 \geq r$, then this the same as the number of r-combinations of the multiset

$$T' = \{\infty \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}.$$

This principle applies in reverse. If we are looking for an r-combination in a multiset T, and x is an object in T with repetition number ∞, then we may assume that x has repetition number r.

2.4. Hence, we have determined the number of r-combinations in two extreme cases:
   - $n_1 = n_2 = \ldots = n_k = 1$ (in that case, T is a set).
   - $n_1 = n_2 = \ldots = n_k = r$ (in that case, we may assume that $n_i = +\infty$ and apply 2.2.

Example 2.2. Determine the number of 10-combinations of the multiset

$$T = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}$$

Solution:

2.1. Set $T^* = \{10 \cdot a, 10 \cdot b, 10 \cdot c\}$ (or $\{\infty \cdot a, \infty \cdot b, \infty \cdot c\}$). Let S be the set of all 10-combinations in T^*. Then,

$$|S| = \binom{10+3-1}{10} = 66.$$

2.2. Let P_1 be the property that a 10-combination of T^* has more than 3 a's. Let P_2 be the property that it has more than 4 b's, and P_3 be the property that it has more than 5 c's. Then, if A_i is the set defined as before, then we we wish to calculate

$$|A_1^c \cap A_2^c \cap A_3^c|.$$

By the Inclusion-Exclusion principle, we wish to find

$$|S| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3|) - |A_1 \cap A_2 \cap A_3|.$$

2.3. Now, if we take an element of A_1, then A_1 is a 10-combination with ≥ 4 a's. If we remove 4 a's, we are left with a 6-combination of T^*. Conversely, if we take a 6-combination of T^* and add 4 a's to it, we get an element of A_1. Hence,

$$|A_1| = \text{ the number of 6-combinations of } T^* = \binom{6+3-1}{6} = 28.$$

Similarly,

$$|A_2| = \text{ the number of 5-combinations of } T^* = \binom{5+3-1}{5} = 21$$
$$|A_3| = \text{ the number of 4-combinations of } T^* = \binom{4+3-1}{4} = 15.$$

2.4. Now, $A_1 \cap A_2$ consists of those 10-combinations with ≥ 4 a's and ≥ 5 b's. Deleting these 4 a's and 5 b's leaves a 1-combination of T^*. As before, we see that

$$|A_1 \cap A_2| = \text{ the number of 1-combinations of } T^* = \binom{1+3-1}{1} = 3.$$

Similarly,

$$\begin{aligned} |A_1 \cap A_3| &= \text{ the number of 0-combinations of } T^* = \binom{0+3-1}{0} = 1 \\ |A_2 \cap A_3| &= 0, \end{aligned}$$

because $A_2 \cap A_3 = \emptyset$. Finally,

$$|A_1 \cap A_2 \cap A_3| = 0$$

Hence,

$$|A_1^c \cap A_2^c \cap A_3^c| = 66 - (28 + 21 + 15) + (3 + 1 + 0) - 0 = 6.$$

□

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Remark 2.3. The number of r-combinations of a multiset $T = \{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$ is equal to the number of integral solutions of the equation

$$x_1 + x_2 + \ldots + x_k = r$$

that satisfy $0 \leq x_i \leq n_i$ for all $1 \leq i \leq k$.

Example 2.4. Find the number of integral solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 18$$

that satisfy

$$1 \leq x_1 \leq 5, -2 \leq x_2 \leq 4, 0 \leq x_3 \leq 5, 3 \leq x_4 \leq 9.$$

Solution: Write

$$y_1 := x_1 - 1, y_2 := x_2 + 2, y_3 := x_3, y_4 := x_4 - 3.$$

Then, we are looking for integral solutions to

$$y_1 + y_2 + y_3 + y_4 = 16$$

that satisfy

$$0 \leq y_1 \leq 4, 0 \leq y_2 \leq 6, 0 \leq y_3 \leq 5, 0 \leq y_4 \leq 6.$$

2.1. Let S be the set of all integral solutions (with $y_i \geq 0$ but no upper bound). Then,

$$|S| = \binom{16+4-1}{16} = 969.$$

2.2. Let P_1 be the property that $y_1 \geq 5$, P_2 be the property that $y_2 \geq 7$, P_3 be the property that $y_3 \geq 6$ and P_4 be the property that $y_4 \geq 7$. Let A_i be the corresponding subsets of S. We wish to evaluate

$$|A_1^c \cap A_2^c \cap A_3^c \cap A_4^c|$$

2.3. To calculate $|A_1|$, we wish to find all integral solutions to the equation

$$y_1 + y_2 + y_3 + y_4 = 16$$

that satisfy

$$y_1 \geq 5, \text{ and } y_2, y_3, y_4 \geq 0.$$

Write $z_1 := y_1 - 5, z_2 = y_2, z_3 = y_3$ and $z_4 = y_4$. Then, we wish to find all integral solutions to

$$z_1 + z_2 + z_3 + z_4 = 11$$

with $z_i \geq 0$ for all $1 \leq i \leq 4$. The number of such solutions is

$$|A_1| = \binom{11+4-1}{11} = 364.$$

2.4. To calculate A_2, we do the same to find all integral solutions to

$$z_1 + z_2 + z_3 + z_4 = 9$$

with $z_i \geq 0$. Thus,

$$|A_2| = \binom{9+4-1}{9} = 220.$$

Similarly,

$$|A_3| = \binom{10+4-1}{10} = 286 \text{ and } |A_4| = \binom{9+3-1}{9} = 220.$$

2.5. To calculate $|A_1 \cap A_2|$, we are ooking for those solutions to

$$y_1 + y_2 + y_3 + y_4 = 16$$

such that $y_1 \geq 5, y_2 \geq 7, y_3 \geq 0, y_4 \geq 0$. As before, this is given by solutions to

$$z_1 + z_2 + z_3 + z_4 = 4$$

with $z_i \geq 0$, and so

$$|A_1 \cap A_2| = \binom{4+4-1}{4} = 35.$$

Similarly,

$$\begin{aligned}
|A_1 \cap A_3| &= \binom{5+4-1}{5} = 56 \\
|A_1 \cap A_4| &= \binom{4+4-1}{4} = 35 \\
|A_2 \cap A_3| &= \binom{3+4-1}{3} = 20 \\
|A_2 \cap A_4| &= \binom{2+4-1}{2} = 10 \\
|A_3 \cap A_4| &= \binom{3+4-1}{3} = 20.
\end{aligned}$$

2.6. One finds that for any three of the sets, $A_{i_1} \cap A_{i_2} \cap A_{i_3}$ is empty. Hence, the inclusion-exclusion principle gives

$$|A_1^c \cap A_2^c \cap A_3^c \cap A_4^c| = 969 - (364 + 220 + 286 + 220) + (35 + 56 + 35 + 20 + 10 + 20) = 55.$$

□

3. Derangements

Example 3.1. At a party, there are n men and n women. In how many ways can the n women choose male partners for the first dance? How many ways are there for the second dance if everyone has to change partners?

Solution: For the first dance, it is clearly $n!$. For the second dance, we make number the men $\{m_1, m_2, \dots, m_n\}$ and the women $\{w_1, w_2, \dots, w_n\}$ and assume that w_i has danced with m_i for $1 \leq i \leq n$ in the first dance. We now need to find a permutation σ of $\{1, 2, \dots, n\}$ such that

$$\sigma(1) \neq 1, \sigma(2) \neq 2, \dots, \sigma(n) \neq n.$$

To do this, we use the Inclusion-Exclusion Principle: Let S be the set of all permutations of $\{1, 2, \dots, n\}$ so that

$$|S| = n!.$$

Let P_i be the property that $\sigma(i) = i$. Then, we wish to compute

$$|A_1^c \cap A_2^c \cap \dots \cap A_n^c|.$$

3.1. To find $|A_1|$, note that if $\sigma(1) = 1$, then σ is effectively a permutation of $\{2, 3, \ldots, n\}$. Hence,

$$|A_1| = (n-1)! = |A_2| = \ldots = |A_5|.$$

3.2. To find $|A_1 \cap A_2|$, note that if $\sigma(1) = 1, \sigma(2) = 2$, then σ is a permutation of $\{3, 4, \ldots, n\}$. Hence,

$$|A_1 \cap A_2| = (n-2)! = |A_i \cap A_j|$$

for all 2-subsets $\{i, j\} \subset \{1, 2, \ldots, 5\}$.

3.3. By the same logic,

$$|A_1 \cap A_2 \cap A_3| = (n-3)! = |A_i \cap A_j \cap A_k|$$

for any 3-subset $\{i, j, k\} \subset \{1, 2, \ldots, n\}$.

3.4. Thus proceeding, for each $1 \leq k \leq n$,

$$|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}| = (n-k)!.$$

By Remark 1.8,

$$\begin{aligned}|A_1^c \cap A_2^c \cap \ldots \cap A_5^c| &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! + \\ &\ldots + (-1)^k \binom{n}{k}(n-k)! + \ldots + (-1)^n \cdot 1 \\ &= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right)\end{aligned}$$

$\square$

Definition 3.2. A <u>derangement</u> of the set $X := \{1, 2, \ldots, n\}$ is a permutation σ of X such that

$$\sigma(i) \neq i \text{ for all } 1 \leq i \leq n.$$

We write D_n for the number of derangements of X, so that

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right) \quad \text{(V.2)}$$

Example 3.3.

3.1. $D_1 = 0$ because there is no derangement of $\{1\}$.

3.2. $D_2 = 1$ because there is only one derangement, namely $\sigma = (1, 2)$.

3.3. There are two derangements of $\{1,2,3\}$, namely $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$. Thus, $D_3 = 2$.

Remark 3.4. Recall that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots.$$

Hence,

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^k \frac{1}{k!} + \ldots$$

Comparing these two expressions, we see that

$$e^{-1} = \frac{D_n}{n!} + (-1)^{n+1} \frac{1}{(n+1)!} + (-1)^{n+2} \frac{1}{(n+2)!} + \ldots$$

Hence,

$$\frac{D_n}{n!} \approx e^{-1}.$$

So, consider the event E of choosing a derangement from the set of all permutations of $\{1, 2, \ldots, n\}$. Then,

$$Prob(E) = \frac{D_n}{n!} \approx e^{-1}.$$

Proposition 3.5. *The D_n satisfy the relation*

$$D_n = (n-1)(D_{n-2} + D_{n-1}).$$

Proof. One can verify this directly from the formula above. However, we may count. Let $n \geq 3$ (the case $n = 2$ can be verified independently). Let T be the set of all derangements of $X = \{1, 2, \ldots, n\}$.

3.1. For each $2 \leq i \leq n$, let T_i be the set of all derangements σ with the property that $\sigma(i) = 2$. Then,

$$T = \bigsqcup_{i=2}^{n} T_i.$$

Moreover, $|T_2| = |T_3| = \ldots = |T_n|$, so if $d_n := |T_1|$, then

$$D_n = |T| = (n-1)d_n.$$

3.2. Now consider elements σ of T_1. Such a permutation is of the form

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 2 & i_2 & i_3 & \ldots & i_n \end{pmatrix}$$

such that $i_2 \neq 2, i_3 \neq 3, \dots, i_n \neq n$. Let T_1' be the set of all derangements in T_1 such that

$$i_2 = 1,$$

and let $T_1'' := T_1 \setminus T_1'$. Let $d_n' := |T_1'|$ and $d_n'' := |T_1''$ so that

$$d_n = d_n' + d_n''.$$

3.3. Now, elements of T_1' are of the form

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 1 & i_3 & \dots & i_n \end{pmatrix}$$

with the condition that $i_3 \neq 3, i_4 \neq 4, \dots, i_n \neq n$. Hence, T_1' consists of all derangements of the set $\{3, 4, \dots, n\}$ so that

$$d_n' = D_{n-2}.$$

3.4. Now, T_1'' consists of derangements of the form

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 2 & i_2 & i_3 & \dots & i_n \end{pmatrix}$$

where $i_2 \neq 1, i_3 \neq 3, \dots, i_n \neq n$. In effect, this is derangement of the set $\{1, 3, 4, \dots, n\}$. Hence,

$$d_n'' = D_{n-1}.$$

Combining all the previous equations gives

$$D_n = (n-1)d_n = (n-1)(d_n' + d_n'') = (n-1)(D_{n-2} + D_{n-1}).$$

□

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Proposition 3.6. *For each $n \geq 2$,*

$$D_n = nD_{n-1} + (-1)^{n-2}.$$

Proof. Once again, this can be computed directly, but we use the recurrence relation from Proposition 3.5. Once again, we assume $n \geq 3$ for convenience. Then,

$$D_n - nD_{n-1} = -[D_{n-1} - (n-1)D_{n-2}]$$

The expression inside the brackets has the same form as the LHS, so

$$\begin{aligned} D_n - nD_{n-1} &= (-1)^2[D_{n-2} - (n-2)D_{n-3}] \\ &= \dots \\ &= (-1)^{n-2}(D_2 - 2D_1). \end{aligned}$$

Since $D_2 = 1$ and $D_1 = 0$, we get the required formula. □

Remark 3.7. Note that Proposition 3.6 can be used to give an alternate proof of Equation V.2, since Proposition 3.6 is independent of our earlier proof.

4. Permutations with Forbidden Positions

Recall from Example 4.9 the problem of placing rooks on a $n \times n$ chessboard in a non-attacking position. One example for $n = 8$ is shown below.

	Φ						
Φ							
			Φ				
		Φ					
					Φ		
				Φ			
						Φ	
							Φ

We now consider a variation of this problem. Now, we wish to place rooks in a non-attacking position, but with further restrictions on where each rook may be placed.

Example 4.1. Let $n = 4$, and consider the picture:

1	$\times$	$\times$		
2		$\times$	$\times$	
3			$\times$	$\times$
4	$\times$			$\times$

Each "$\times$" represents a *forbidden position*. In other words, we wish to place the rooks in non-attacking positions, with the conditions that

4.1. Rook 1 cannot be placed in $X_1 = \{1, 2\}$.

4.2. Rook 2 cannot be placed in $X_2 = \{2, 3\}$.

4.3. Rook 3 cannot be placed in $X_3 = \{3, 4\}$.

4.4. Rook 4 cannot be placed in $X_4 = \{1, 4\}$.

How many such positions are possible?

Solution: There are two solutions depicted below:

1	$\times$	$\times$	Φ	
2		$\times$	$\times$	Φ
3	Φ		$\times$	$\times$
4	$\times$	Φ		$\times$

and

1	$\times$	$\times$		Φ
2	Φ	$\times$	$\times$	
3		Φ	$\times$	$\times$
4	$\times$		Φ	$\times$

□

Notice that each such configuration corresponds to a permutation σ of $S = \{1,2,3,4\}$ where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ i_2 & i_2 & i_3 & i_4 \end{pmatrix}$$

where $i_1 \notin X_1, i_2 \notin X_2, i_3 \notin X_3$ and $i_4 \notin X_4$.

Definition 4.2. Let $X_1, X_2, \ldots, X_n$ be (possibly empty) subsets of $\{1,2,\ldots,n\}$. We let

$$P(X_1, X_2, \ldots, X_n)$$

be the set of all permutations σ of $\{1,2,\ldots,n\}$ such that

$$\sigma(j) \notin X_j$$

for all $1 \leq j \leq n$. We write

$$p(X_1, X_2, \ldots, X_n) = |P(X_1, X_2, \ldots, X_n)|.$$

Example 4.3.

4.1. If $X_1 = \{1\}, X_2 = \{2\}, \ldots, X_n = \{n\}$, then an element of $P(X_1, X_2, \ldots, X_n)$ is a *derangement* of S as in Definition 3.2. Hence,

$$p(X_1, X_2, \ldots, X_n) = D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right).$$

4.2. If $n = 4$, $X_1 = \{1,2\}, X_2 = \{2,3\}, X_3 = \{3,4\}, X_4 = \{1,4\}$, then $P(X_1, X_2, X_3, X_4) = \{\sigma_1, \sigma_2\}$ where

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \text{ and } \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

and thus $p(X_1, X_2, X_3, X_4) = 2$.

Remark 4.4. To compute $p(X_1, X_2, \ldots, X_n)$ where use the Inclusion-Exclusion Principle as follows: Let S denote the set of all permutations of $T := \{1,2,\ldots,n\}$. Let A_1 be the set of all permutations $\sigma \in S$ such that $\sigma(1) \in X_1$, let A_2 be the set of all $\sigma \in S$ with $\sigma(2) \in X_2$, and so on. We now wish to compute

$$|A_1^c \cap A_2^c \cap \ldots \cap A_n^c|$$

4.1. Note that $|S| = n!$.

4.2. To compute $|A_1|$, we need to place $\sigma(1)$ in X_1 and the remaining $\sigma(j)$ may take any values. Hence,

$$|A_1| = |X_1|(n-1)!.$$

Similarly, $|A_j| = |X_j|(n-1)!$ for all $1 \leq j \leq n$. Hence,

$$|A_1| + |A_2| + \ldots + |A_n| = r_1(n-1)!$$

where $r_1 = |X_1| + |X_2| + \ldots + |X_n|$. Note that r_1 is the number of forbidden squares on the board.

4.3. To compute $|A_1 \cap A_2|$, we need to place $\sigma(1) \in X_1, \sigma(2) \in X_2$ and the remaining $\sigma(j)$ may take any values. As before, let r_2 denote the number of ways to place two non-attacking rooks in forbidden positions. Then,

$$\sum |A_i \cap A_j| = r_2(n-2)!.$$

4.4. More generally, for each $1 \leq k \leq n$, let r_k denote the number of ways to place k non-attacking rooks in forbidden positions. Then,

$$\sum |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}| = r_k(n-k)!$$

where the sum on the left is taken over all k-subsets of $\{1, 2, \ldots, n\}$.

Theorem 4.5. *The number of ways to place n non-attacking, indistinguishable rooks on an $n \times n$ board with forbidden positions $X_1, X_2, \ldots X_n$ is*

$$n! - r_1(n-1)! + r_2(n-2)! + \ldots + (-1)^k r_k(n-k)! + \ldots (-1)^n r_n.$$

This formula is not computable in general, except for simple examples.

Example 4.6. Determine the number of ways to place 6 non-attacking rooks on a 6×6 board with forbidden positions as shown below:

1	$\times$					
2	$\times$	$\times$				
3			$\times$	$\times$		
4			$\times$	$\times$		
5						
6						

Solution: We wish to compute $r_1, r_2, \ldots, r_6$. Note that $X_1 = \{1\}, X_2 = \{1, 2\}, X_3 = \{3, 4\} = X_4, X_5 = \emptyset = X_6$.

4.1. $r_1 = |X_1| + |X_2| + \ldots + |X_6| = 7.$

4.2. To compute r_2: Let F_1 be the forbidden positions in the three positions in the top left corner and F_2 be the four remaining positions. Then, the ways to place 2 rooks in these positions, we need to compute:

   (i) If both rooks are in F_1: We get 1 possible position.

   (ii) If both rooks are in F_2: We get 2 possible positions.

   (iii) If one rook is in F_1 and the other is in F_2: We get $3 \times 4 = 12$ possible positions.

   Hence,
$$r_2 = 1 + 2 + 12 = 15.$$

4.3. To compute r_3: We need to place three rooks in non-attacking and forbidden positions.

   - If one rook is in F_1 and the other two in F_2: We get 3 positions for the first rook and 2 positions for the other two, so we get 6 positions.
   - If one rook is in F_2 and the other two are in F_1: We get 4 positions for the first rook and one position for the other two, so we get 4 positions.

   Hence,
$$r_3 = 6 + 4 = 10.$$

4.4. To compute r_4: We need to place four rooks in non-attacking and forbidden positions: Therefore, two rooks must be in F_1 and two in F_2. The two in F_1 yield 1 possible position, and the two in F_2 yield two possible positions. Hence,
$$r_4 = 2.$$

4.5. Finally, $r_5 = r_6 = 0$.

Hence,
$$p(X_1, X_2, \dots, X_6) = 6! - (7 \times 5!) + (15 \times 4!) - (10 \times 3!) + (2 \times 2!) + 0 + 0 = 184.$$

□

(End of Day 18)

5. Another Forbidden Position Problem

Example 5.1. Suppose 8 boys take a walk every day, walking in a line (single file). On the first day, we number them
$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8.$$

On day 2, we would like to re-arrange the boys in such a way that no boy in $\{2, 3, \ldots, 8\}$ should be preceded by the same boy as Day 1. In other words,

$$8\ 7\ 6\ 5\ 4\ 3\ 2\ 1$$

is allowed. So is

$$3\ 2\ 1\ 5\ 4\ 6\ 8\ 7$$

However,

$$3\ 1\ 2\ 5\ 4\ 6\ 8\ 7$$

is not allowed because 1 precedes 2.

Once again, this determines a permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & 8 \\ i_1 & i_2 & \ldots & i_8 \end{pmatrix}$$

of $\{1, 2, \ldots, 8\}$ where certain positions are forbidden. Specifically, the patterns $[1\ 2], [2\ 3], \ldots, [7\ 8]$ are forbidden in the bottom row.

Definition 5.2. For $n \geq 2$, let Q_n denote the number of all permutations of $\{1, 2, \ldots, n\}$ such that the patterns $[1\ 2], [2\ 3], \ldots, [n-1\ n]$ do not occur.

Example 5.3.

5.1. If $n = 2$, then $\sigma = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is the only possible permutation, so $Q_2 = 1$.

5.2. If $n = 3$, then the possible permutations are

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Hence, $Q_3 = 3$.

5.3. If $n = 4$: There are 11 possible positions. Can you list them?

Theorem 5.4. *For $n \geq 2$,*

$$\begin{aligned} Q_n = n! - \binom{n-1}{1}(n-1)! + \binom{n-1}{2}(n-2)! \\ - \binom{n-1}{3}(n-3)! + \ldots + (-1)^{n-1}\binom{n-1}{n-1}1!. \end{aligned}$$

Proof. We use the Inclusion-Exclusion Principle on the set S of all permutations of the set $\{1, 2, \ldots, n\}$. As before, for $j \in \{1, 2, \ldots, n-1\}$, let A_j be the set of all permutations in S such that the pair $[j\ (j+1)]$ occurs. We wish to compute

$$|A_1^c \cap A_2^c \cap \ldots \cap A_{n-1}^c|.$$

5.1. Note that $|S| = n!$.

5.2. To compute $|A_1|$: A permutation σ is in A_1 iff [1 2] occurs in σ. This amounts to a permutation of the symbols $\{[1\ 2], 3, 4, \ldots, n\}$. Hence,

$$|A_1| = (n-1)!$$

Similarly, $|A_2| = |A_3| = \ldots = |A_{n-1}| = (n-1)!$.

5.3. To compute $|A_1 \cap A_2|$: A permutation σ is in $A_1 \cap A_2$ iff [1 2 3] occurs in σ. This amounts to a permutation of the symbols $\{[1\ 2\ 3], 4, 5, \ldots, n\}$. Hence,

$$|A_1 \cap A_2| = (n-2)!.$$

Similarly, $|A_i \cap A_j| = (n-2)!$ for any 2-subset $\{i, j\} \subset \{1, 2, \ldots, n\}$.

5.4. In general, if $\{i_1, i_2, \ldots, i_k\}$ is a k-subset of $\{1, 2, \ldots, n-1\}$, then an element σ in $A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}$ may be thought of as a permutation of $(n-k)$ symbols

$$\{[i_1\ i_2\ \ldots\ i_k], \ldots\}$$

where "…" contains all numbers in $\{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_k\}$. Hence,

$$|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}| = (n-k)!.$$

There are precisely $\binom{n-1}{k}$ such terms occuring the Inclusion-Exclusion principle formula.

By the Inclusion-Exclusion principle, we get the required formula. □

Example 5.5. If $n = 4$,

$$\begin{aligned} Q_4 &= 4! - \binom{3}{1}3! + \binom{3}{2}2! - \binom{3}{3}1! \\ &= 24 - 18 + 6 - 1 \\ &= 11. \end{aligned}$$

VI. Recurrence Relations and Generating Functions

Many combinatorial problems involve solving for a number h_n that depends on a parameter $n = 0, 1, 2, \ldots$. Thus, we obtain a sequence

$$h_0, h_1, h_2, \ldots$$

Example 0.1.

0.1. Let h_n be the number of permutations of the set $\{1, 2, \ldots, n\}$. Then, $h_n = n!$.

0.2. For $n \geq 0$, let g_n denote the number of non-negative integral solutions to the equation

$$x_1 + x_2 + x_3 = n.$$

We know from earlier sections that

$$g_n = \binom{n+3}{3}.$$

We wish to obtain general methods to solve such counting problems by

- Recurrence relations: If the number h_n is related to the number h_{n-1} or may be obtained from all the numbers $\{h_0, h_1, \ldots, h_{n-1}\}$ occurring before it.
- Generating functions: If there is a function g which has a power series expansion of the form $g(x) = h_0 + h_1 x + h_2 x^2 + \ldots$ so that the coefficient of x^n is h_n.

(End of Day 19)

1. Some Number Sequences

Let $h_0, h_1, \ldots, h_n, \ldots$ denote a sequence of numbers. The symbol h_n is called the <u>general term</u> or <u>generic term</u> of the sequence.

Example 1.1.

1.1. Arithmetic Sequence: A sequence of the form

$$h_0, h_0 + q, h_0 + 2q, \ldots, h_0 + nq, \ldots$$

Here, the generic term is $h_n = h_0 + nq$. Moreover, we have a recurrence relation

$$h_n = h_{n-1} + q.$$

1.2. Geometric Sequence: A sequence of the form

$$h_0, qh_0, q^2h_0, \ldots, q^nh_0, \ldots$$

Here, the generic term is $h_n = q^n h_0$ and we have a recurrence relation

$$h_n = qh_{n-1}.$$

1.3. Given a sequence $\{h_0, h_1, \ldots\}$, the partial sums of the sequence form a new sequence $\{s_0, s_1, \ldots, \}$ where

$$\begin{aligned} s_0 &= h_0 \\ s_1 &= h_0 + h_1 \\ \vdots &= \vdots \\ s_n &= h_0 + h_1 + \ldots + h_n \end{aligned}$$

For instance, if $\{h_0, h_1, \ldots, \}$ is an arithmetic sequence as above, then

$$s_n = \sum_{k=0}^{n} (h_0 + kq) = (n+1)h_0 + \frac{n(n+1)}{2}q.$$

While if $\{h_0, h_1, \ldots\}$ is a geometric sequence as above, then

$$s_n = \sum_{k=0}^{n} q^k h_0 = \begin{cases} (n+1)h_0 & : \text{ if } q = 1 \\ \frac{q^{n+1}-1}{q-1}h_0 & : \text{ if } q \neq 1. \end{cases}$$

1.4. For $n \geq 1$, let D_n denote the number of derangements of the set $\{1, 2, \ldots, n\}$ as in Definition 3.2. Then, we computed

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + (-1)^n \frac{1}{n!}\right)$$

We also obtained some recurrence relations:

$$D_n = (n-1)(D_{n-2} + D_{n-1}) \text{ and } D_n = nD_{n-1} + (-1)^{n-2}.$$

Definition 1.2. The Fibonacci sequence is a sequence $\{f_0, f_1, f_2, \ldots\}$ defined by the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \qquad (n \geq 2)$$

with initial conditions $f_0 = 0, f_1 = 1$.

We may thus compute the sequence as

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

Lemma 1.3. *If* $s_n = f_0 + f_1 + \ldots + f_n$*, then*

$$s_n = f_{n+2} - 1.$$

Proof. We prove this by induction. For $n = 0$, we have $s_0 = f_0 = f_2 - 1$, which holds.

Now assume that the result holds for n, and consider

$$\begin{aligned}
s_{n+1} &= f_0 + f_1 + \ldots + f_n + f_{n+1} \\
&= s_n + f_{n+1} \\
&= f_{n+2} - 1 + f_{n+1} \text{ (by induction hypothesis)} \\
&= f_{n+3} - 1
\end{aligned}$$

□

Lemma 1.4. *The Fibonacci number* f_n *is even if and only if* n *is divisible by* 3*.*

Proof. Note that the triple $(f_0, f_1, f_2) = (0, 1, 1)$ are in the form $(\text{even}, \text{odd}, \text{odd})$. Now suppose by induction that $(f_{3n}, f_{3n+1}, f_{3n+2})$ are in the same form, then

$$\begin{aligned}
f_{3(n+1)} &= f_{3n+3} = \text{odd} + \text{odd} = \text{even} \\
f_{3(n+1)+1} &= f_{3n+4} = \text{odd} + \text{even} = \text{odd} \\
f_{3(n+1)+2} &= f_{3n+5} = \text{odd} + \text{even} = \text{odd}.
\end{aligned}$$

Hence, the pattern repeats at $(n + 1)$ and the proof is complete. □

Remark 1.5. We wish to find explicit values for f_n by solving the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \qquad (n \geq 2) \tag{VI.1}$$

with initial conditions $f_0 = 0, f_1 = 1$.

1.1. Suppose we are looking for a solution of the form

$$f_n = q^n$$

for some $q \neq 0$. Then this would not satisfy the initial conditions (because $q^0 = 1 \neq f_0$). However, if $f_n = q^n$ satisfied the recurrence relation, then we would get

$$q^n - q^{n-1} - q^{n-2} = 0. \tag{VI.2}$$

Dividing by q^{n-2} gives

$$q^2 - q - 1 = 0.$$

Using the quadratic formula, we get two roots

$$q_1 = \frac{1 + \sqrt{5}}{2} \text{ and } q_2 = \frac{1 - \sqrt{5}}{2}.$$

Thus,

$$f_n = \left(\frac{1+\sqrt{5}}{2}\right)^n \text{ and } f_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$$

are both solutions to Equation VI.1.

1.2. Now observe that any expression of the form

$$f_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

is also a solution to Equation VI.1. Now we input the initial conditions $f_0 = 0$ and $f_1 = 1$ and obtain

$$\begin{aligned} c_1 + c_2 &= 0. \\ c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) &= 0. \end{aligned} \tag{VI.3}$$

Solving this system gives

$$c_1 = \frac{1}{\sqrt{5}} \text{ and } c_2 = \frac{-1}{\sqrt{5}}.$$

1.3. Note that

$$\det\begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} = -\sqrt{5} \neq 0.$$

Hence, the solution obtained above to Equation VI.3 is unique.

Theorem 1.6. *The Fibonacci numbers satisfy the formula*

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Example 1.7. Let $\{g_0, g_1, \ldots\}$ be a sequence of numbers satisfying the Fibonacci recurrence relations

$$g_n = g_{n-1} + g_{n-2} \qquad (n \geq 2)$$

with the initial conditions $g_0 = 2, g_1 = -1$. Determine the generic term g_n.

Solution: We know that

$$g_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

where c_1, c_2 satisfy

$$c_1 + c_2 = 2.$$
$$c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = -1. \tag{VI.4}$$

Solving this system gives

$$c_1 = \frac{\sqrt{5}-2}{\sqrt{5}}, c_2 = \frac{\sqrt{5}+2}{\sqrt{5}}.$$

As before, the solution is unique and this gives the formula for g_n. □

Example 1.8. Determine the number h_n of ways to perfectly cover a $2 \times n$ board with dominoes.

Solution: Define $h_0 = 1$ (there is exactly one way to cover an empty board), and it is easy to see that

$$h_1 = 1.$$

To compute h_n for $n \geq 2$, we divide the set of all perfect covers into two collections. Let A be the collection where there is a vertical domino on the top-left corner (and hence the bottom left corner as well). Let B be the collection where there is a horizontal domino at that spot. Note that

$$|A| = h_{n-1}.$$

Also, for a perfect cover in B, there must be a horizontal domino on the bottom left corner as well, so

$$|B| = h_{n-2}.$$

Hence, $h_n = h_{n-1} + h_{n-2}$. It follows that the sequence $\{h_0, h_1, h_2, \ldots\}$ is the same as the Fibonacci sequence starting at 1, i.e. $\{f_1, f_2, f_3, \ldots\}$. □

Theorem 1.9. *For each $n \geq 0$,*

$$f_n = \binom{n-1}{0} + \binom{n-2}{1} + \ldots + \binom{n-t}{t-1}$$

where $t = \lfloor \frac{n+1}{2} \rfloor$. These are the numbers occuring along a 'diagonal' on Pascal's triangle, starting at the $(n-1)^{th}$ row and moving up and to the right.

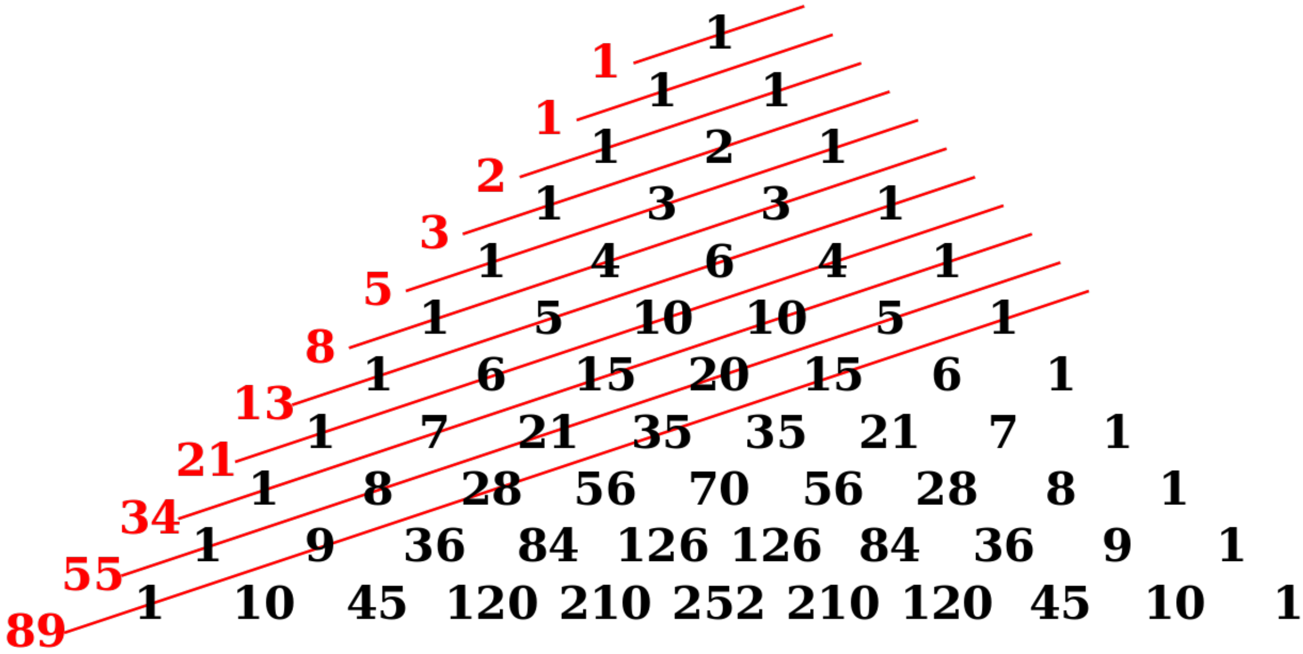

Figure VI.1.: Fibonacci Sequence on Pascal's Triangle

Proof. Define a sequence $\{g_0, g_1, \ldots\}$ as follows: Set $g_0 = 0$ and

$$g_n := \binom{n-1}{0} + \binom{n-2}{1} + \ldots + \binom{n-t}{t-1}$$

for $n \geq 1$. Note that $g_1 = 1$, so it suffices to show that (g_n) satisfies the Fibonacci recurrence relation. Since $\binom{m}{p} = 0$ if $p > m$, we have

$$\begin{aligned} g_n &= \binom{n-1}{0} + \binom{n-2}{1} + \ldots + \binom{n-t}{t-1} + \ldots + \binom{0}{n-1} \\ &= \sum_{p=0}^{n-1} \binom{n-1-p}{p}. \end{aligned}$$

Hence,

$$g_2 = \binom{1}{0} + \binom{0}{1} = 1.$$

And if $n \geq 3$, then

$$\begin{aligned}
g_{n-1} + g_{n-2} &= \sum_{k=0}^{n-2} \binom{n-2-k}{k} + \sum_{j=0}^{n-3} \binom{n-3-j}{j} \\
&= \binom{n-2}{0} + \sum_{k=1}^{n-2} \binom{n-2-k}{k} + \sum_{p=1}^{n-2} \binom{n-3-(p-1)}{p-1} \\
&= \binom{n-2}{0} + \sum_{k=1}^{n-2} \left(\binom{n-2-k}{k} + \binom{n-2-k}{k-1} \right) \\
&= \binom{n-1}{0} + \sum_{k=1}^{n-2} \binom{n-1-k}{k} \\
&= \sum_{k=0}^{n-2} \binom{n-1-k}{k} \\
&= g_n.
\end{aligned}$$

Hence, $\{g_0, g_1, g_2, \ldots\}$ is precisely the Fibonacci sequence. □

(End of Day 20)

2. Generating Functions

Definition 2.1. The generating function for a sequence $\{h_0, h_1, h_2, \ldots\}$ is

$$g(x) = h_0 + h_1 x + h_2 x^2 + \ldots$$

Remark 2.2.

2.1. We do not treat this as a series, but merely as a formal sum (i.e. We do not care about convergence), so we write $g(x)$ and not g.

2.2. If the sequence consists of finitely many non-zero terms, then $g(x)$ is a polynomial.

Example 2.3.

2.1. The generating function for $\{1, 1, 1, \ldots\}$ is

$$g(x) = 1 + x + x^2 + \ldots = \frac{1}{1-x}.$$

2.2. If $m \in \mathbb{N}$, the generating function for

$$\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}$$

is

$$g(x) = \binom{m}{0} + \binom{m}{1} x + \ldots + \binom{m}{m} x^m = (1+x)^m.$$

2.3. If $\alpha \in \mathbb{R}$, the generating function for the sequence

$$\binom{\alpha}{0}, \binom{\alpha}{1}, \binom{\alpha}{2}, \ldots$$

is given by Newton's Binomial theorem by

$$g(x) = (1+x)^{\alpha}.$$

2.4. Fix $k \in \mathbb{N}$. For each $n \geq 0$, let h_n denote the number of non-negative integral solutions to the equation

$$e_1 + e_2 + \ldots + e_k = n \tag{VI.5}$$

From section 5, we know that

$$h_n = \binom{n+k-1}{k-1}.$$

The generating function for this sequence is

$$g(x) = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n = \frac{1}{(1-x)^k}$$

To see why this works, we wrote it as

$$\begin{aligned}\frac{1}{(1-x)^k} &= \frac{1}{(1-x)} \times \frac{1}{(1-x)} \times \ldots \times \frac{1}{(1-x)} \\ &= \left(1 + x + x^2 + \ldots\right) \times \ldots \times \left(1 + x + x^2 + \ldots\right) \\ &= \left(\sum_{e_1=0}^{\infty} x^{e_1}\right)\left(\sum_{e_2=0}^{\infty} x^{e_2}\right) \ldots \left(\sum_{e_k=0}^{\infty} x^{e_k}\right).\end{aligned}$$

Therefore, the coefficient of x^n corresponds to the number of solutions to Equation VI.5.

2.5. We may use this idea again: For what sequence (h_n) is the function

$$g(x) = (1 + x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2)(1 + x + x^2 + x^3 + x^4)$$

the generating function?

Solution: The coefficient h_n of x^n is obtained as

$$x^n = x^{e_1} x^{e_2} x^{e_3}$$

where $0 \leq e_1 \leq 5, 0 \leq e_2 \leq 2, 0 \leq e_3 \leq 4$. Thus, h_n is the number of integral solutions to the equation

$$e_1 + e_2 + e_3 = n$$

satisfying $0 \leq e_1 \leq 5, 0 \leq e_2 \leq 2, 0 \leq e_3 \leq 4$. □

2.6. Determine the number of n-combinations of apples, bananas, oranges and pears where

- The number of apples is even.
- The number of bananas is odd.
- The number of oranges is between 0 and 4.
- There is at least one pear.

Solution: We are looking for integral solutions to the equation

$$e_1 + e_2 + e_3 + e_4 = n$$

where e_1 is even, e_2 is odd, $0 \leq e_3 \leq 4$ and $e_4 \geq 1$. Each factor contributes

$$\begin{aligned}
g_1(x) &= 1 + x^2 + x^4 + x^6 + \dots \\
&= \frac{1}{1-x^2} \\
g_2(x) &= x + x^3 + x^5 + \dots \\
&= \frac{x}{1-x^2} \\
g_3(x) &= 1 + x + x^2 + x^3 + x^4 \\
g_4(x) &= x + x^2 + x^3 + \dots \\
&= \frac{x}{1-x}
\end{aligned}$$

The generating function for h_n is therefore

$$g(x) = g_1(x)g_2(x)g_3(x)g_4(x).$$

The n^{th} coefficient of the Maclaurin series of $g(x)$ is h_n. □

2.7. Find the number h_n of fruit that can be made out of apples, bananas, oranges and pears where

- The number of apples is even.
- The number of bananas is a multiple of 5.
- The number of oranges is atmost 4.
- The number of pears is either 0 or 1.

Solution: As before, we get the generating function of h_n to be

$$\begin{aligned}
g(x) &= \frac{1}{(1-x^2)}(1+x^5+x^{10}+x^{15}+\ldots)(1+x+x^2+x^3+x^4)(1+x) \\
&= \frac{1}{(1-x^2)}\frac{1}{(1-x^5)}\frac{(1-x^5)}{(1-x)}(1+x) \\
&= \frac{1}{(1-x)^2} \\
&= \sum_{n=0}^{\infty}\binom{n+2-1}{1}x^n = \sum_{n=0}^{\infty}(n+1)x^n.
\end{aligned}$$

Therefore, $h_n = (n+1)$ for all $n \geq 0$. $\square$

2.8. Fix $k \in \mathbb{N}$. Determine the generating function for the sequence (h_n) of non-negative integral solutions to the equation

$$e_1 + e_2 + \ldots + e_k = n$$

under the condition that each e_i is odd.

Solution: As before, the generating function is given by

$$\begin{aligned}
g(x) &= (x+x^3+x^5+\ldots)^k \\
&= x^k(1+x^2+x^4+\ldots)^k \\
&= \frac{x^k}{(1-x^2)^k}.
\end{aligned}$$

$\square$

3. Exponential Generating Functions

Definition 3.1. The exponential generating function for the sequence $\{h_0, h_1, h_2, \ldots\}$ is defined as

$$g^{(e)}(x) = h_0 + h_1\frac{x}{1!} + h_2\frac{x^2}{2!} + h_3\frac{x^3}{3!} + \ldots$$

This function is sometimes more useful than a generating function because $x \mapsto e^x$ is a homomorphism

$$e^{x+y} = e^x \times e^y.$$

Example 3.2.

3.1. The exponential generating function for the sequence $\{1, 1, 1, \ldots\}$ is

$$g^{(e)}(x) = e^x.$$

3.2. More generally, the exponential generating function for the sequence $\{1, a, a^2, \ldots\}$ is

$$g^{(e)}(x) = e^{ax}.$$

3.3. For $n \in \mathbb{N}$ and $0 \leq k \leq n$, let $P(n,k)$ denote the number of k-permutations of an n-element set. Then,

$$P(n,k) = \frac{n!}{(n-k)!}.$$

The exponential generating function for the sequence $\{P(n,0), P(n,1), \ldots, P(n,n)\}$ is

$$\begin{aligned} g^{(e)}(x) &= P(n,0) + P(n,1)\frac{x}{1!} + \ldots + P(n,n)\frac{x^n}{n!} \\ &= 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n \\ &= (1+x)^n. \end{aligned}$$

(End of Day 21)

Theorem 3.3. *Let S be the multiset $\{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$ where $n_1, n_2 \ldots, n_k$ are non-negative integers. Let h_n denote the number of n-permutations of S. Then, the exponential generating function for (h_n) is*

$$g^{(e)}(x) = f_{n_1}(x) f_{n_2}(x) \ldots f_{n_k}(x) \tag{VI.6}$$

where

$$f_t(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^t}{t!}.$$

Proof. Note that $h_n = 0$ if $n > n_1 + n_2 + \ldots + n_k$. If $n \leq n_1 + n_2 + \ldots + n_k$, then there exist $0 \leq m_1 \leq n_1, 0 \leq m_2 \leq n_2, \ldots, 0 \leq m_k \leq n_k$ such that $n = m_1 + m_2 + \ldots + m_k$. In that case, the number of n-permutations of the multiset $S' = \{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_k \cdot a_k\}$ is given by

$$\frac{n!}{m_1! m_2! \ldots m_k!}$$

by Theorem 4.5. Therefore,

$$h_n = \sum \frac{n!}{m_1! m_2! \ldots m_k!}$$

where the sum is taken over all tuples $(m_1, m_2, \ldots, m_k)$ such that $0 \leq m_i \leq n_i$ and $m_1 + m_2 + \ldots + m_k = n$.

Now consider the coefficient of x^n in Equation VI.6. If $n > m_1 + m_2 + \ldots m_k$, both sides yield 0. If $n \leq m_1 + m_2 + \ldots + m_k$, on the LHS, it is

$$\frac{h_n}{n!} = \sum \frac{1}{m_1! m_2! \ldots m_k!}$$

This is the same as the coefficient of x^n on the RHS, so the two polynomials are equal. □

Example 3.4. 3.1. Let h_n denote the number of n-digit numbers with digits in $\{1, 2, 3\}$ such that

- The number of 1s is even.
- The number of 2s is at least three.
- The number of 3s is atmost four.

Find the exponential generating function for (h_n).

Solution: By Theorem 3.3,

$$g^{(e)}(x) = h_1(x) h_2(x) h_3(x)$$

where

$$\begin{aligned} h_1(x) &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \\ h_2(x) &= \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \\ h_3(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}. \end{aligned}$$

□

3.2. Determine the number of ways to colour the squares of a $1 \times n$ chessboard using red, white and blue such that an even number of squares are to be coloured red.

Solution: The exponential generating function for the corresponding sequence (h_n) is given by

$$\begin{aligned}
g^{(e)}(x) &= h_r(x)h_w(x)h_b(x) \\
&= \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\right)\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right)^2 \\
&= \frac{e^x+e^{-x}}{2}e^x e^x \\
&= \frac{e^{3x}+e^x}{2} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}(3^n+1)\frac{x^n}{n!}.
\end{aligned}$$

Therefore,

$$h_n = \frac{3^n+1}{2}.$$

□

3.3. Determine the number h_n of n-digit numbers with each digit odd, where the digits 1 and 3 occur an even number of times.

Solution: h_n equals the number of n-permutations of the multiset $S = \{\infty \cdot 1, \infty \cdot 3, \infty \cdot 5, \infty \cdot 7, \infty \cdot 9\}$ in which 1 and 3 occur an even number of times. By Theorem 3.3, the exponential generating function is given by

$$\begin{aligned}
g^{(e)}(x) &= h_1(x)h_3(x)h_5(x)h_7(x)h_9(x) \\
&= \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\right)^2 e^{3x} \\
&= \frac{1}{4}(e^x+e^{-x})^2 e^{3x} \\
&= \frac{(e^{2x}+e^{-2x}+2)e^{3x}}{4} \\
&= \frac{e^{5x}+e^x+2e^{3x}}{4} \\
&= \frac{1}{4}\sum_{n=0}^{\infty}(5^n+1+2\times 3^n)\frac{x^n}{n!}.
\end{aligned}$$

Therefore,

$$h_n = \frac{5^n+1+2\times 3^n}{4}.$$

□

3.4. Determine the number h_n of ways to colour the squares of a $1 \times n$ chessboard with red, white and blue such that the number of red squares is even and there is at least one blue square.

Solution: The exponential generating function for (h_n) is given by

$$\begin{aligned} g^{(e)}(x) &= h_r(x)h_w(x)h_b(x) \\ &= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)e^x \\ &= \frac{e^x + e^{-x}}{2} \times (e^x - 1)(e^x) \\ &= \frac{e^{3x} - e^{2x} + e^x - 1}{2} \\ &= 0 + \frac{1}{2}\sum_{n=1}^{\infty}(3^n - 2^n + 1)\frac{x^n}{n!} \end{aligned}$$

Therefore, $h_0 = 0$ and

$$h_n = \frac{3^n - 2^n + 1}{2}.$$

for $n \geq 1$. □

4. Solving Linear Homogeneous Recurrence Relations

Definition 4.1. A sequence $\{h_0, h_1, \dots\}$ is said to satisfy a linear recurrence relation (LRR) of order k if for each $n \geq k$, there exist constants $a_1, a_2, \dots, a_k, b_n$ (which may depend on n) such that $a_k \neq 0$ and

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \dots + a_k h_{n-k} + b_n. \tag{VI.7}$$

Example 4.2.

4.1. The Fibonacci sequence (f_n) satisfies $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$, so this is a LRR of order 2 with *constant* coefficients: $a_1 = 1, a_2 = 1, b_n = 0$.

4.2. The factorial sequence $h_n = n!$ satisfies $h_n = nh_{n-1}$, so this is an LRR of order 1 with $a_1 = n, b_n = 0$.

4.3. A geometric sequence $h_n = q^n$ satisfies $h_n = qh_{n-1}$, so this is an LRR of order 1 with $a_1 = q, b_n = 0$.

4.4. The derangement numbers (D_n) satisfy $D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$ for $n \geq 2$, so this is an LRR order 2 with $a_1 = (n-1), a_2 = (n-1)$ and $b_n = 0$. Note that they also satisfy $D_n = nD_{n-1} + (-1)^n$ which is an LRR of order 1 with $a_1 = n$ and $b_n = (-1)^n$.

Definition 4.3. The LRR Equation VI.7 is said to be homogeneous if $b_n = 0$ for all $n \in \mathbb{N}$, and is said to have constant coefficients if $a_1, a_2, \ldots, a_k$ do not depend on n.

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Theorem 4.4. *Consider a homogeneous LRR with constant coefficients*

$$h_n - a_1 h_{n-1} - a_2 h_{n-2} - \ldots - a_k h_{n-k} \qquad (n \geq k). \tag{VI.8}$$

with $a_k \neq 0$.

4.1. *If $q \neq 0$, then $h_n = q^n$ is a solution to this equation if and only if q is a root of the polynomial*
$$z^k - a_1 z^{k-1} - a_2 z^{k-2} - \ldots - a_k = 0 \tag{VI.9}$$

4.2. *If this polynomial equation has roots $\{q_1, q_2, \ldots, q_j\}$, then for any constants $c_1, c_2, \ldots, c_j$*
$$h_n = c_1 q_1^n + c_2 q_2^n + \ldots + c_j q_j^n$$
is a solution to Equation VI.8.

4.3. *If this polynomial equation has k distinct roots $\{q_1, q_2, \ldots, q_k\}$, then for any initial values of $\{h_0, h_1, \ldots, h_{k-1}\}$, there are constants $c_1, c_2, \ldots, c_k$ such that*
$$h_n = c_1 q_1^n + c_2 q_2^n + \ldots + c_k q_k^n$$
is a solution to Equation VI.8 satisfying these initial conditions. Moreover, this solution is unique.

Proof.

4.1. If $h_n = q^n$ is a solution to Equation VI.8, then
$$q^n - a_1 q^{n-1} - \ldots - a_k q^{n-k} = 0.$$
holds for all $n \geq k$. Since $q \neq 0$, q must satisfy the polynomial Equation VI.9. Conversely, if q is a root of Equation VI.9, $h_n = q^n$ is a solution.

4.2. This follows from the fact that Equation VI.8 is linear.

4.3. Suppose Equation VI.9 has k distinct roots $\{q_1, q_2, \ldots, q_k\}$, and we are given initial conditions
$$h_0 = b_0, h_1 = b_1, \ldots, h_{k-1} = b_{k-1}$$
To find $(c_1, c_2, \ldots, c_k)$ such that $h_n = \sum_{i=1}^k c_i q_i^n$ is a solution satisfying the initial conditions, we need to solve the system of equations
$$\begin{aligned} c_1 + c_2 + \ldots + c_k &= b_0 \\ c_1 q_1 + c_2 q_2 + \ldots + c_k q_k &= b_1 \\ \vdots &= \vdots \\ c_1 q_1^{k-1} + c_2 q_2^{k-1} + \ldots + c_k q_k^{k-1} &= b_{k-1} \end{aligned}$$

Now observe that

$$det \begin{pmatrix} 1 & 1 & \dots & 1 \\ q_1 & q_2 & \dots & q_k \\ q_1^2 & a_2^2 & \dots & q_k^2 \\ \vdots & \vdots & \vdots & \vdots \\ q_1^{k-1} & q_2^{k-1} & \dots & q_k^{k-1} \end{pmatrix} = \prod_{1 \le i < j \le k} (q_j - q_i) \neq 0$$

since the roots are distinct. Therefore, the system has a unique solution.

□

Definition 4.5. Given a homogeneous linear recurrence relation

$$h_n - a_1 h_{n-1} - a_2 h_{n-2} - \dots - a_k = 0$$

with constant coefficients, the polynomial equation

$$z^k - a_1 z^{k-1} - a_2 z^{k-2} - \dots - a_k = 0$$

is called the characteristic equation of the LRR and its roots are called the characteristic roots.

Example 4.6. Solve the recurrence relation

$$h_n = 2h_{n-1} + h_{n-2} - 2h_{n-3} \qquad (n \ge 3)$$

subject to the initial values $h_0 = 1, h_1 = 2, h_2 = 0$.

Solution: The characteristic equation is

$$x^3 - 2x^2 - x + 2 = 0$$

Factoring, we get $x(x^2-1) - 2(x^2-1) = (x-2)(x-1)(x+1)$. So the characteristic roots are $1, -1$ and 2. By Theorem 4.4, the solution to the LRR is given by

$$h_n = c_1(1)^n + c_2(-1)^n + c_3(2)^n$$

The initial conditions give

$$\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ c_1 - c_2 + 2c_3 &= 2 \\ c_1 + c_2 + 4c_3 &= 0 \end{aligned}$$

This yields

$$c_1 = 2, c_2 = \frac{-2}{3}, c_3 = \frac{-1}{3}.$$

So that

$$h_n = 2 - \frac{2}{3}(-1)^n - \frac{1}{3}2^n.$$

□

Example 4.7. Words of length n, using only the three letters a, b, c, are to be transmitted over a communication channel subject to the condition that no word in which two as appear consecutively is to be transmitted. Determine the number of words allowed by the communication channel.

Solution: Let h_n denote the number of allowed words of length n. We have

$$h_0 = 1 \text{ (the empty word) and } h_1 = 3.$$

If $n \geq 2$, and w is word of length n, consider the cases:

- If w starts with a b or c, then deleting that alphabet yields any words of length $(n-1)$.
- If w starts with a a, then the second letter is a b or c. Deleting these two alphabets yields any word of length $(n-2)$.

Therefore, the recurrence relation satisfied by h_n is

$$h_n = 2h_{n-1} + 2h_{n-2}$$

The characteristic equation is

$$x^2 - 2x - 2 = 0$$

whose roots are $q_1 = (1+\sqrt{3})$ and $q_2 = (1-\sqrt{3})$. Therefore, the general solution of the LRR is given by

$$h_n = c_1(1+\sqrt{3})^n + c_2(1-\sqrt{3})^n$$

With the initial conditions, we get

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1(1+\sqrt{3}) + c_2(1-\sqrt{3}) &= 3. \end{aligned}$$

Solving gives

$$c_1 = \frac{2+\sqrt{3}}{2\sqrt{3}} \text{ and } c_2 = \frac{-2+\sqrt{3}}{2\sqrt{3}}.$$

Therefore,

$$h_n = \frac{2+\sqrt{3}}{2\sqrt{3}}(1+\sqrt{3})^n + \frac{-2+\sqrt{3}}{2\sqrt{3}}(1-\sqrt{3})^n$$

□

If the characteristic roots $\{q_1, q_2, \ldots, q_k\}$ of the characteristic equation are not distinct, then $h_n = c_1q_1^n + c_2q_2^n + \ldots + c_kq_k^n$ is not a general solution.

Example 4.8. The recurrence relation

$$h_n = 4h_{n-1} - 4h_{n-2} \qquad (n \geq 2)$$

with $h_0 = 1, h_1 = 3$ has characteristic equation

$$x^2 - 4x - 4 = 0$$

which as a repeated root 2. If $h_n = c2^n$ were the general solution, we would require

$$c = 1 \text{ and } 2c = 3.$$

This is impossible, so $h_n = c2^n$ is not a general solution.

Example 4.9. Consider the differential equation

$$y'' - 4y' - 4 = 0.$$

with initial conditions $y(0) = a, y'(0) = b$. If $y = e^{qx}$ is a solution, then q must satisfy $q^2 - 4q - 4 = 0$. Here, $y = e^{2x}$ is one solution, but so is $y = xe^{2x}$. Indeed, the general solution is given by

$$y = c_1 e^{2x} + c_2 x e^{2x}.$$

To solve for the initial conditions, we get

$$\begin{aligned} c_1 &= a \\ 2c_1 + c_2 &= b. \end{aligned}$$

This system has a unique solution $c_1 = a, c_2 = b - 2a$.

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We try the same trick for LRRs whose characteristic equation has repeated roots.

Example 4.10. Consider the LRR

$$h_n = 4h_{n-1} - 4h_{n-2} \qquad (n \geq 2)$$

with $h_0 = a, h_1 = b$. As before, the characteristic root is 2, so $h_n = 2^n$ is a solution. Consider $h_n = n2^n$, then

$$\begin{aligned} 4h_{n-1} - 4h_{n-2} &= 4(n-1)2^{n-1} - 4(n-2)2^{n-1} \\ &= 42^{n-2}((n-1)2 - (n-2)) \\ &= 42^{n-2}(n) \\ &= n2^n = h_n. \end{aligned}$$

Hence,

$$h_n = c_1 2^n + c_2 n 2^n$$

is a solution to the LRR. Now imposing the initial conditions gives

$$\begin{aligned} c_1 &= a \\ 2c_1 + c_2 &= b \end{aligned}$$

which has a unique solution $c_1 = a, c_2 = b - 2a$. Therefore,

$$h_n = a2^n + (b - 2a)n2^n$$

is the solution to the LRR.

Theorem 4.11. *Suppose the characteristic equation for an LRR is*

$$(x - q)^s = 0.$$

Then, the general solution to the LRR is

$$h_n = c_1 q^n + c_2 n q^n + c_3 n^2 q^n + \ldots + c_s n^{s-1} q^n.$$

Theorem 4.12. *Let $q_1, q_2, \ldots, q_t$ be the distinct roots of the characteristic equation of a homogeneous LRR with constant coefficients:*

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \ldots + a_k h_{n-k} \qquad (n \geq k)$$

with $a_k \neq 0$. Suppose each q_i has multiplicity s_i as a root of the characteristic equation. For each $1 \leq i \leq t$, define

$$H_n^{(i)} = c_{i,1} q_i^n + c_{i,2} n q_i^n + \ldots + c_{i,s_i} n^{s_i - 1} q_i^n.$$

The general solution to the LRR is given by

$$h_n = H_n^{(1)} + H_n^{(2)} + \ldots + H_n^{(t)}.$$

Example 4.13. Solve the recurrence relation

$$h_n = -h_{n-1} + 3h_{n-2} + 5h_{n-3} + 2h_{n-4} \qquad (n \geq 4)$$

subject to the initial values $h_0 = 1, h_1 = 0, h_2 = 1, h_3 = 2$.

Solution: The characteristic equation is given by

$$x^4 + x^3 - 3x^2 - 5x - 2 = 0.$$

This has roots $-1, -1, -1, 2$. So the general solution is given by

$$h_n = c_1(-1)^n + c_2 n(-1)^n + c_3 n^2 (-1)^n + c_4 (2)^n.$$

The initial conditions give

$$\begin{aligned}
c_1 + c_4 &= 1 \\
-c_1 - c_2 - c_3 + 2c_4 &= 0 \\
c_1 + 2c_2 + 4c_3 + 4c_4 &= 1 \\
-c_1 - 3c_2 - 9c_3 + 8c_4 &= 2.
\end{aligned}$$

The unique solution is given by

$$c_1 = \frac{7}{9}, c_2 = \frac{-3}{9}, c_3 = 0, c_4 = \frac{2}{9}.$$

Thus the general solution is given by

$$h_n = \frac{7}{9}(-1)^n - \frac{1}{3}n(-1)^n + \frac{2}{9}2^n.$$

$\square$

a. Using the Generating Function

Recall that if $n \in \mathbb{N}$ and $r \neq 0$, then

$$\frac{1}{1-rx}^{n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-rx)^k$$

which converges if $|rx| < 1$. Note that

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k-1}$$

by Example 5.2, so this is equivalent to

$$\frac{1}{(1-rx)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} r^k x^k$$

which converges if $|x| < \frac{1}{|r|}$.

Example 4.14. Solve the recurrence relation

$$h_n = 5h_{n-1} - 6h_{n-2} \qquad (n \geq 2)$$

with initial values $h_0 = 1, h_1 = -2$.

Solution: Let $g(x) = h_0 + h_1 x + h_2 x^2 + \dots$ be the generating function. Comparing with the equation

$$h_n - 5h_{n-1} + 6h_{n-2} = 0$$

makes us try

$$\begin{aligned} g(x) &= h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \dots \\ 5xg(x) &= 5h_0 x + 5h_1 x^2 + 5h_2 x^3 + 5h_3 x^4 + \dots \\ 6x^2 g(x) &= 6h_0 x^2 + 6h_1 x^3 + 6h_2 x^4 + 6h_3 x^5 + \dots \end{aligned}$$

So consider

$$\begin{aligned} g(x) - 5xg(x) + 6x^2 g(x) &= h_0 + (h_1 - 5h_0)x + (h_2 - 5h_1 + 6h_0)x^2 + (h_3 - 5h_2 + 6h_1)x^3 + \dots \\ &= h_0 + (h_1 - 5h_0)x \\ &= 1 + (-2 - 5)x = 1 - 7x. \end{aligned}$$

Hence, the generating function is

$$g(x) = \frac{1 - 7x}{1 - 5x + 6x^2}$$

Observe that $(1 - 5x + 6x^2) = (1 - 2x)(1 - 3x)$, so

$$g(x) = \frac{c_1}{(1 - 2x)} + \frac{c_2}{(1 - 3x)}.$$

Equivalently, $(1 - 3x)c_1 + (1 - 2x)c_2 = (1 - 7x)$, so comparing coefficients gives

$$\begin{aligned} c_1 + c_2 &= 1 \\ -3c_1 - 2c_2 &= -7. \end{aligned}$$

Solving this system gives $c_1 = 5, c_2 = -4$. Therefore

$$\begin{aligned} g(x) &= \frac{5}{(1 - 2x)} - \frac{4}{(1 - 3x)} \\ &= 5\left(\sum_{k=0}^{\infty} 2^k x^k\right) - 4\left(\sum_{k=0}^{\infty} 3^k x^k\right). \end{aligned}$$

Therefore,

$$h_n = 5 \times 2^n - 4 \times 3^n.$$

□

Example 4.15. Let (h_n) be a sequence satisfying the recurrence relation

$$h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0 \qquad (n \geq 3)$$

subject to the initial conditions $h_0 = 0, h_1 = 1, h_2 = -1$. Find a general formula for h_n.

Solution: Let $g(x) = h_0 + h_1x + h_2x^2 + \ldots$ be the associated generating function and consider

$$\begin{aligned}
(1 + x - 16x^2 + 20x^3)g(x) &= h_0 + h_1x + h_2x^2 + h_3x^3 + \ldots \\
&\quad + 0 + h_0x + h_1x^2 + h_2x^3 + h_4x^4 + \ldots \\
&\quad + 0 + 0 - 16h_0x^2 - 16h_1x^3 - 16h_2x^3 - \ldots \\
&\quad + 0 + 0 + 0 + 20h_0x^3 + 20h_1x^4 + \ldots \\
&= h_0 + (h_1 + h_0)x + (h_2 + h_1 - 16h_0)x^2 \\
&= 0 + (1 + 0)x + (-1 + 1 - 0)x^2 \\
&= x.
\end{aligned}$$

Therefore,

$$g(x) = \frac{x}{1 + x - 16x^2 + 20x^3}.$$

Note that $(1 + x - 16x^2 + 20x^3) = (1 - 2x)^2(1 + 5x)$, so

$$g(x) = \frac{c_1}{(1 - 2x)} + \frac{c_2}{(1 - 2x)^2} + \frac{c_3}{(1 + 5x)}$$

Multiplying, we get

$$x = (1 - 2x)(1 + 5x)c_1 + (1 + 5x)c_2 + (1 - 2x)^2c_3.$$

Hence, we get

$$\begin{aligned}
c_1 + c_2 + c_3 &= 0 \\
3c_1 + 5c_2 - 4c_3 &= 1 \\
-10c_1 + 4c_3 &= 0.
\end{aligned}$$

Solving gives us

$$c_1 = \frac{-2}{49}, c_2 = \frac{7}{49}, c_3 = \frac{-5}{49}$$

so that

$$\begin{aligned}
g(x) &= \frac{-2}{49}\frac{1}{(1 - 2x)} + \frac{7}{49}\frac{1}{(1 - 2x)^2} - \frac{5}{49}\frac{1}{(1 + 5x)} \\
&= \frac{-2}{49}\left(\sum_{k=0}^{\infty} 2^kx^k\right) + \frac{1}{7}\left(\sum_{k=0}^{\infty}\binom{k+1}{k}2^kx^k\right)
\end{aligned}$$

$$-\frac{5}{49}\left(\sum_{k=0}^{\infty}(-5)^kx^k\right)$$

Therefore,

$$h_n = \frac{-2}{49}2^n + \frac{7}{49}(n+1)2^n - \frac{5}{49}(-5)^n.$$

$\square$

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Remark 4.16.

4.1. Given an LRR of order k with constant coefficients, the generating function $g(x)$ is always a rational function of the form $p(x)/q(x)$ where $q(x)$ is a polynomial of degree k.

4.2. We may then express $g(x)$ as a sum, where each term is of the form

$$\frac{c}{(1-rx)^t}$$

where $r \neq 0$ and c is a constant. We may then use the power series expansion

$$\frac{1}{(1-rx)^t} = \sum_{k=0}^{\infty} \binom{t+k-1}{k-1} r^k x^k$$

and combine like terms to obtain the terms of the sequence.

4.3. In the preceding example, the LRR is given by

$$h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0 \qquad (n \geq 3).$$

The characteristic equation is given by $r(x) = 0$ where

$$r(x) = x^3 + x^2 - 16x + 20$$

while the generating function is given by

$$g(x) = \frac{x}{1 + x - 16x^2 + 20x^3} = \frac{p(x)}{q(x)}.$$

Observe that

$$x^3 r(1/x) = q(x).$$

Thus, the roots of the characteristic equation $r(x) = 0$ are $2, -2, -5$ because $r(x) = (x-2)^2(x+5)$. It follows that

$$q(x) = x^3 r(1/x) = x^3 \left(\frac{1}{x} - 2\right)^2 \left(\frac{1}{x} + 5\right) = (1-2x)^2(1+5x)$$

4.4. This relationship holds in general: Consider a sequence (h_n) satisfying a LRR given by

$$h_n + a_1 h_{n-1} + a_2 h_{n-2} + \ldots + a_k h_{n-k} = 0 \qquad (n \geq k)$$

of order k with initial values $h_0, h_1, \ldots, h_{k-1}$. Let $g(x)$ be the generating function for the sequence, then there are polynomials $p(x)$ and $q(x)$ such that

$$g(x) = \frac{p(x)}{q(x)}.$$

Moreover, $q(x)$ has degree k and $p(x)$ has degree less than k. Indeed,

$$q(x) = 1 + a_1 x + a_2 x^2 + \ldots + a_k x^k$$

and

$$p(x) = h_0 + (h_1 + a_1 h_0)x + (h_2 + a_1 h_1 + a_2 h_0)x^2 + \ldots$$
$$+ (h_{k-1} + a_1 h_{k-2} + \ldots + a_{k-1} h_0)x^{k-1}.$$

The characteristic equation for this sequence is

$$r(x) = x^k + a_1 x^{k-1} + \ldots + a_k = 0.$$

Hence, $q(x) = x^k r(1/x)$. Thus, if the roots of $r(x) = 0$ are $q_1, q_2, \ldots, q_k$, then

$$r(x) = (x - q_1)(x - q_2)\ldots(x - q_k)$$

then $q(x) = (1 - q_1 x)(1 - q_2 x)\ldots(1 - q_k x)$

4.5. The converse is also true: Given two polynomials $p(x)$ and $q(x)$ where $q(x)$ has degree k and non-zero constant term and $p(x)$ has degree less than k, then there is a sequence (h_n) satisfying a LRR with constant coefficients of degree k whose generating function is given by

$$g(x) = \frac{p(x)}{q(x)}.$$

5. Nonhomogeneous Recurrence Relations

We wish to solve recurrence relations of the form

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \ldots + a_k h_{n-k} + b_n$$

where $b_n \neq 0$.

Example 5.1. The Towers of Hanoi Puzzle: There are three pegs and n circular disks of increasing size on one peg with the largest disk at the bottom. We need to transfer the disks onto another peg with the condition that at no time is one allowed to place a larger disk on a smaller one. How many moves are needed?

Solution: Let h_n be the number of moves to transfer n disks. Then, to transfer n disks, one first needs to transfer $(n-1)$ disks onto Peg 2, place the largest disk on Peg 3, and then transfer all the $(n-1)$ disks onto Peg 3. Therefore,

$$h_n = 2h_{n-1} + 1.$$

Note that $h_0 = 0$. Hence, we enumerate

$$h_n = 2(2h_{n-2} + 1) + 1 = 2^2 h_{n-2} + 2 + 1 = 2^3 h_{n-3} + 2^2 + 2 + 1$$

and so on. Therefore, with $h_0 = 0$,

$$h_n = 2^{n-1} + 2^{n-2} + \ldots + 1 = \frac{2^n - 1}{2 - 1} = 2^n - 1.$$

□

We may try another method using generating functions.

Solution: Suppose $g(x) = \sum_{n=0}^{\infty} h_n x^n$ is the generating function for (h_n). Then,

$$\begin{aligned}
g(x) &= h_0 + h_1 x + h_2 x^2 + \ldots \\
-2xg(x) &= -2h_0 x - 2h_1 x^2 - 2h_2 x^3 - \ldots \\
\Rightarrow (1-2x)g(x) &= h_0 + x + x^2 + x^3 + \ldots = \frac{x}{1-x} \\
\Rightarrow g(x) &= \frac{x}{(1-x)(1-2x)} \\
&= \frac{c_1}{(1-x)} + \frac{c_2}{(1-2x)}.
\end{aligned}$$

Solving for c_1, c_2, we get

$$g(x) = \frac{1}{1-2x} - \frac{1}{1-x} = \sum_{n=0}^{\infty} (2^n - 1) x^n.$$

Therefore, $h_n = 2^n - 1$. □

Remark 5.2. To solve a nonhomogeneous recurrence relation as above, we use a method from ODEs:

5.1. Solve the corresponding homogeneous recurrence relation.
5.2. Find a particular solution to the nonhomogeneous relation.
5.3. Add the two and determine constants to satisfy the initial conditions.

The difficulty here is in Step (2), and we use the following heuristic:

5.1. If b_n is a polynomial of degree k in n, then try h_n to also be a polynomial of degree k. For instance,
   - If b_n is constant, try $h_n = r$.
   - If $b_n = an + b$, try $h_n = rn + s$.
   - If $b_n = an^2 + bn + c$, try $h_n = rn^2 + sn + t$.

5.2. If $b_n = d^n$, try $h_n = pd^n$.

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Example 5.3. Solve $h_n = 3h_{n-1} - 4n$ for $n \geq 1$ with $h_0 = 2$.

Solution:

5.1. The homogeneous equation is $h_n = 3h_{n-1}$ whose characteristic equation is

$$(x-3) = 0.$$

Hence, $h_n = c3^n$ is a solution to the homogeneous equation.

5.2. Suppose $h_n = rn + s$ is a particular solution, then

$$rn + s = 3(r(n-1) + s) - 4n = (3r-4)n + (3s-3r).$$

Hence,

$$\begin{aligned} r &= 3r - 4 \\ s &= 3s - 3r. \end{aligned}$$

Solving, we get $r = 2, s = 3$ so that $h_n = 2n + 3$.

5.3. Hence, the general solution is given by

$$h_n = c3^n + 2n + 3.$$

If $n = 0$, we get $2 = c + 3$ so that $c = -1$. Therefore,

$$h_n = -3^n + 2n + 3$$

is a solution.

□

Example 5.4. Solve $h_n = 2h_{n-1} + 3^n$ for $n \geq 1$, with $h_0 = 2$.

Solution:

5.1. The homogeneous equation is $h_n = 2h_{n-1}$ whose general solution is

$$h_n = c2^n.$$

5.2. For a particular solution, we try $h_n = p3^n$. Then,

$$\begin{aligned} p3^n &= 2p3^{n-1} + 3^n \\ \Rightarrow 3p &= 2p + 3 \\ \Rightarrow p &= 3. \end{aligned}$$

5.3. Therefore, a general solution is given by

$$h_n = c2^n + 3^{n+1}.$$

At $n = 0$, $h_0 = 2$, so $c = -1$. Therefore,

$$h_n = -2^n + 3^{n+1}$$

is a solution.

□

We may try another solution using a generating function.

Solution: Let $g(x) = \sum_{n=0}^{\infty} h_n x^n$ be the generating function for (h_n). Then,

$$\begin{aligned} g(x) &= h_0 + h_1 x + h_2 x^2 + \ldots \\ 2xg(x) &= 2h_0 x + 2h_1 x^2 + 2h_2 x^3 + \ldots \\ \Rightarrow (1-2x)g(x) &= h_0 + (h_1 - 2h_0)x + (h_2 - 2h_1)x^2 + \ldots \\ &= 2 + 3x + 3^2x^2 + \ldots + 3^n x^n \\ &= 1 + \sum_{n=0}^{\infty} (3x)^n \\ &= 1 + \frac{1}{1-3x} \\ \Rightarrow g(x) &= \frac{1}{(1-2x)} + \frac{1}{(1-2x)(1-3x)} \\ &= \frac{-1}{1-2x} + \frac{3}{1-3x} \\ &= \sum_{n=0}^{\infty} (-2^n + 3^{n+1})x^n. \end{aligned}$$

Therefore, $h_n = -2^n + 3^{n+1}$. $\square$

Example 5.5. Solve $h_n = 3h_{n-1} + 3^n$ for $n \geq 1$ with $h_0 = 2$.

Solution: We use a generating function, as before:

$$\begin{aligned}
g(x) &= h_0 + h_1 x + h_2 x^2 + \dots \\
3xg(x) &= 3h_0 x + 3h_1 x^2 + 3h_2 x^3 + \dots \\
\Rightarrow (1-3x)g(x) &= h_0 + (h_1 - 3h_0)x + (h_2 - 3h_1)x^2 + \dots \\
&= 2 + 3x + 3^2 x^2 + \dots \\
&= 1 + \frac{1}{1-3x} \\
\Rightarrow g(x) &= \frac{1}{1-3x} + \frac{1}{(1-3x)^2} \\
&= \sum_{n=0}^{\infty} (3^n + \binom{n+1}{n} 3^n) x^n. \\
&= \sum_{n=0}^{\infty} (n+2) 3^n x^n.
\end{aligned}$$

Therefore, $h_n = (n+2)3^n$. $\square$

VII. Introduction to Graph Theory

1. Basic Properties

Definition 1.1. A graph is a pair $G = (V, E)$ where V is a set, and

$$E \subset \{(x, y) \in V \times V : x \neq y\}.$$

1.1. Elements of V are called vertices.

1.2. The cardinality $|V|$ is called the order of the graph.

1.3. Elements of E is called edges.

1.4. If $\alpha = (x, y) \in E$, then we say that α joins x to y. Moreover, we say that x and y are adjacent, and we say that α is incident on both x and y.

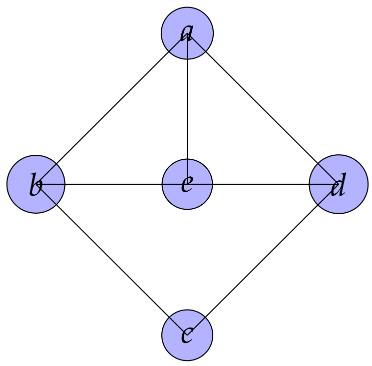
1.5. If $(y, x) \in E$ whenever $(x, y) \in E$, then we say that the graph is undirected, otherwise it is called directed.

For the moment, we will only consider undirected graphs of finite order (finite graphs). Therefore, we will write $\alpha = \{x, y\} = \{y, x\}$ for an edge.

Example 1.2. Let $V = \{a, b, c, d, e\}$ and

$$E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}, \{a, e\}, \{b, e\}, \{d, e\}\}.$$

This may be represented as a diagram



Example 1.3. Let V be any set, and E be the set of all 2-subsets of V (In other words, any two distinct points in V are adjacent to each other). Then, $G = (V, E)$ is called the complete graph of order $n := |V|$, and is denoted by K_n (See Definition 3.2).

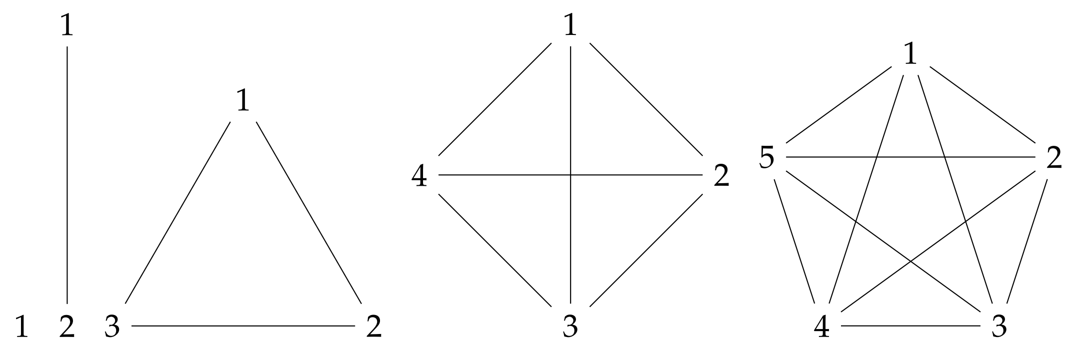
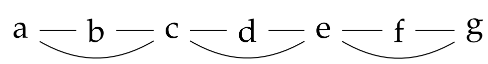


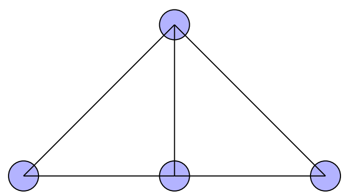
Figure VII.1.: K_1, K_2, K_3, K_4 and K_5

Remark 1.4.

1.1. When drawing a graph on paper, it is not necessary that edges be drawn as straight lines. For instance, this is a graph:



1.2. Note that it is possible to draw K_4 as follows:



Here, no two edges intersect (except at vertices).

1.3. Is it possible to draw K_5 like this?

Definition 1.5. A graph is called planar if it can be represented as a drawing on the plane such that no two edges intersect (except at vertices). Such a drawing is called a planar representation of the graph.

Definition 1.6. 1.1. E can contain sets of the form $\{x, x\}$.

1.2. Given a point $x \in V$, the degree (or valence) of x, denoted $\deg(x)$, is the number of edges that are incident on x. Note that $\{x, x\}$ contributes 2 to $\deg(x)$.

1.3. Given a graph G of order n, we associate a non-increasing sequence $d_1, d_2, \ldots, d_n$ of degrees of vertices. When ordered such that $d_1 \geq d_2 \geq \ldots \geq d_n$, this is called the degree sequence of G.

Theorem 1.7. *Let G be a general graph. Then, the sum $(d_1 + d_2 + \ldots + d_n)$ is an even number.*

Proof. Each edge contributes 2 to the sum. Therefore, the sum is $2|E|$. $\square$

Definition 1.8. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. A morphism is a pair $\theta = (\theta_V, \theta_E)$ where

$$\theta_V : V \to V' \text{ and } \theta_E : E \to E'$$

are two functions such that

$$\theta_E(\{x, y\}) = \{\theta_V(x), \theta_V(y)\}.$$

We say that $\theta : G \to G'$ is an isomorphism if both θ_V and θ_E are bijective. In that case, $\theta^{-1} := (\theta_V^{-1}, \theta_E^{-1}) : G' \to G$ is also a graph morphism. If this happens, we write

$$G \cong G'.$$

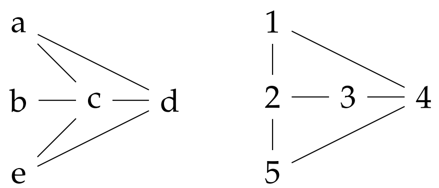
(End of Day 26)

Remark 1.9. Moreover, if $\theta = (\theta_V, \theta_E) : G \to G'$ is an isomorphism, then

- $|V| = |V'|$ and $|E| = |E'|$.
- If $x \in V$, then $\deg(\theta_V(x)) = \deg(x)$.
- Hence, the degree sequences of G and G' must be the same.

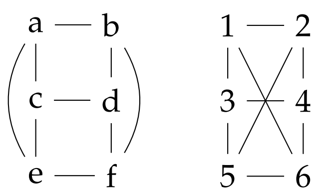
Example 1.10.

1.1. The two graphs shown below have the same number of vertices and edges, but are not isomorphic.



Proof. $\deg(b) = 1$ but there is no vertex in V' with this property. $\square$

1.2. Consider the two graphs shown below



Both graphs have degree sequences $(3, 3, 3, 3, 3, 3)$.

Proof. Note that (a, c, e) forms a 'triangle' in that any two pairs are adjacent. There is no such triangle in G', so G and G' are not isomorphic. $\square$

Definition 1.11. Let $G = (V, E)$ be a graph of order n, and write $V = \{a_1, a_2, \ldots, a_n\}$ be ordered. For each $1 \leq i, j \leq n$, set

$$a_{i,j} = \begin{cases} 1 & : \text{ if } a_i \text{ is adjacent to } a_j \\ 0 & : \text{ otherwise.} \end{cases}$$

The matrix $A = A_G = (a_{i,j})$ is called the adjacency matrix of G.

Remark 1.12. Note that

1.1. $a_{i,j} = a_{j,i}$ for all $1 \leq i, j \leq n$. In other words, $A^t = A$.

1.2. The definition depends on the ordering on V.

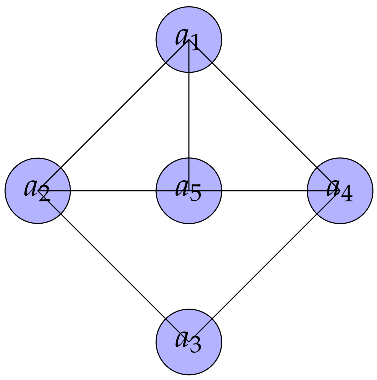
1.3. $\deg(a_i) = \sum_{j=1}^{n} a_{i,j}$.

1.4. $e = (1, 1, \ldots, 1)$, then

$$Ae = \begin{pmatrix} \deg(a_1) \\ \deg(a_2) \\ \vdots \\ \deg(a_n) \end{pmatrix}.$$

So with some ordering of vertices, Ae is the degree sequence of G.

Example 1.13. Consider the graph



The adjacency matrix is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Definition 1.14. Let $\sigma \in S_n$ be a permutation on $\{1, 2, \ldots, n\}$. The permutation matrix associated to σ is given by

$$P_\sigma = \begin{pmatrix} e_{\sigma^{-1}(1)} & e_{\sigma^{-1}(2)} & \cdots & e_{\sigma^{-1}(n)} \end{pmatrix}$$

Note that P sends the i^{th} column of the identity matrix I to the column $\sigma(i)$.

Remark 1.15.

1.1. If $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, then

$$P_\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

1.2. Note that any such matrix is orthogonal. i.e. P is invertible and $P^t = P^{-1}$. Hence, $\det(P) = \pm 1$.

Theorem 1.16. *Let G be a graph with adjacency matrix A. If P is a permutation matrix then P^tAP is the adjacency matrix of some graph that is isomorphic to G. Conversely, for any graph H that is isomorphic to G there exists a permutation matrix P such that P^tAP is the adjacency matrix of H.*

Proof. Let $G = (V, E)$ and write $V = \{1, 2, \ldots, n\}$ and $A = A_G$. If $\sigma : V \to V$ is a permutation, then let $P = P_\sigma$ as above. Then,

$$Pe_j = e_{\sigma^{-1}(j)}$$

for all $1 \leq j \leq n$. Let $B := P^{-1}A_GP$, then observe that

$$B^t = B.$$

Also, if $1 \leq i, j \leq n$, and $k = \sigma(i), \ell = \sigma(j)$, then

$$\begin{aligned} B_{k,\ell} &= e_k^t B e_\ell \\ &= e_k^t P^t A P e_\ell \\ &= (Pe_k)^t A e_{\sigma^{-1}(\ell)} \\ &= e_{\sigma^{-1}(k)}^t A e_{\sigma^{-1}(\ell)} \\ &= e_i^{-1} A e_j = a_{i,j}. \end{aligned}$$

Hence,

$$b_{\sigma(i),\sigma(j)} = a_{i,j} \tag{VII.1}$$

for all $1 \leq i, j \leq n$. Thus, B has entries 0 and 1, and 0 along the diagonal, so is the adjacency matrix of some graph H. Moreover, by Equation VII.1, $\{i, j\}$ is an edge of G if and only if $\{\sigma(i), \sigma(j)\}$ is an edge of H. Hence, $G \cong H$.

Conversely, if $G \cong H$, we maay assume that $H = (V, E')$ and that the isomorphism $\theta = (\theta_V, \theta_E)$ is induced by a permutation $\theta_V : V \to V$. Let P be the permutation matrix associated to $\sigma := \theta_V$. Then, by the above calculation

$$(P^tAP)_{\sigma(i),\sigma(j)} = a_{i,j}.$$

Hence, the $0-1$ matrix P^tAP has a non-zero entry at $(\sigma(i), \sigma(j))$ if and only if A has a non-zero entry at (i, j), which happens iff G has $\{i, j\}$ as an edge. Hence, P^tAP is the adjacency matrix of H. $\square$

Corollary 1.17. *For two graphs G and H, $G \cong H$ if and only if there is a permutation matrix P such that $P^{-1} A_G P = A_H$.*

(End of Day 27)

Definition 1.18. A sequence of m edges of the form

$$\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{m-1}, x_m\}$$

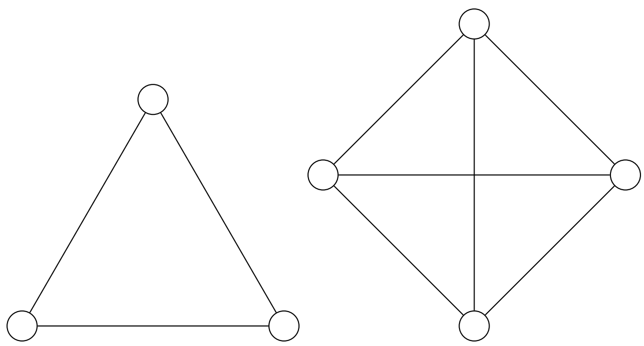
is called a <u>walk</u> of length m that <u>joins</u> x_0 to x_m. We denote this by

$$[x_0 - x_1 - x_2 - \ldots - x_m].$$

1.1. We say that the walk is <u>closed</u> if $x_0 = x_m$. Otherwise, we say that it is <u>open</u>.
1.2. A walk with distinct edges is called a <u>trail</u>.
1.3. A walk with distinct vertices (except perhaps $x_0 = x_m$) is called a <u>path</u>.
1.4. A closed path is called a <u>cycle</u>.
1.5. The graph is said to be <u>connected</u> if for each pair of vertices x and y, there is a walk (equivalently, a path) joining x to y. Otherwise, G is said to be <u>disconnected</u>.
1.6. If $G = (V, E)$ is a connected graph, for any pair $x, y \in V$, define $d(x, y)$ to be the length of the shortest walk from x to y (with the understanding that $d(x, x) = 0$ for all $x \in V$). Note that $d(x, y)$ must be the length of a path.

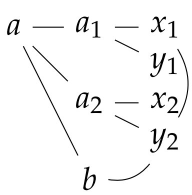
Example 1.19.

1.1. Consider the graph



This is disconnected.

1.2. Consider the graph



Then,
$$d(b, x_1) = \min\{[b - a - a_1 - x_1], [b - y_2 - x_1]\} = 3.$$

Theorem 1.20. *The $(i, j)^{th}$ entry of A^m is the number of walks of length m from i to j.*

Proof. We induct on m. If $m = 1$, this is clear by definition. Suppose we have proved it for A^j for $j \leq m - 1$. Write
$$A^r = (a^r_{i,j}).$$

Since $A^m = A^{m-1}A$, we have
$$a^m_{i,j} = \sum_{k=1}^{m} a^{m-1}_{i,k} a_{k,j}.$$

Now, the number of walks of length m from i to j is given by
$$\sum_{k=1}^{n} (\text{Number of walks from } i \text{ to } k \text{ of length } m - 1) \times a_{k,j}$$

By induction, this is precisely the RHS of the previous equation. □

Definition 1.21.

1.1. A multigraph is a pair $G = (V, E)$ where V is a set and E is multiset, each element of which is a 2-subset of V.

1.2. The multiplicity of an edge $\alpha = \{x, y\}$ is the number of times $\{x, y\}$ occurs in E.

1.3. If we allow a multigraph to have loops (an edge of the form $\{x, x\}$), then such a multigraph is called a general graph.

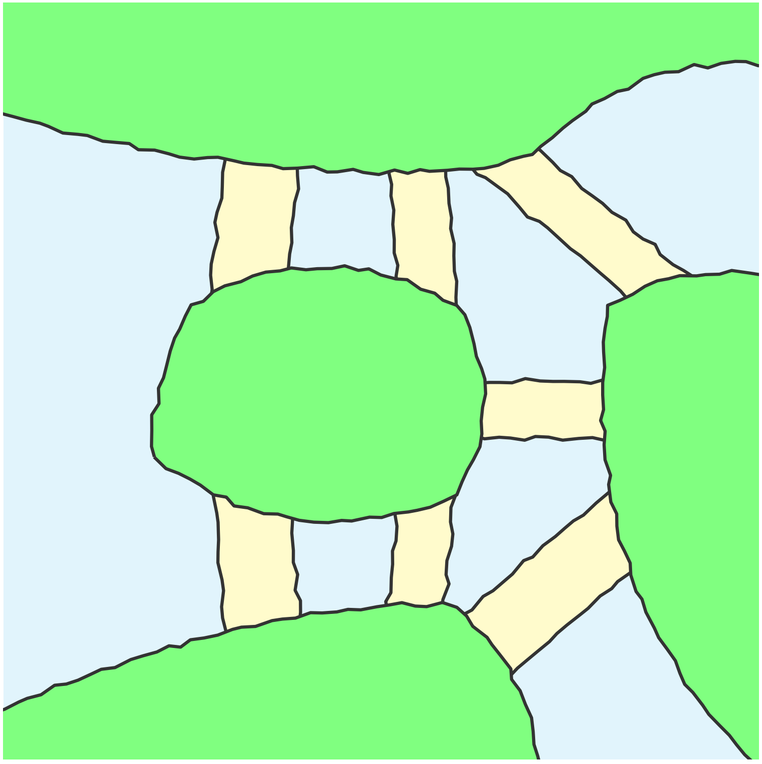
1.4. Given a general graph, we may define the adjacency matrix as above, except we define
$$a_{i,j} = \text{the number of edges joining } a_i \text{ to } a_j.$$

Note that:

- The corresponding matrix $A = A_G$ is once again symmetric $(A^t = A)$ and has non-negative integer entries.
- However, it may happen that
$$a_{i,i} \neq 0$$
for some $1 \leq i \leq n$.
- Moreover, it is no longer a $0 - 1$ matrix.
- An analogue of Theorem 1.16 still holds for general graphs.

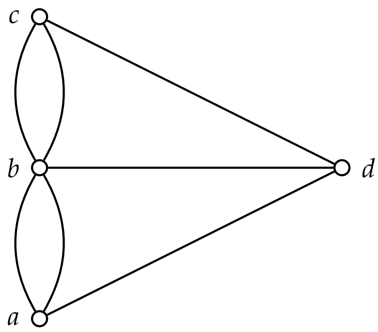
2. Eulerian Trails

Remark 2.1. In 1736, Euler solved the *Königsberg Bridge Problem*: The city of Königsberg is located on the banks and on two islands of the Pregel River, with the four parts of the city connected by seven bridges.



Question: was to find a 'walk' around the town so that each bridge if crossed once and only once, ending the walk where it began.

The Königsberg bridge problem can be represented as a general graph as follows:



Definition 2.2. A trail (a walk with distinct edges) in a general graph is called Eulerian provided it contains every edge of G.

Lemma 2.3. *If a general graph G admits an Eulerian trail, then each vertex must have even degree.*

Proof. Suppose an Eulerian trail is of the form

$$[x_0 \xrightarrow{e_0} x_1 \xrightarrow{e_1} x_2 \xrightarrow{e_2} \dots \xrightarrow{e_{n-1}} x_n \xrightarrow{e_n} x_0].$$

with possible repetition amongst the x_i, but not amongst the e_i. For a given edge $e = \{x, y\}$, there must be two other (distinct from e) edges $f = \{a, x\}$ and $g = \{y, b\}$. Hence, for each vertex x, there is one edge to 'enter' it, and one edge to 'leave' it. Each such pairing introduces 2 to $\deg(x)$, so $\deg(x)$ is even. □

Example 2.4. In the Königsberg bridge problem, $\deg(d) = 3$ so there is no Eulerian trail.

Lemma 2.5. *Let $G = (V, E)$ be a general graph in which each vertex has even degree. Then each edge of G belongs to a closed trail, and hence to a cycle.*

Note: If there is a closed trail, then one can construct a cycle by simply deleting edges (shortening the trail) until no vertex repeats.

Proof. Fix an edge $\alpha_1 = \{x_0, x_1\}$. We proceed by means of an algorithm.

2.1. Set $i = 1, W = \{x_0, x_1\}$ and $F = \{\alpha_1\}$.

2.2. If $x_i = x_0$, end the algorithm. If $x_i \neq x_0$, then do the following:
   (i) Locate an edge $\alpha_{i+1} = \{x_i, x_{i+1}\}$ not in F.
   (ii) Add x_{i+1} to W, add α_{i+1} to F and increase i by 1.

   We can continue this algorithm until we reach the k^{th} step where $x_k = x_0$. In that case, $W = \{x_0, x_1, \dots, x_k\}$ and $F = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$, and

$$\alpha_1 - \alpha_2 - \dots - \alpha_k$$

   must be a closed trail.

Of course, in order for 2.2(i) to work, we must be able to find such an edge α_{i+1}. To see that it does, suppose $x_i \neq x_0$, consider the subgraph

$$H = (W, F)$$

where $W = \{x_0, x_1, \dots, x_i\}$ and $F = \{\alpha_1, \alpha_2, \dots, \alpha_i\}$. Notice that

$$\deg_H(x_i) \text{ is odd.}$$

Since $\deg_G(x_i)$ is even, α_{i+1} must exist. □

Example 2.6. In the graph below, we start with $W = \{G, D\}$ and $F = \{\alpha_1\}$ as shown. The algorithm then proceeds as follows:

2.1. Take $W = \{G, D, C\}$ and $F = \{\alpha_1, \alpha_2\}$.

2.2. Take $W = \{G, D, C, A\}$ and $F = \{\alpha_1, \alpha_2, \alpha_3\}$.

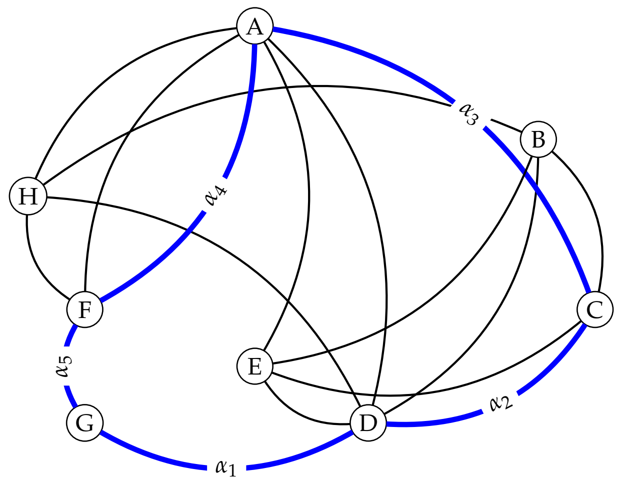


Figure VII.2.: A General Graph and Closed Trail

2.3. Take $W = \{G, D, C, A, F\}$ and $F = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

2.4. Take $W = \{G, D, C, A, F, G\}$ and $F = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.

The closed trail (indeed cycle) is then $G - D - C - A - F - G$.

(End of Day 28)

Theorem 2.7. *A connected general graph G has a closed Eulerian trail if and only if every vertex has even degree.*

Proof. If G has a closed Eulerian trail, then each vertex has even degree by Lemma 2.5. Conversely, suppose each vertex has even degree, and we proceed by an algorithm (an example is also embedded within the proof):

2.1. Set $G_1 = (V, E) = G$ and set $E_1 = E$.

2.2. Choose an edge α_1 of G_1 and construct a closed trail γ_1 containing α_1.

2.3. Let $G_2 = (V, E_2)$ where $E_2 = E_1 \setminus \gamma_1$. Note that

   (i) Every vertex in the trail γ_1 has even degree, so every vertex in G_2 has even degree.

   (ii) Since G is connected, there is an edge α_2 in G_2 that is incident on γ_1, say at a point z_1.

2.4. Now choose a closed trail γ_2' containing this edge α_2.

2.5. Patch the two trails γ_1 and γ_2' together to construct a closed trail

$$\gamma_2 := \gamma_2' *^{z_1} \gamma_1.$$

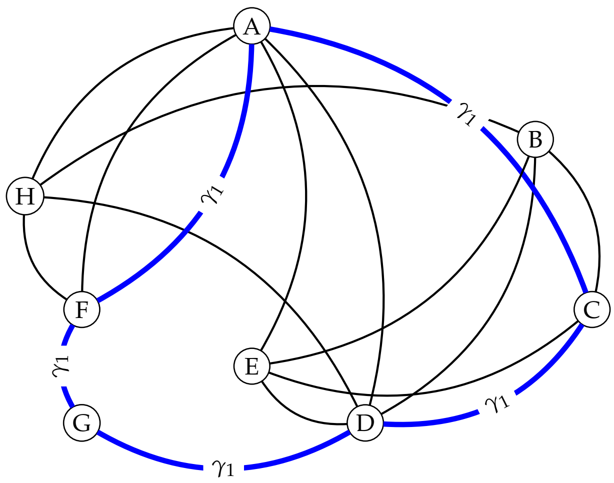


Figure VII.3.: $G_1 = (V, E_1)$ and Closed Trail γ_1

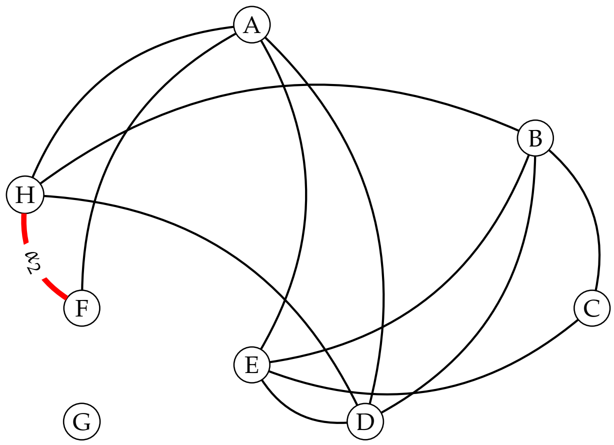


Figure VII.4.: $G_2 = (V, E_2)$ and the edge α_2 is marked. Here, $z_1 = F$.

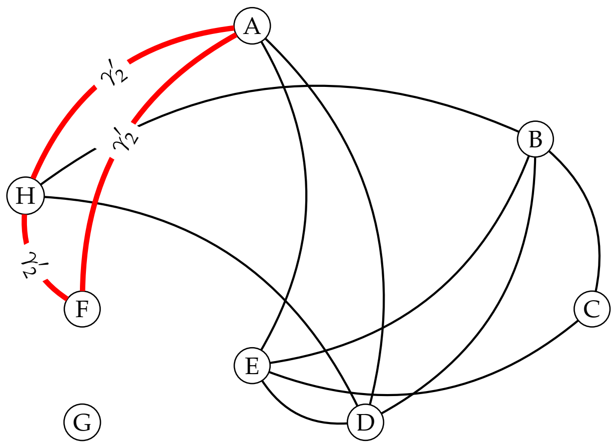


Figure VII.5.: $G_2 = (V, E_2)$ and Closed Trail γ_2'.

containing all the edges of γ_1 and γ_2'. This is done by inserting γ_2' into γ_1 at the point z_1. In our example, the resultant trail is

$$\overbrace{A - \underbrace{(F - H - A - F)}_{\text{Step 2}} - G - D - C - A}^{\text{Step 1}}$$

2.6. Let $G = (V, E_3)$ where $E_3 = E \setminus \gamma_2$ and make an edge α_3 incident on γ_2 at a point z_2.

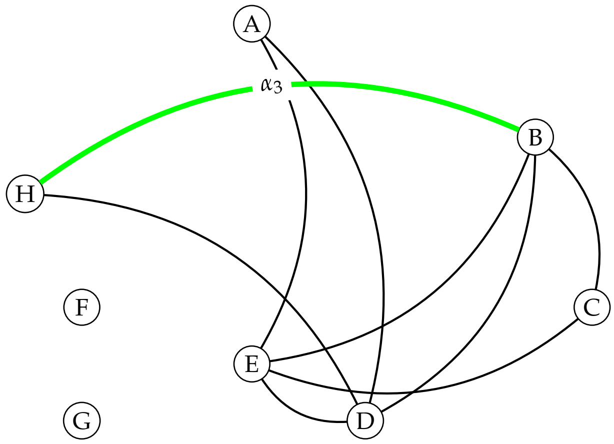


Figure VII.6.: $G_3 = (V, E_3)$ and the edge α_3 is marked. Here, $z_2 = H$.

2.7. Continue this process until we arrive at a closed trail

$$\gamma_k = \gamma_k' *_{z_{k-1}} \gamma_{k-1}$$

which contains all the edges of G. This is a closed Eulerian trail.

We complete the process in our example.

At this point, the trail looks like

$$\overbrace{A - \underbrace{(F - \overbrace{(H - B - C - E - D - H)}^{\text{Step 3}} - A - F)}_{\text{Step 2}} - G - D - C - A}^{\text{Step 1}}$$

The Eulerian trail thus obtained is

$$\overbrace{A - \underbrace{(F - \overbrace{(H - B - C - \underbrace{(E - B - D - A - E)}_{\text{Step 4}} - D - H)}^{\text{Step 3}} - A - F)}_{\text{Step 2}} - G - D - C - A}^{\text{Step 1}}$$

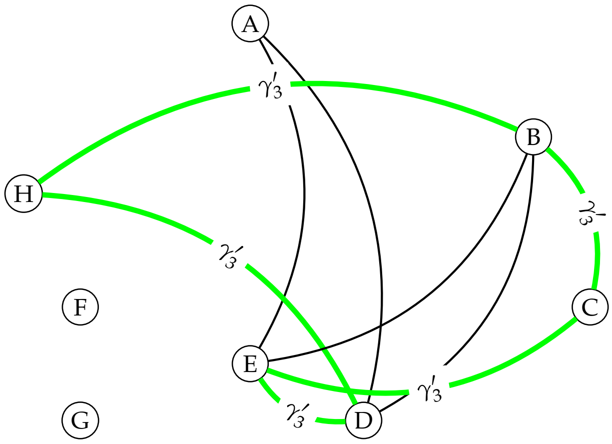


Figure VII.7.: $G_3 = (V, E_3)$ and the closed trail γ_3' is marked.

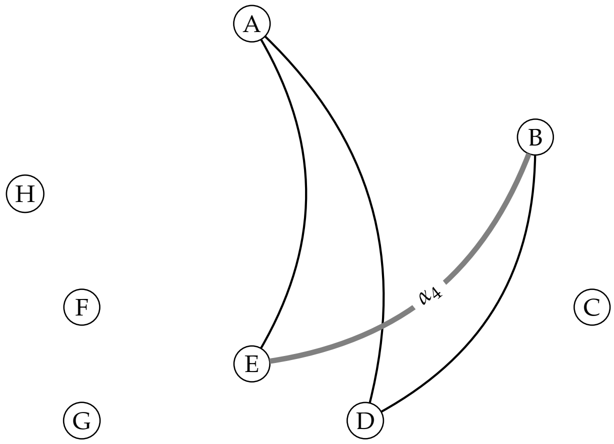


Figure VII.8.: $G_4 = (V, E_4)$ and the edge α_4 is marked. Here, $z_3 = E$.

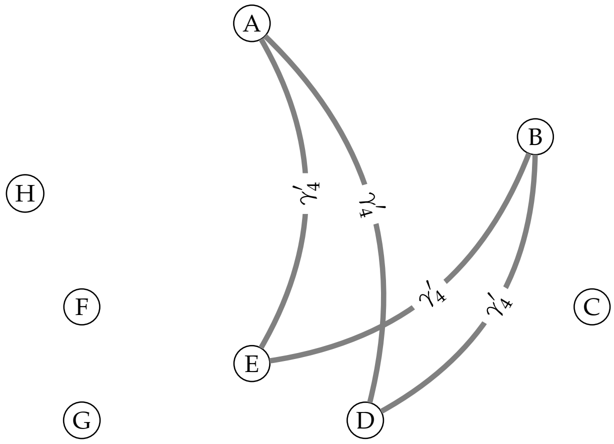


Figure VII.9.: $G_4 = (V, E_4)$ and the closed trail γ_4' is marked.

$\square$

3. Trees

Lemma 3.1. *If G is a connected graph, then removing an edge from a cycle will not disconnect the graph.*

Proof. Exercise. □

Lemma 3.2. *If G is a connected graph with v vertices and e edges, then $v \leq e + 1$.*

Proof. We induct on e. If $e = 1$, then $v \leq 2$ because G is connected.

Now suppose the result is true for any graph H that has $\leq e - 1$ edges, and let G be a connected graph with e edges and v vertices.

3.1. If G contains a cycle, then delete one edge of that cycle to obtain a connected graph G with $(e-1)$ edges. Now, the number of vertices in H is $v \leq e - 1 + 1 = e$ by induction hypothesis. Therefore, $v \leq e + 1$.
3.2. Suppose G does not contain any cycle, then choose the longest path in G and write it as
$$a - x_1 - x_2 - \ldots - x_n - b.$$
Note that $\deg(a) = 1$ because otherwise it would be possible to find a longer path, or there would be a cycle. Now, let H be the subgraph obtained by deleting the vertex a and the edge $\{a, x_1\}$. Then, H is connected and has $(v-1)$ vertices and $(e-1)$ edges. By induction hypothesis,
$$v - 1 \leq (e-1) + 1 = e \Rightarrow v \leq e + 1.$$
This completes the proof.

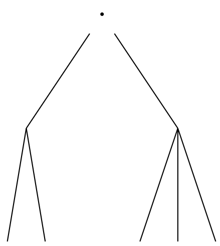
□

Definition 3.3.

3.1. A <u>tree</u> is a connected graph that contains no subgraph isomorphic to a cycle.
3.2. A <u>forest</u> is a graph that does not contain a cycle.

Remark 3.4.

3.1. A path of length n is a tree. It is denoted P_n.
3.2. A connected forest is a tree.
3.3. A subgraph of a forest is a forest. A subgraph of a tree is a forest.
3.4. Every tree is a forest.



(End of Day 29)

Proposition 3.5. *If a tree has v vertices and e edges, then $v = e + 1$.*

Proof. We induct on e. If $e = 1$, then this is clearly true.

Now suppose the result is true for any tree with $\leq e - 1$ edges, and suppose G has e edges. Once again, choose the longest path and write it as

$$a - x_1 - x_2 - \ldots - x_n - b.$$

Once again, $\deg(a) = 1$, so if we subtract a and the edge $\{a, x_1\}$, we obtain a tree H with $(v - 1)$ vertices and $(e - 1)$ edges. By induction hypothesis,

$$v - 1 = e - 1 + 1 \Rightarrow v = e + 1.$$

□

Theorem 3.6. *If G is a connected graph such that $v = e + 1$, then G is a tree.*

Proof. Suppose there is a connected graph G with $v = e + 1$ that is not a tree. Then G must contain a cycle. Deleting one edge from that cycle will give us a graph H which is connected, has v vertices and $(e - 1)$ edges. By Lemma 3.2,

$$v \leq (e - 1) + 1 = e.$$

This contradiction proves that G must have been a tree. □

Theorem 3.7. *A graph is a tree if and only if there is exactly one path between any two given vertices.*

Proof.

3.1. Suppose $G = (V, E)$ is a tree. Let $v_1, v_2 \in V$. Suppose there are two paths P_1 and P_2 connecting these two vertices, we may write them as

$$v_1 = x_0 - x_1 - x_2 - \ldots - x_n = v_2 \text{ and } v_1 = y_0 - y_1 - y_2 - \ldots - y_m = x_2.$$

Assume that $n \leq m$, and choose the minimal $1 \leq i \leq n$ (possibly $i = 1$) such that $x_i \neq y_i$. In other words,

$$x_j = y_j \text{ for all } 1 \leq j \leq i - 1 \text{ and } x_i \neq y_i.$$

Now choose the $k > i$ (possibly $k = n$) such that

$$x_j \neq y_j \text{ for all } i \leq j \leq k-1 \text{ and } x_k = y_k.$$

Now observe that

$$x_{i-1} - x_i - x_{i+1} - \ldots - x_k - y_{k-1} - \ldots - y_i - y_{i-1} = x_{i-1}.$$

This defines a cycle in G, which is impossible since G is a tree.

3.2. Suppose G is a graph such that there is exactly one path between any two vertices. Then, G must be connected. Moreover, suppose G has a cycle

$$v_1 - v_2 - \ldots - v_n - v_1.$$

Then, there are two paths from v_1 to v_n, viz.

$$[v_1 - v_2 - \ldots - v_n] \text{ and } [v_1 - v_n].$$

This contradiction proves that G cannot have any cycles and G is a tree.

□

Definition 3.8. For a graph $G = (V, E)$, a subgraph $H = (W, F)$ is said to be spanning if $W = V$.

Theorem 3.9. *Every connected graph contains a spanning tree.*

Proof. We induct on the number q of edges of G. If $q = 1$, then $p \leq q + 1 = 2$, so G is itself a tree. Now suppose that the result is true for any graph that has $(q-1)$ edges, and let G be a connected graph with q edges.

If G is a tree, there is nothing to prove. If not, then G has a cycle. Deleting one edge of the cycle gives us a new graph H with the same vertex set. By induction hypothesis, H has a spanning tree, which is also a spanning tree for G. □

4. Planar Graphs

Definition 4.1. A planar graph is one that can be drawn on the plane in such a way that no two edges cross. Such a drawing of the graph is called a planar representation.

Remark 4.2.

4.1. The graphs K_1, K_2 and K_3 are planar.

4.2. Every tree is planar.

4.3. The graph K_4 is planar.

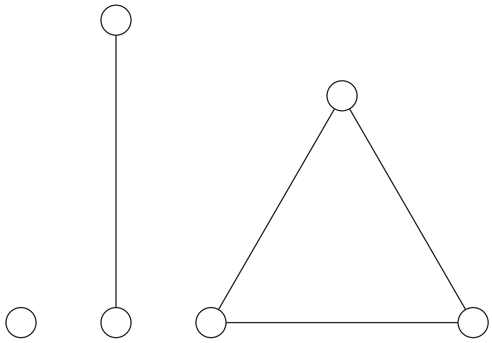
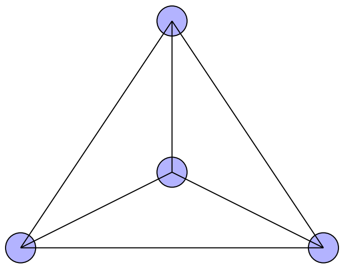
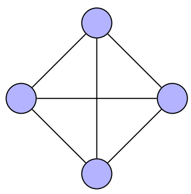


Figure VII.10.: K_1, K_2 and K_3

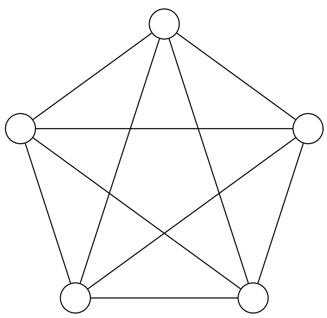


4.4. Note that K_4 can also be drawn as



This is not a planar representation of K_4, while part (i) is.

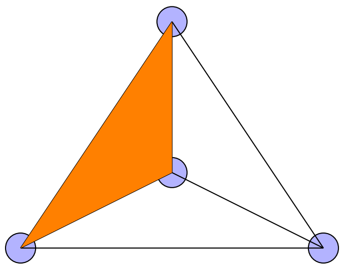
4.5. The following is not a planar representation of K_5:



But we do not know if K_5 admits a planar representation or not.

Definition 4.3. Given a planar graph G expressed as a subset of $\mathbb{R}^2$, a *face* of G is the region enclosed by a family of edges, which cannot be further subdivided. In other words, it is a connected component of $\mathbb{R}^2 \setminus G$.

Example 4.4. A face marked out in K_4 is shown below.



Remark 4.5.

4.1. The set of faces may change depending on how G is represented as a subset of $\mathbb{R}^2$.

4.2. Every graph has at least one face, its *exterior* (If a graph has only that one face, it is a tree).

4.3. Let $G = (V, E)$ be a connected planar graph, thought of as a subset of $\mathbb{R}^2$. Write

$$
\begin{aligned}
v &= \text{the number of vertices,}\\
e &= \text{number of edges and}\\
f &= \text{the number of faces in } G.
\end{aligned}
$$

Theorem 4.6 (Euler's Formula)**.** *For a connected planar graph G, we have*

$$v - e + f = 2.$$

Proof. If G is a tree, then $f = 1$ and $v = e + 1$. This gives the desired formula.

Now we induct on the number of cycles in G. If G has a cycle, then deleting one edge of the cycle G gives a connected graph G' with one less cycle. If $e' =$ the number of edges of G', and v' and f' are defined analogously, then

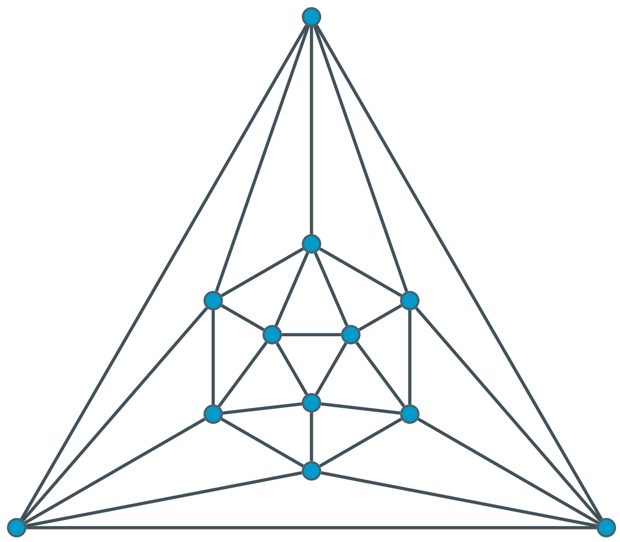
$$e' = e - 1 \text{ and } f' = f - 1 \text{ and } v' = v.$$

By induction, we have $v' - e' + f' = 2$. Substituting gives the result. □

(End of Day 30)

Definition 4.7. A planar graph is said to be maximal planar (or just maximal) if adding any edge to G results in a nonplanar graph.

Example 4.8. The Icosahedral graph is maximal planar.



Lemma 4.9. *Let $G = (V, E)$ be a maximal planar graph with $v \geq 3$. Then, in any planar representation of G, each face is a triangle (including the exterior face).*

Proof. Note that G must be connected (otherwise, adding an edge connecting the two components will not break planarity). Let C be a face with boundary $H = (W, F)$ as a subgraph of G.

4.1. Suppose H does not contain a cycle, then H is a tree (by definition) and has ≥ 3 vertices. Adding one edge to such a tree does not break planarity, so G could not be maximal. Therefore, H must contain a cycle.

4.2. Let $H' = (W', F')$ be a cycle contained in H. If $H' \neq H$, then there is a vertex $v_0 \in W \setminus W'$. If $v_1 \in W'$, then adding the edge $\{v_0, v_1\}$ does not break planarity (because C is a face). This contradicts the maximality of G, so H is itself a cycle.

4.3. Suppose C is not a triangle, then there are vertices $\{v_0, v_1, v_2\} \subset W$ such that $\{v_0, v_1\} \notin F$. Once again, adding $\{v_0, v_1\}$ does not break planarity, so this contradicts the maximality of G.

Hence each face is a triangle. □

Lemma 4.10. *If G is a maximal planar graph with $v \geq 3$, then*

4.1. $3f = 2e$.

4.2. $e = 3v - 6$.

Proof. Every face has three edges. So if you walk along the edge of each face (including the exterior face) and add up the perimeters, you get $3f$. However, each edge is on the boundary between two faces, so $3f = 2e$.

Now use Euler's formula:

$$\begin{aligned} v - e + f &= 2 \\ \Rightarrow v - e + \frac{2}{3}e &= 2 \\ \Rightarrow 3v - e &= 6. \end{aligned}$$

□

Proposition 4.11. *If G is a connected planar graph with $v \geq 3$, then $e \leq 3v - 6$.*

Proof. Add edges until you obtain a maximal planar graph. Then apply Lemma 4.10. □

Example 4.12. K_5 is not planar because $e = 10$ and $v = 5$.

Definition 4.13.

4.1. A graph $G = (V, E)$ is said to be bipartite if V can be partitioned into two sets

$$V = X \sqcup Y$$

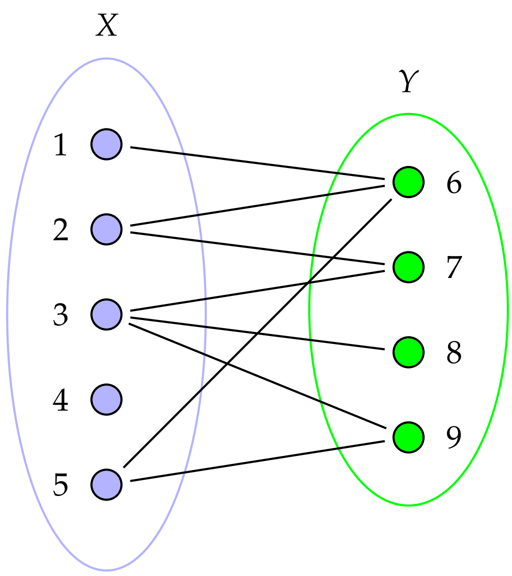
such that for each edge $\alpha = \{x, y\} \in E$, either $(x \in X, y \in Y)$ or $(y \in X, x \in Y)$.

4.2. If G is a bipartitite graph, then the pair $\{X, Y\}$ as above is called a bipartition of G.

4.3. A bipartite graph is said to be complete if there is a bipartition $\{X, Y\}$ such that each vertex of X is adjacent to each vertex of Y.

Example 4.14.

4.1.



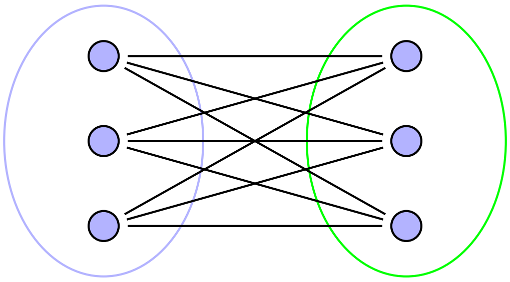


Figure VII.11.: The complete bipartite graph $K_{3,3}$

4.2. If G is a complete bipartitite graph with bipartition $\{X, Y\}$, then G has $|X| \times |Y|$ edges. If $|X| = m$ and $|Y| = n$, then such a graph is unique and is denoted by $K_{m,n}$.

4.3. Note that in a planar representation of a bipartite graph, every face has ≥ 4 edges surrounding it (i.e. there are no triangles).

Lemma 4.15. *If G is a planar bipartite graph with $v \geq 3$ vertices and e edges, then*

$$e \leq 2v - 4.$$

Proof. By Euler's formula, it suffices (check!) to prove that

$$2f \leq e.$$

Observe that every face has ≥ 4 edges surrounding it. For each $i \geq 4$, let n_i be the number of faces with i edges. Then,

$$\sum_{i \geq 4} i n_i$$

counts the total number of edges of all faces. However, each edge is incident on two faces, so

$$\sum_{i \geq 4} i n_i = 2e.$$

Now, $\sum_{i \geq 4} n_i = f$, so we conclude that

$$4f \leq \sum_{i \geq 4} i n_i \leq 2e.$$

□

Example 4.16.

4.1. The complete bipartite graph $K_{3,3}$ is not planar because $e = 9$ and $v = 6$.

4.2. If a graph contains either K_5 or $K_{3,3}$, then it is not planar.

4.3. A theorem of Kuratowski states that the converse is true: If a graph is not planar, then it must contain either K_5 or $K_{3,3}$ as a subgraph (or a *subdivision* of K_5 or $K_{3,3}$).

(End of Day 31)

5. Graph Colouring

Definition 5.1. Let $G = (V, E)$ be a graph and $n \in \mathbb{N}$. An n-colouring of G is a function

$$\rho : V \to \{1, 2, \ldots, n\}$$

such that if $\alpha = \{x, y\} \in E$, then $\rho(x) \neq \rho(y)$.

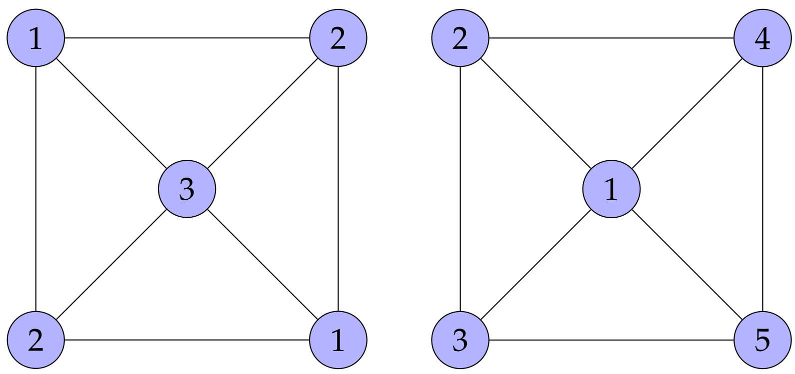
Remark 5.2.

5.1. The function ρ is thought of as assigning *colours* to each vertex of G in such a way that adjacent vertices have different colours.

5.2. If $G = (V, E)$ is any graph with $V = \{1, 2, \ldots, n\}$, then $\mathrm{id}_V : V \to V$ is a n-colouring of V.

5.3. If $G = K_m$, the complete graph on m vertices, and $\rho : V \to \{1, 2, \ldots, n\}$ is a n-colouring, then ρ must be injective (for any distinct vertices $x, y \in V, \rho(x) \neq \rho(y)$). Therefore, $n \geq m$.

5.4. Consider the following colourings (of the Wheel Graph W_4)



Notice that the number of colours can be different.

Definition 5.3. For a graph G, the chromatic number of G is the least $n \in \mathbb{N}$ such that G admits an n-colouring. This is denoted by $\chi(G)$.

In other words, if $\chi(G) = n$, then G admits an n-colouring, but *does not admit* an $(n-1)$-colouring.

Example 5.4.

5.1. If $G = K_m$, then $\chi(G) = m$.

5.2. If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

5.3. If G is a tree, then $\chi(G) \leq 2$.

Proof. We induct on v, the number of vertices of G. If $v = 1$, there is nothing to prove. Suppose that the result is true for any tree with $\leq v - 1$ vertices, and let G be a tree with v vertices. Choose the longest path in G

$$a - x_1 - x_2 - \ldots - x_n - b.$$

As before, $\deg(a) = 1$, so $H := (V \setminus \{a\}, E \setminus \{\{a, x_1\}\})$ is a tree. By induction hypothesis, $\chi(H) \leq 2$, so there is a 2-colouring

$$\rho : V \setminus \{a\} \to \{1, 2\}.$$

Now assume that $\rho(x_1) = 1$, and define $\eta : V \to \{1, 2\}$ by

$$\eta(v) = \begin{cases} \rho(v) & : v \neq a \\ 2 & : v = a. \end{cases}$$

Since $\{a, x_1\}$ is the only edge incident on a, this is a 2-colouring of G, so $\chi(G) \leq 2$. □

5.4. If G is bipartite, then $\chi(G) \leq 2$.

Lemma 5.5. *If G is a planar graph, then G has a vertex p with* $\deg(p) \leq 5$.

Proof. Assume that $v \geq 3$ (else the lemma is trivial) Suppose that $\deg(q) \geq 6$ for each $q \in V$, then by Theorem 1.7,

$$2e = \sum_{q \in V} \deg(q) \geq 6v \Rightarrow e \geq 3v.$$

This contradicts the inequality $e \leq 3v - 6$ from Proposition 4.11. □

Theorem 5.6. *If G is a planar graph, then $\chi(G) \leq 6$.*

Proof. We induct on the number of vertices in G. If $v \leq 2$ there is nothing to prove, so assume $v \geq 3$, and assume by induction hypothesis that for any planar graph H with $\leq (v - 1)$ vertices, $\chi(H) \leq 6$. Let G be a planar graph with v vertices, and choose a vertex $p \in V$ with $\deg(p) \leq 5$ by the previous lemma. Deleting p and all the edges incident on it, we get a planar graph H with $(v - 1)$ vertices. By hypothesis, $\chi(H) \leq 6$. Choose a 6-coloring $\rho : V \setminus \{p\} \to \{1, 2, \ldots, 6\}$. Assume that $x_1, x_2, \ldots, x_5$ are the vertices in G adjacent to p (there may be less than 5). Then, there exists $1 \leq i \leq 6$ such that

$$i \notin \{\rho(x_j) : 1 \leq j \leq 5\}.$$

Define $\eta : V \to \{1, 2, \ldots, 6\}$ by

$$\eta(q) = \begin{cases} i & : q = p \\ \rho(q) & : \text{otherwise.} \end{cases}$$

By construction, this defines a 6-colouring of G. Hence, $\chi(G) \leq 6$. □

Definition 5.7.

5.1. Let $G = (V, E)$ be a graph and $W \subset V$ be any set. The subgraph of G induced by W is the graph $H = (W, F)$, where

$$F = \{\{x, y\} \in E : x, y \in W\}.$$

5.2. Given a graph G, a connected component of G is a subgraph H such that H is connected, and if K is a connected subgraph of G containing H, then $H = K$. In other words, H is a *maximal* connected subgraph of G.

(End of Day 32)

Lemma 5.8. *Let $G = (V, E)$ be a graph and $\rho : V \to \{1, 2, \ldots, k\}$ be a k-colouring of G. Fix two colours, say 1 and 2, and set*

$$W := \rho^{-1}(\{1, 2\}).$$

Let $H = H_{1,2}$ be the subgraph of G induced by the vertices in W. Let $C = (V_0, F_0)$ denote a connected component of H. Define $\eta : V \to \{1, 2, \ldots, k\}$ by

$$\eta(p) = \begin{cases} \rho(p) & : \text{if } p \notin C \\ 1 & : \text{if } p \in C \text{ and } \rho(p) = 2 \\ 2 & : \text{if } p \in C \text{ and } \rho(p) = 1. \end{cases}$$

Then, η is a k-colouring of G.

In other words, if we interchange the two colours on a connected component of $H_{1,2}$, then we get a k-colouring of G.

Proof. Suppose η is not a k-colouring of G, then there exist adjacent points $x, y \in G$ such that $\eta(x) = \eta(y)$. Since ρ is a k-colouring of G, it must happen that

$$(x \in V_0 \text{ and } y \notin V_0) \text{ or } (x \notin V_0 \text{ and } y \in V_0).$$

Assume without loss of generality that $x \in V_0$ and $y \notin V_0$, and assume that

$$\rho(x) = 1 \text{ and } \rho(y) = 2.$$

In particular, $x, y \in W$. Since x and y are adjacent and C is a connected component, if $x \in C$, then it must happen that $y \in C$. This is a contradiction. $\square$

Theorem 5.9 (Five Colour Theorem). *If G is a planar graph, then $\chi(G) \leq 5$.*

Proof. Suppose $G = (V, E)$ is a planar graph of order n, and fix a planar representation of G. If $n \geq 5$, there is nothing to prove, so assume $n > 5$. We now induct on n.

Choose a vertex x of G with $\deg(x) \leq 5$. Let H be the subgraph of G induced by $V \setminus \{x\}$. Then, H has order $(n-1)$, so by induction hypothesis,

$$\chi(H) \leq 5.$$

Choose a 5-colouring $\rho : V \setminus \{e\} \to \{1, 2, \ldots, 5\}$ of H. We wish to define a 5-colouring $\eta : V \to \{1, 2, \ldots, 5\}$.

5.1. If $\deg(x) \leq 4$, then we may assign $\rho(x)$ to a value different from all four vertices adjacent to x.

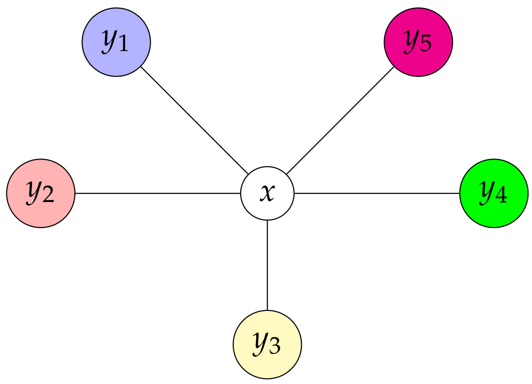
5.2. Assume $\deg(x) = 5$, and suppose $\{y_1, y_2, \ldots, y_5\}$ are the five vertices adjacent to x in G.

   (i) If any two y_i are assigned the same colour, then we may once again assign $\rho(x)$ to a value different from $\{\rho(y_i) : 1 \leq i \leq 5\}$.

   (ii) Now suppose $\{\rho(y_i) : 1 \leq i \leq 5\}$ are all distinct, and arrange it so that

$$\rho(y_i) = i$$

   for all $1 \leq i \leq 5$. Moreover, we may arrange them 'anti-clockwise' as shown below.



   We now consider the graph $H_{1,3}$ of H induced by vertices of colour 1 and 3.

   (a) If y_1 and y_3 belong to two different connected components of $H_{1,3}$, then we may apply Lemma 5.8 to obtain a 5-colouring of G in which y_1 and y_3 are the same colour. This returns us to the case 5.2(i) and we are done.

   (b) Suppose y_1 and y_3 belong to the same connected component of $H_{1,3}$. Then, there is a path connecting y_1 and y_3 which lies entirely in $H_{1,3}$, say

$$y_1 = v_0 - v_1 - v_2 - \ldots - v_n = y_3.$$

Extending this path to

$$x - v_0 - v_1 - v_2 - \ldots - v_n - x$$

yields a closed curve γ in $\mathbb{R}^2$. Therefore, it must happen that either (y_2 lies in the interior of γ and y_4, y_5 lie in the exterior) or (y_2 lies in the exterior of γ and y_4, y_5 lie in the interior). Assume without loss of generality that

$$y_2 \text{ lies in the interior of } \gamma \text{ and } y_4, y_5 \text{ lie in the exterior.}$$

Now consider the subgraph $H_{2,4}$ of H induced by vertices of colour 2 and 4. Then, y_2 and y_4 do not belong to the same component of $H_{2,4}$. Therefore, if we choose a connected component C of $H_{2,4}$ containing y_2, then y_4 does not lie in C. By Lemma 5.8, we may recolour G such that y_2 and y_4 have the same colour. Once again, we return to the case considered in 5.2(i).

Therefore $\chi(G) \leq 5$. $\square$

Remark 5.10. It is a fact that if G is a planar graph, then $\chi(G) \leq 4$. This is the Four Colour Theorem, which has been proved using computers.

VIII. Instructor Notes

0.1. Overall, the combinatorics material lacks bite (barring perhaps Ramsey theory). While the graph theory is fun and interesting, it does not get enough time to shine. It probably demands a stand-alone course for itself.

0.2. The syllabus for the course is just right. I was not able to cover directed graphs or network flows, but that is because I lost a number of classes to holidays.

0.3. The student response continues to be disappointing after Covid.