MTH 304: General Topology

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Dr. Prahlad Vaidyanathan

Contents

I. Continuous Functions

1. Functions of a Real Variable

Let $S \subset \mathbb{R}$. A function in this section will be a real-valued function whose domain is S.

Remark 1.1.

(i) Consider two graphs (one continuous and other discontinuous at $x = 1$). Continuity means that we can draw the graph of f without lifting our pencil. i.e. If we approach a point on the x axis from either direction, the value of $f(x)$ should be 'predicted' by the values of $f(y)$ where y is near x.

(ii) Continuity is a 'local' property. Continuity at one point does not tell you anything about continuity at another point.

Definition 1.2. A function $f : S \to \mathbb{R}$ is said to be sequentially continuous at $a \in S$ if, for any sequence $(x_n) \subset S$ such that $x_n \to a$, we have $f(x_n) \to f(a)$.

Example 1.3. $f(x) = x/|x|$ for $x \neq 0$ and $f(0) = 1$

(i) If we choose $a = 0$ and $x_n = 1/n$, then $f(a) = \lim f(x_n)$

(ii) However, if we choose $x_n = -1/n$, then $f(a) \neq \lim f(x_n)$.

So f is not sequentially continuous.

Definition 1.4. A function $f : S \to \mathbb{R}$ is said to be continuous at a if, for every $\epsilon > 0, \exists \delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon \tag{I.1}$$

Example 1.5.

(i) $f(x) = x^2$ is continuous at 2

  (i) If $a = 0, \epsilon = 1$, we want $\delta > 0$ such that Equation I.1 holds. ie. We want

$$|x| < \delta \Rightarrow |x^2| < 1$$

  Since $|x^2| = |x|^2$, we may choose $\delta = 1$.

  (ii) If $a = 2, \epsilon = 1$, we want $\delta > 0$ such that Equation I.1 holds. ie. We want

$$|x - 2| < \delta \Rightarrow |x^2 - 2^2| < 1$$

  Notice that $\delta = 1$ does not work, because if $x = 2.9$ then $x^2 \approx 9$. However,

$$|x^2 - 2^2| = |x - 2||x + 2|$$

  So $\exists \delta > 0$ that works.

(ii) $f(x) = x^2$ if $x \neq 0$ and $f(0) = 0.5$ is discontinuous at 1.

   (i) If $\epsilon = 1$, then $\delta = 0.5$ works because if

$$|x| < 0.5 \Rightarrow |x^2| < 0.25 < 1, \text{ and } |f(0)| = 0.5 < 1$$

   (ii) However, if $\epsilon = 0.2$, then no $\delta > 0$ works because if $|x| < \delta$, then we may choose small enough x so that $|x| < 0.5$, so that $|x^2| < 0.25$ and hence

$$|x^2 - 0.5| > 0.25$$

   So f is discontinuous at 0.

Theorem 1.6. *f is continuous at a if and only if it is sequentially continuous at a.*

Proof. (i) Suppose f is continuous at a and $(x_n) \subset S$ is a sequence such that $x_n \to a$. WTS: $f(x_n) \to f(a)$, so choose $\epsilon > 0$, then $\exists \delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

For this $\delta > 0, \exists N \in \mathbb{N}$ such that $|x_n - a| < \delta$ for all $n \geq N$. Hence,

$$|f(x_n) - f(a)| < \epsilon \quad \forall n \geq N$$

This is true for any $\epsilon > 0$ so $f(x_n) \to f(a)$

(ii) Suppose f is sequentially continuous at a, but it is not continuous at a, then $\exists \epsilon > 0$ for which no δ works. Hence, $\delta = 1/n$ does not work, so $\exists x_n \in S$ such that

$$|x_n - a| < 1/n, \text{ but } |f(x_n) - f(a)| \geq \epsilon$$

Clearly, $x_n \to a$, but $f(x_n)$ does not converge to $f(a)$. Hence, f is not sequentially continuous - a contradiction.

□

2. Open Sets

Remark 2.1. Definition 1.4 (The '$\epsilon - \delta$' definition) says that f is continuous at a if and only if, for any $\epsilon > 0, \exists \delta > 0$ such that

$$x \in (a - \delta, a + \delta) \Rightarrow f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$$

Definition 2.2.

(i) An open interval in $\mathbb{R}$ is a set of the form $(a, b) := \{x \in \mathbb{R} : a < x < b\}$ for some $a, b \in \mathbb{R}$.

(ii) A set $U \subset \mathbb{R}$ is said to be open if is a union of open intervals. (Note: We are not restricting ourselves to finite unions. i.e. We are referring to 'arbitrary' unions)

Proposition 2.3. *A set $U \subset \mathbb{R}$ is open iff for all $x \in U, \exists \delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \subset U$*

Note: The value of δ_x depends on x.

Proof.

(i) Suppose that, for any $x \in U, \exists \delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \subset U$, then

$$U = \bigcup_{x \in U} (x - \delta_x, x + \delta_x)$$

so U is open.

(ii) Conversely, if U is open, then write $U = \bigcup_{\alpha \in J} I_\alpha$, where each I_α is an open interval. If $x \in U$, then $\exists \alpha \in J$ such that $x \in I_\alpha$. Write $I_\alpha = (a, b)$, then $a < x < b$, so

$$\delta_x = \min\{|x - a|/2, |b - x|/2\}$$

works.

□

Example 2.4.

(i) (a, b)

(ii) A closed interval (or even a half-open interval) is not open.

(iii) $\{0\}$ is not open. A finite set is not open.

Proposition 2.5.

(i) An arbitrary union of open sets is open.

(ii) A finite intersection of open sets is open.

Proof. (i) is obvious, so we prove (ii): By induction, it suffices to consider the case of two sets, U_1, U_2 say. WTS: $U_1 \cap U_2$ is open, so fix $x \in U_1 \cap U_2$, then $\exists \delta_1, \delta_2 > 0$ such that $(x - \delta_i, x + \delta_i) \subset U_i, i = 1, 2$. Then if $\delta = \min\{\delta_1, \delta_2\}$, then $(x - \delta, x + \delta) \subset U_1 \cap U_2$, which verifies Theorem 2.3. □

Example 2.6. A countable intersection of open sets may not be open. If $U_n = (-1/n, 1/n)$, then $\bigcap_{n=1}^{\infty} U_n = \{0\}$.

Definition 2.7. A set $F \subset \mathbb{R}$ is closed if F^c is open.

Example 2.8.

(i) Closed interval

(ii) $[2, \infty)$ is closed.

(iii) Arbitrary intersection of closed sets is closed.

(iv) Finite union of closed sets is closed.

(v) $[1, 2)$ is neither open nor closed.

3. Continuity by Open Sets

Definition 3.1. Let $f : X \to Y$ be a function between two sets and $A \subset Y$, then

$$f^{-1}(A) = \{x \in X : f(x) \in A\}$$

Note: This definition does not imply that f^{-1} exists as a function. It is simply notation.

Example 3.2. $f(x) = x^2 - x = x(x-1)$

(i) $f^{-1}(\mathbb{R}) = \mathbb{R}$

(ii) $f^{-1}(\emptyset) = \emptyset$

(iii) $f^{-1}[-1, \infty) = \mathbb{R}$

(iv) $f^{-1}[0, \infty) = \mathbb{R} \setminus (0, 1)$

(v) $f^{-1}(\{0\}) = \{0, 1\}$

Proposition 3.3. *Let $f : X \to Y$ and $\{A_\alpha : \alpha \in J\}$ be a collection of subset of Y, then*

(i) $f^{-1}(\emptyset) = \emptyset$

(ii) $f^{-1}(Y) = X$

(iii) $f^{-1}(\bigcap_{\alpha \in J} A_\alpha) = \bigcap_{\alpha \in J} f^{-1}(A_\alpha)$

(iv) $f^{-1}(\bigcup_{\alpha \in J} A_\alpha) = \bigcup_{\alpha \in J} f^{-1}(A_\alpha)$

Proof. HW. □

Theorem 3.4. *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}(U)$ is open whenever U is open.*

Proof.

(i) Suppose f is continuous and U is open in $\mathbb{R}$. WTS: $f^{-1}(U)$ is open, so fix $x \in f^{-1}(U)$. So that $f(x) \in U$, so $\exists \epsilon > 0$ such that

$$(f(x) - \epsilon, f(x) + \epsilon) \subset U$$

By definition of continuity, $\exists \delta > 0$ such that

$$|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon$$

So if $y \in (x - \delta, x + \delta)$, then $f(y) \in (f(x) - \epsilon, f(x) + \epsilon) \subset U$. Hence,

$$(x - \delta, x + \delta) \subset f^{-1}(U)$$

This is true for any $x \in f^{-1}(U)$. By Proposition 2.3, $f^{-1}(U)$ is open.

(ii) Suppose $f^{-1}(U)$ is open whenever U is open. Fix $a \in \mathbb{R}, \epsilon > 0$. Then

$$U = (f(a) - \epsilon, f(a) + \epsilon)$$

is open in $\mathbb{R}$ so $f^{-1}(U)$ is open. Since $a \in f^{-1}(U), \exists \delta > 0$ such that

$$(a - \delta, a + \delta) \subset f^{-1}(U)$$

Hence, if $x \in \mathbb{R}$ such that $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

□

II. Topological Spaces

1. Definition and Examples

Definition 1.1. Let X be a set. A collection τ of subsets of X is called a topology on X if

(i) $\emptyset, X \in \tau$

(ii) If $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$

(iii) If $\{U_\alpha : \alpha \in J\}$ is an arbitrary collection of sets in τ, then $\bigcup_{\alpha \in J} U_\alpha \in \tau$

The pair (X, τ) is called a topological space, and members of τ are called open sets in (X, τ).

Example 1.2.

(i) $X = \mathbb{R}$ and $\tau =$ the collection of open sets in $\mathbb{R}$ (as defined in the previous section) is a topological space. This is called the usual topology on $\mathbb{R}$

(ii) Let $X = \mathbb{R}^2$.

  (i) Fix $\overline{a} := (a_1, a_2) \in X, r > 0$. An open disc in X centered at x of radius r is the set

$$B(\overline{a}, r) := \{(x_1, x_2) \in \mathbb{R}^2 : \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} < r\}$$

  (ii) A set $U \subset \mathbb{R}^2$ is said to be open if it is a union of open discs. As in Proposition 2.3, a set $U \subset \mathbb{R}^2$ is open if and only if, for any $\overline{a} \in U, \exists r > 0$ such that $B(\overline{a}, r) \subset U$.

  (iii) As in Proposition 2.5, an arbitrary union of open sets is open, and a finite intersection of open sets is open. Hence, this collection of open sets forms a topology on $\mathbb{R}^2$. This is called the Euclidean topology on $\mathbb{R}^2$.

(iii) Let X be any set and $\tau = \{\emptyset, X\}$. This is called the indiscrete topology on X.

(iv) Let X be any set and $\tau = \mathcal{P}(X)$. This is called the discrete topology on X.

Definition 1.3. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is said to be continuous if $f^{-1}(U) \in \tau_X$ whenever $U \in \tau_Y$. i.e. The inverse image of an open set is open.

Note: We think of continuity as a global property here, and don't care if a function is continuous at all but one point.

Example 1.4.

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$ is continuous, but $f(x) = x/|x|$ if $x \neq 0$ and $f(0) = 1$ is discontinuous.

(ii) Let (X, τ_d) be a discrete topological space, and (Y, τ_Y) any topological space. If $f : X \to Y$ is any function, then f is continuous.

(iii) Similarly, if (X, τ_X) is any topological space and (Y, τ_i) is an indiscrete topological space, then any function $f : X \to Y$ is continuous.

(iv) Let $f : X \to Y$ be a constant function, then f is continuous.

Proof. Suppose $f(x) = y_0$ for all $x \in X$. Let U be an open set in Y, then

$$f^{-1}(U) = \begin{cases} \emptyset & : \text{ if } y_0 \notin U \\ X & : \text{ if } y_0 \in U \end{cases}$$

In either case, $f^{-1}(U)$ is open. □

(v) Let $A : \mathbb{R}^2 \to \mathbb{R}$ be the addition map $A(x, y) = x + y$. Then A is continuous.

Proof. Let $U \subset \mathbb{R}$ be open. We WTS: $A^{-1}(U)$ is open. As mentioned above, it suffices to show that, for any point $(a, b) \in A^{-1}(U), \exists r > 0$ such that $B((a, b), r) \subset A^{-1}(U)$. So fix $(a, b) \in A^{-1}(U)$. Then $a+b \in U$, so $\exists \epsilon > 0$ such that $(a+b-\epsilon, a+b+\epsilon) \subset U$. Note that $A^{-1}((a + b - \epsilon, a + b + \epsilon))$ describes the region enclosed by (but not including) the two lines

$$x + y = a + b - \epsilon \text{ and } x + y = a + b + \epsilon$$

and (a, b) lies in this region. Now the distance of a point (x_0, y_0) from a line of the form $\alpha x + \beta y + \gamma = 0$ is given by

$$d = \frac{|\alpha x_0 + \beta y_0 + \gamma|}{\sqrt{\alpha^2 + \beta^2}}$$

In this case, we get

$$d = \frac{|a + b + (-a - b - \epsilon)|}{\sqrt{2}} = \frac{\epsilon}{\sqrt{2}}$$

Hence, if $(x, y) \in B((a, b), \epsilon/\sqrt{2})$, then $(x, y) \in A^{-1}((a+b-\epsilon, a+b+\epsilon))$, and hence $B((a, b), \epsilon/\sqrt{2}) \subset A^{-1}(U)$, and so $A^{-1}(U)$ is open. Hence, A is continuous. □

(vi) Similarly, the multiplication map $M : \mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto xy$ is also continuous [We will give a simpler proof later]

(vii) Let $d : \mathbb{R} \to \mathbb{R}^2$ be the diagonal map $d(x) = (x, x)$. Then d is continuous.

Proof. Once again, fix an open set $U \subset \mathbb{R}^2$ and a point $x \in d^{-1}(U)$. WTS: $\exists \delta > 0$ such that $(x - \delta, x + \delta) \subset d^{-1}(U)$. Since $(x, x) \in U$ and U is open, $\exists \epsilon > 0$ such that $B((x, x), \epsilon) \subset U$. Consider the part of the line $y = x$ inside this disc, and project it onto the X-axis. Note that if $\delta = \epsilon/\sqrt{2}$, then for any $y \in (x - \delta, x + \delta)$, we have

$$\sqrt{(x - y)^2 + (x - y)^2} < \epsilon \Rightarrow (y, y) \in B((x, x), \epsilon)$$

Hence, $(x - \delta, x + \delta) \subset d^{-1}(U)$ $\square$

(End of Week 1)

(viii) Let $f : \mathbb{R} \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$ be continuous. Define $h : \mathbb{R}^2 \to \mathbb{R}^2$ by $h(x, y) = (f(x), g(y))$. Then h is continuous.

Proof. Let $U \subset \mathbb{R}^2$ be open, $(a, b) \in h^{-1}(U)$, and $\epsilon > 0$ such that $B((f(a), g(b)), \epsilon) \subset U$. Choose $\delta_1 > 0$ such that

$$|x - a| < \delta_1 \Rightarrow |f(x) - f(a)| < \epsilon/\sqrt{2}$$

and similarly choose $\delta_2 > 0$ for g at b. Then if $\delta = \min\{\delta_1, \delta_2\}$, consider $(x, y) \in B((a, b), \delta)$. Then

$$|x - a| \leq \sqrt{(x - a)^2 + (y - b)^2} < \delta \Rightarrow |f(x) - f(a)| < \epsilon/\sqrt{2}$$

Similarly, $|g(y) - g(b)| < \epsilon/\sqrt{2}$, so

$$\sqrt{(f(x) - f(a))^2 + (g(x) - g(b))^2} < \epsilon \Rightarrow (f(x), g(y)) \in U$$

Hence, $B((a, b), \delta) \subset h^{-1}(U)$, so this is an open set and h is continuous. $\square$

(ix) Let X, Y, Z be topological spaces and $f : X \to Y, g : Y \to Z$ be continuous, then $g \circ f : X \to Z$ is continuous [HW]

(x) Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial, then f is continuous.

Proof. We induct on $\deg(f)$. If $\deg(f) = 0$, then f is constant, so continuous. So suppose $\deg(f) = n$ and the result is true for polynomials of degree $\leq n-1$. Then write

$$f(x) = g(x) + ax^n$$

Then f is a composition of

$$x \xrightarrow{d} (x, x) \xrightarrow{h} (g(x), ax^n) \xrightarrow{A} g(x) + ax^n$$

where $h(x, y) = (g(x), ay^n)$. By the previous steps, it suffices to show that $x \mapsto ax^n$ is continuous. Once again $y \mapsto ay$ is continuous for any $a \in \mathbb{R}$, so it suffices to show that $x \mapsto x^n$ is continuous. Once again we induct on n. If $n = 1$, then this is the identity map, so continuous. So suppose it is true for $n - 1$, then $x \mapsto x^n$ is the composition

$$x \xrightarrow{d} (x, x) \xrightarrow{h} (x, x^{n-1}) \xrightarrow{M} x^n$$

where $h(x, y) = (x, y^{n-1})$. This is continuous by all the previous steps. $\square$

Theorem 1.5. *Let (X, τ_X) be a topological space and $Y \subset X$. Define*

$$\tau_Y := \{U \cap Y : U \in \tau_X\}$$

Then τ_Y is a topology on Y, and is called the subspace topology on Y.

Proof. HW. □

Example 1.6.

(i) $\mathbb{Z} \subset \mathbb{R}$. We claim that every subset of $\mathbb{Z}$ is open in the subspace topology (i.e. $\mathbb{Z}$ with the subspace topology is discrete). It suffices to show that every singleton is open. To do this, fix $n \in \mathbb{N}$, then $(n - 1/2, n + 1/2)$ is open in $\mathbb{R}$ and

$$(n - 1/2, n + 1/2) \cap \mathbb{Z} = \{n\}$$

(ii) $\mathbb{Q} \subset \mathbb{R}$. Here the subspace topology is not discrete because if U is an open set in $\mathbb{R}$, then $U \cap \mathbb{Q}$ contains infinitely many points. In particular, singleton sets are not open in $\mathbb{Q}$.

(iii) $S^1 \subset \mathbb{R}^2$: An example of an open set is the intersection of any disc in $\mathbb{R}^2$ with S^1. This will give arcs in S^1. Hence, every arc in S^1 is an open set. Furthermore, since every open set in $\mathbb{R}^2$ is a union of discs, every open set in S^1 is a union of arcs.

(iv) $[0, 1] \subset \mathbb{R}$: Here, $[0, 1]$ is itself an open set since

$$[0, 1] = \mathbb{R} \cap [0, 1]$$

Furthermore, $[0, 1/2)$ is also an open set in $[0, 1]$.

(v) If $Y = [0, 1] \cup [2, 3] \subset \mathbb{R}$, then $[0, 1]$ is an open set in Y because

$$[0, 1] = (1/2, 3/2) \cap Y$$

Similarly, $[2, 3]$ is also an open set. Hence, $[0, 1]$ is both open and closed in Y.

2. Metric Spaces

Definition 2.1. Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a metric on X if

(i) $d(x, y) \geq 0$ for all $(x, y) \in X \times X$

(ii) $d(x, y) = 0$ if and only if $x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (Triangle Inequality)

The pair (X, d) is called a metric space.

Example 2.2.

(i) $\mathbb{R}$ with $d(x,y) = |x-y|$

(ii) Similarly, $\mathbb{C}$ with $d(z,w) = |z-w|$

(iii) $\mathbb{R}^n$ with

$$d(\overline{x},\overline{y}) = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$$

Proof. Clearly, the first three axioms are satisfied, so it suffices to prove the triangle inequality. For this, note that

$$\begin{aligned} d(x,y)^2 &= \sum_{i=1}^{n}(x_i-y_i)^2 \\ &= \sum_{i=1}^{n}(x_i-z_i+z_i-y_i)^2 \\ &= \sum_{i=1}^{n}(x_i-z_i)^2+(z_i-y_i)^2+2(x_i-z_i)(z_i-y_i) \end{aligned}$$

But by Cauchy-Schwartz inequality,

$$\sum_{i=1}^{n}(x_i-z_i)(z_i-y_i) \leq \sqrt{\sum_{i=1}^{n}(x_i-z_i)^2}\sqrt{\sum_{i=1}^{n}(z_i-y_i)^2} = d(x,z)d(z,y)$$

Hence,

$$d(x,y)^2 \leq d(x,z)^2 + d(y,z)^2 + 2d(x,z)d(z,y) = [d(x,z)+d(y,z)]^2$$

which gives the triangle inequality. □

(iv) $\mathbb{R}^n$ with

$$d(\overline{x},\overline{y}) = \max_{1\leq i\leq n}|x_i-y_i|$$

This is called the uniform or supremum metric on $\mathbb{R}^n$, and the metric is written as d_∞.

(v) $\mathbb{R}^n$ with

$$d(\overline{x},\overline{y}) = \sum_{i=1}^{n}|x_i-y_i|$$

This is called the L^1 metric on $\mathbb{R}^n$, and is written as d_1.

(vi) Let X be any set. Define $d : X\times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0 & : x = y \\ 1 & : x \neq y \end{cases}$$

This is called the discrete metric on X.

Definition 2.3. Let (X, d) be a metric space.

(i) An open ball of radius $r > 0$ centered at a point $a \in X$ is the set

$$B(a, r) := \{x \in X : d(x, a) < r\}$$

(ii) A set $U \subset \mathbb{R}$ is said to be open if it is a union of open balls. Equivalently, if, for each $a \in U, \exists \delta_a > 0$ such that $B(a, \delta_a) \subset U$

Theorem 2.4. *Let (X, d) be a metric space, and τ_d be the collection of open sets as defined above. Then τ_d is a topology on X. This is called the metric topology on X induced by d.*

Proof.

(i) Clearly, $\emptyset \in \tau_d$ and $X \in \tau_d$

(ii) τ_d is closed under arbitrary union by definition.

(iii) If $U_1, U_2 \in \tau_d$, WTS: $U_1 \cap U_2 \in \tau_d$, so fix $a \in U_1 \cap U_2$. Then $\exists \delta_i > 0$ such that $B(a, \delta_i) \subset U_i$. Let $\delta = \min\{\delta_1, \delta_2\}$, then if $x \in B(a, \delta)$, then $d(x, a) < \delta \leq \delta_1 \Rightarrow x \in B(a, \delta_1) \subset U_1$. Similarly, $x \in U_2$, so $B(a, \delta) \subset U_1 \cap U_2$.

□

Definition 2.5. Let (X, d) be a metric space. We say that a sequence $(x_n) \subset X$ converges to a point $a \in X$ if, for each $\epsilon > 0, \exists N \in \mathbb{N}$ such that $d(x_n, a) < \epsilon$ for all $n \geq N$. If this happens, we write $x_n \to a$.

Theorem 2.6. *Let (X, d_X) and (Y, d_Y) be two metric space, $f : X \to Y$ a function. Then TFAE:*

(i) For any $a \in X$ and any sequence $(x_n) \subset X$ such that $x_n \to a$ implies $f(x_n) \to f(a)$

(ii) For any $a \in X$ and each $\epsilon > 0, \exists \delta > 0$ such that

$$d_X(x, a) < \delta \Rightarrow d_Y(f(x), f(a)) < \epsilon$$

(iii) $f^{-1}(U)$ is open in X whenever U is open in Y (with respect to the metric topologies on each).

Proof.

(i) $\Rightarrow$ (ii): Suppose (i) holds and $a \in X$ is fixed and $\epsilon > 0$ given. Suppose no $\delta > 0$ works, then for each $n \in \mathbb{N}$, $\delta = 1/n$ does not work. So $\exists x_n \in X$ such that

$$d_X(x_n, a) < 1/n, \text{ but } d_Y(f(x_n), f(a)) \geq \epsilon$$

So $x_n \to a$ and $f(x_n)$ does not converge to $f(a)$ contradicting (i).

(ii) $\Rightarrow$ (iii): Suppose U is open in X. WTS: $f^{-1}(U)$ is open in Y, so choose $a \in f^{-1}(U)$. Then $f(a) \in U$ and U is open, so $\exists \epsilon > 0$ such that

$$B_Y(f(a), \epsilon) \subset U$$

Now by (ii), choose $\delta > 0$ such that

$$d_X(x, a) < \delta \Rightarrow d_Y(f(x), f(a)) < \epsilon$$

Then clearly $B_X(a, \delta) \subset f^{-1}(U)$, so that $f^{-1}(U)$ is open.

(iii) $\Rightarrow$ (i) Suppose $a \in X$ and $x_n \to a$. WTS: $f(x_n) \to f(a)$. So fix $\epsilon > 0$, then $U = B_Y(f(a), \epsilon)$ is open so $f^{-1}(U)$ is an open set containing a. Hence, $\exists \delta > 0$ such that $B_X(a, \delta) \subset f^{-1}(U)$. Since $x_n \to a, \exists N \in \mathbb{N}$ such that

$$d_X(x_n, a) < \delta \quad \forall n \geq N$$

Hence, $x_n \in f^{-1}(U)$ so that $f(x_n) \in U$, whence

$$d_Y(f(x_n), f(a)) < \epsilon \quad \forall n \geq N$$

Hence, $f(x_n) \to f(a)$.

□

Example 2.7.

(i) Let $M : \mathbb{R}^2 \to \mathbb{R}$ be the multiplication map $(x, y) \mapsto xy$. Then M is continuous.

Proof. Choose a sequence $(x_n, y_n) \to (a, b)$. Then

$$|x_n - a| \leq \sqrt{|x_n - a|^2 + |y_n - b|^2} = d((x_n, y_n), (a, b)) \to 0$$

So $x_n \to a$ in $\mathbb{R}$. Similarly, $y_n \to b$ in $\mathbb{R}$. Hence,

$$|x_n y_n - ab| \leq |x_n y_n - a y_n| + |a y_n - ab| = |x_n - a||y_n| + |a||y_n - b|$$

Since $y_n \to b$, (y_n) is bounded, so $\exists M > 0$ such that $|y_n| \leq M$ for all $n \in \mathbb{N}$. Hence,

$$|x_n y_n - ab| \leq M|x_n - a| + |a||y_n - b| \to 0$$

Hence, M is sequentially continuous, so it is continuous by the previous theorem. □

(ii) Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function

$$P(x_1, x_2, \ldots, x_n) = \sum_{\text{finite}} a_{i_1, i_2, \ldots, i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$$

Then P is continuous.

Proof. Similar to part (x) of Example 1.4. $\square$

Theorem 2.8. *Let (X, d_X) be a metric space and $Y \subset X$. Define $d_Y : Y \times Y \to \mathbb{R}$ by $d_Y(y_1, y_2) = d_X(y_1, y_2)$. Then*

(i) d_Y is a metric on Y, and

(ii) the metric topology induced on Y by d_Y coincides with the subspace topology induced on Y from (X, τ_{d_X})

Proof. Part (i) is trivial. To check part (ii), let η denote the subspace topology on Y and τ denote the metric topology on Y induced by d_Y.

(i) To show $\eta \subset \tau$: So fix an open set $V \in \eta$, then $\exists U$ open in (X, d_X) such that $V = U \cap Y$. To show that $V \in \tau$, we fix a point $a \in V$. WTS: $\exists \delta > 0$ such that $B_Y(a, \delta) \subset V$. Since U is open, $\exists \delta > 0$ such that

$$B_X(a, \delta) \subset U$$

Then note that $B_Y(a, \delta) = B_X(a, \delta) \cap Y \subset U \cap Y = V$.

(ii) To show $\tau \subset \eta$: It suffices to show that every open ball $B_Y(a, r) \in \eta$. But once again this follows from the fact that

$$B_Y(a, r) = B_X(a, r) \cap Y$$

$\square$

Example 2.9. Any subset of $\mathbb{R}^n$ inherits a metric topology from $\mathbb{R}^n$, so is, in particular, a metric space. For instance, this applies to

(i) (The circle) $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

(ii) (The n-sphere) $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$

(iii) (The cylinder) $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 0 \leq z \leq 1\}$

(iv) (The Torus) $T = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - 4\sqrt{x^2 + y^2} + 3 = 0\}$

Proposition 2.10. *Let $f : X \to Y$ be an injective function and d_Y is a metric on Y. Define $d_X : X \times X \to \mathbb{R}$ by*

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$$

Then d_X is a metric on X, called the metric <u>induced</u> by f. [HW]

Note: $f : X \to Y$ is automatically continuous in this situation.

Lemma 2.11. *Let $f : X \to Y$ be a bijective function and d_Y be a metric on Y. Let d_X be the metric on X induced by f. Then a function $g : X \to Z$ (some other topological space) is continuous if and only if $g \circ f^{-1} : Y \to Z$ is continuous.*

Proof. Note that in the above situation, f^{-1} is automatically continuous from $Y \to X$. Hence, if g is continuous, so is $g \circ f^{-1}$. Conversely, if $g \circ f^{-1}$ is continuous, then

$$g = g \circ f^{-1} \circ f$$

is also continuous. $\square$

Example 2.12.

(i) Let $M_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with real entries. There is a map

$$f : M_n(\mathbb{R}) \to \mathbb{R}^{n^2}$$

that expands a matrix into a tuple. This map is clearly injective. Thus, $M_n(\mathbb{R})$ is a metric space with the metric induced by f. i.e. we have

$$d((a_{i,j}), (b_{i,j})) = \sqrt{\sum_{i,j} (a_{i,j} - b_{i,j})^2}$$

(ii) Consider the determinant map $\det : M_n(\mathbb{R}) \to \mathbb{R}$. Note that $\det \circ f^{-1} : \mathbb{R}^{n^2} \to \mathbb{R}$ is a polynomial map which is continuous. Hence, by the previous lemma, det is continuous.

(iii) Note that $GL_n(\mathbb{R})$, the set of invertible $n \times n$ matrices is the set

$$GL_n(\mathbb{R}) = det^{-1}(\mathbb{R} \setminus \{0\})$$

Hence, $GL_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$ and is a metric space in its own right.

Definition 2.13. Let X be a set and d_1, d_2 be two metrics on X. We say that d_1 and d_2 are equivalent (In symbols, $d_1 \sim d_2$) if $\exists K, M > 0$ such that

$$K d_1(x, y) \leq d_2(x, y) \leq M d_1(x, y) \quad \forall x, y \in X$$

Example 2.14. Let $X = \mathbb{R}^n$ and d_1, d_2 be the uniform and Euclidean metrics respectively. Then $d_1 \sim d_2$

Proof.

$$d_1(\overline{x}, \overline{y}) = \max\{|x_i - y_i|\} \leq d_2(\overline{x}, \overline{y})$$

$$d_2(\overline{x}, \overline{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \leq \sqrt{n} d_1(\overline{x}, \overline{y})$$

$\square$

Theorem 2.15. *Let d_1 and d_2 be equivalent metrics on a set X, then $\tau_{d_1} = \tau_{d_2}$.*

Proof. By symmetry, it suffices to show that $\tau_{d_1} \subset \tau_{d_2}$. So let $K, M > 0$ such that

$$Kd_1(x,y) \leq d_2(x,y) \leq Md_1(x,y) \quad \forall x, y \in X$$

So fix $U \in \tau_{d_1}$ and $a \in U$. Then $\exists r > 0$ such that $B_{d_1}(a,r) \subset U$. Now if $x \in B_{d_2}(a, rK)$, then

$$d_1(x,a) \leq \frac{d_2(x,a)}{K} < r$$

So $B_{d_2}(a, rK) \subset B_{d_1}(a,r) \subset U$. Hence, $U \in \tau_{d_2}$ as required. □

Example 2.16. (The converse of the previous theorem is not true) Let d be the usual metric on $\mathbb{R}$ and

$$\rho(x,y) := \min\{|x-y|, 1\}$$

Then

(i) $\tau_\rho = \tau_d$

Proof. Since $\rho(x,y) \leq d(x,y)$, it follows as above that

$$B_d(a,r) \subset B_\rho(a,r)$$

Hence, $\tau_\rho \subset \tau_d$ [Check!]. Conversely, if $U \in \tau_d$ and $a \in U$, then $\exists r > 0$ such that $B_d(a,r) \subset U$. We may assume that $r < 1$, but in that case,

$$B_\rho(a,r) = B_d(a,r) \subset U$$

so that $U \in \tau_\rho$ as well. Hence, $\tau_d \subset \tau_\rho$ as required. □

(ii) ρ is not equivalent to d

Proof. Note that $\rho(x,y) \leq 1$ for all $x, y \in \mathbb{R}$. If $\exists M > 0$ such that

$$d(x,y) \leq M\rho(x,y)$$

Then this would imply that $d(x,y) \leq M$ for all $x, y \in \mathbb{R}$. This is not true because $d(n, 0) = n$ for all $n \in \mathbb{N}$. □

3. Basis for a topology

Definition 3.1. Let (X, τ) be a topological space. A collection $\mathcal{B} \subset \tau$ of open sets is called a basis for τ if every member of τ is a union of elements from $\mathcal{B}$. Equivalently, $U \in \tau$ if and only if, for each $x \in U, \exists B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

Example 3.2.

(i) Let $X = \mathbb{R}$ with the usual topology and $\mathcal{B}$ be the collection of open intervals.

(ii) Similarly, if (X, d) is any metric space with τ the metric topology. Then $\mathcal{B}$ may denote the set of all balls (of various centers and radii).

(End of Week 2)

Proposition 3.3. *Let $f : X \to Y$ be a function between two topological spaces, and suppose $\mathcal{B}$ is a basis for τ_Y. Then f is continuous if and only if $f^{-1}(B) \in \tau_X$ for all $B \in \mathcal{B}$.*

Proof. One direction is clear, so suppose $f^{-1}(B) \in \tau_X$ for all $B \in \mathcal{B}$. WTS: f is continuous, so fix an open set $U \in \tau_Y$ and we want to show $f^{-1}(U) \in \tau_X$. Fix $x \in f^{-1}(U)$, then $f(x) \in U$, so $\exists B_x \in \mathcal{B}$ such that $x \in B_x$, and $B_x \subset U$. Hence,

$$V_x := f^{-1}(B_x) \in \tau_X \text{ and } V_x \subset f^{-1}(U)$$

This is true for any $x \in f^{-1}(U)$ so

$$f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} V_x$$

Hence, $f^{-1}(U) \in \tau_X$ as required. □

Lemma 3.4. *Let $\mathcal{C}$ be a collection of subset of X. Then there is a unique topology τ on X such that*

(i) $\mathcal{C} \subset \tau$

(ii) If η is any other topology on X such that $\mathcal{C} \subset \eta$, then $\tau \subset \eta$.

ie. τ is the smallest topology containing $\mathcal{C}$. This is called the topology generated by $\mathcal{C}$.

Proof. Let $\mathcal{F}$ be the set set of all topologies η on X such that $\mathcal{C} \subset \eta$. Then $\mathcal{F} \neq \emptyset$ because $\mathcal{P}(X) \in \mathcal{F}$. Now set

$$\tau = \bigcap_{\eta \in \mathcal{F}} \eta$$

Then check that τ is a topology that satisfies the required conditions. □

Theorem 3.5. *Let X be a set and $\mathcal{B}$ be a collection of subsets of X such that*

(a) For each $x \in X, \exists B \in \mathcal{B}$ such that $x \in B$

(b) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

Let τ denote the topology generated by $\mathcal{B}$. Then $\mathcal{B}$ is a basis for τ.

Proof. Let η be the collection of all subsets of X that are unions of members of $\mathcal{B}$. Claim: η is a topology on X. The first three axioms hold trivially, and the last one follows from property (b) of $\mathcal{B}$.

Now clearly, $\mathcal{B} \subset \eta$, so that $\eta \in \mathcal{F}$ of the previous proof. Hence, $\tau \subset \eta$. Furthermore, if μ is any topology that contains $\mathcal{B}$, then $\eta \subset \mu$ because μ is closed under arbitrary unions. Hence, $\eta \subset \tau$ as required. □

4. The Product Topology on $X \times Y$

Theorem 4.1. *Let (X, τ_X) and (Y, τ_Y) be two topological spaces. Then there is a unique topology on $X \times Y$ whose basis are sets of the form*

$$U \times V$$

where $U \in \tau_X$ and $V \in \tau_Y$. This is called the product topology on $X \times Y$, denoted by $\tau_{X\times Y}$.

Proof. Let $\mathcal{B} = \{U \times V : U \in \tau_X, V \in \tau_Y\}$. We check that $\mathcal{B}$ satisfies the conditions of Theorem 3.5.

(i) Clearly, $X \times Y \in \mathcal{B}$

(ii) If $U_1, U_2 \in \tau_X$ and $V_1, V_2 \in \tau_Y$, then

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}$$

□

Theorem 4.2. *Suppose (X, d_X) and (Y, d_Y) are metric spaces. Define $d : (X\times Y)^2 \to \mathbb{R}$ by*

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

Then

(i) d is a metric on $X \times Y$.

(ii) The metric topology induced by d coincides with the product topology on $X \times Y$

Proof. Part (i) is trivial, so we prove (ii). Let τ_d denote the metric topology and $\tau_{X\times Y}$ denote the product topology.

- WTS: $\tau_{X\times Y} \subset \tau_d$: If $U = B_X(a, \delta_1)$ and $V = B_Y(b, \delta_2)$ are open balls in X and Y respectively, consider

  $$W = U \times V$$

  We claim that $W \in \tau_d$. To see this, fix $(x, y) \in W$, then $x \in U, y \in V$, so

  $$d_X(x, a) < \delta_1 \text{ and } d_Y(y, b) < \delta_2$$

  Let $r = \min\{\delta_1 - d_X(x, a), \delta_2 - d_Y(y, b)\} > 0$. We claim that

  $$B_d((x, y), r) \subset W$$

  So choose $(u, v) \in B_d((x, y), r)$, then $d((u, v), (x, y)) < r$, so that

  $$d_X(u, x) < r, \text{ and } d_Y(v, y) < r$$

Hence,

$$d_X(u,a) \le d_X(u,x) + d_X(x,a) < r + d_X(x,a) \le \delta_1 - d_X(x,a) + d_X(x,a) = \delta_1$$

Hence, $u \in U$. Similarly, $v \in V$, so that $(u,v) \in W$, proving the claim. Hence,

$$U \times V \in \tau_d$$

for any open ball $U \in \tau_{d_X}$ and $V \in \tau_{d_Y}$. But these open balls form a basis for τ_{d_X} and τ_{d_Y} respectively. Hence, by Lemma 4.2,

$$\tau_{X\times Y} \subset \tau_d$$

- WTS: $\tau_d \subset \tau_{X\times Y}$: Let $(a,b) \in X \times Y$ and $r > 0$. It suffices to show that

$$B_d((a,b),r) \subset \tau_{X\times Y}$$

Note that $(x,y) \in B_d((a,b),r)$ iff

$$d_X(x,a) < r \text{ and } d_Y(y,b) < r$$

Hence,

$$B_d((a,b),r) = B_X(a,r) \times B_Y(b,r) \in \tau_{X\times Y}$$

This is true for any open d-ball in $X \times Y$, so $\tau_d \subset \tau_{X\times Y}$.

□

Remark 4.3.

(i) The metric d defined on $X \times Y$ certainly induces the product topology. However, it is not the only metric that does so. For instance, the metric $\rho : (X \times Y)^2 \to \mathbb{R}$ given by

$$\rho((x_1,y_1),(x_2,y_2)) := d_X(x_1,x_2) + d_Y(y_1,y_2)$$

is also a metric on $X \times Y$, and the metric topology on $X \times Y$ induced by ρ is also the product topology. [HW]

(ii) Let X_1, X_2, X_3 be three topological spaces, then we may define the product topology inductively as the product topology on $(X_1 \times X_2) \times X_3$ where $X_1 \times X_2$ has the product topology. Thus, basic open sets in $X_1 \times X_2 \times X_3$ are of the form

$$U_1 \times U_2 \times U_3$$

where U_i are open in X_i. The same can be done for finitely many spaces $X_1, X_2, \ldots, X_n$.

Corollary 4.4. *The metric topology on $\mathbb{R}^n$ induced by the Euclidean metric is the same as the product topology.*

Proof. By Example 2.14, the Euclidean metric on $\mathbb{R}^n$ is equivalent to the supremum metric. By Theorem 2.15, the two metrics induce the same topology on $\mathbb{R}^n$. However, by Theorem 4.2, the supremum metric induces the product topology. Hence the result. $\square$

Definition 4.5. Let X, Y be sets. The maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ given by

$$\pi_X(x, y) = x \text{ and } \pi_Y(x, y) = y$$

are called the projection maps.

Lemma 4.6.

(i) The maps π_X and π_Y are continuous if $X \times Y$ is equipped with the product topology.

(ii) If η is a topology on $X \times Y$ such that π_X and π_Y are both continuous, then $\tau_{X\times Y} \subset \eta$.

Proof.

(i) If $U \subset X$ is open, then

$$\pi_X^{-1}(U) = U \times Y \in \tau_{X\times Y}$$

and similarly for π_Y.

(ii) If η is a topology such that π_X and π_Y are continuous, then for any U, V open in X, Y respectively,

$$U \times V = \pi_X^{-1}(U) \cap \pi_Y^{-1}(V) \in \eta$$

Hence, $\tau_{X\times Y} \subset \eta$.

$\square$

Theorem 4.7. *Let $f : Z \to X \times Y$ be a function. Then f is continuous if and only if $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous.*

Proof. If f is continuous then $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous since π_X and π_Y are continuous by the previous lemma, and part (ix) of Example 1.4. Conversely, suppose $f_1 := \pi_X \circ f$ and $f_2 := \pi_Y \circ f$ are continuous, and WTS: f is continuous. By Proposition 3.3, it suffices to show that $f^{-1}(W)$ is open when $W \subset X \times Y$ is a basic open set. So write $W = U \times V$ where U and V are open in X and Y respectively. Then

$$f^{-1}(W) = \{z \in Z : f(z) \in U \times V\} = f_1^{-1}(U) \cap f_2^{-1}(V)$$

which is open by hypothesis. $\square$

5. The Product Topology on $\prod X_\alpha$

Fix topological spaces $(X_\alpha, \tau_\alpha), \alpha \in J$, where J is a possibly infinite set.

Remark 5.1. The product topology on $X \times Y$ has two definitions:

(i) The basis sets are of the form $U \times V$ where $U \in \tau_X, V \in \tau_Y$ (Theorem 4.1).

(ii) It is the smallest topology that maps π_X and π_Y continuous (Lemma 4.6).

Theorem 5.2. *Let (X_α, τ_α) be a family of topological spaces, and let $X = \prod X_\alpha$. Let $\pi_\alpha : X \to X_\alpha$ be the projection map. Let $\mathcal{B}$ be the collection of finite intersections of the form*

$$\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_i)$$

for some finite set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset J$ and open sets $U_i \in \tau_{\alpha_i}$. Then there is a unique topology τ_p on X which has $\mathcal{B}$ as a basis. This is called the product topology on X.

Proof. We once again check the conditions of Theorem 3.5.

(i) If $x \in X$ then $x \in \prod X_\alpha = \pi_{\alpha_1}^{-1}(X_{\alpha_1})$

(ii) If $B_1 := \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_i)$ and $B_2 = \bigcap_{j=1}^{m} \pi_{\beta_j}^{-1}(V_j)$, then $B_1 \cap B_2 \in \mathcal{B}$

□

Lemma 5.3. *Let $\{X_\alpha\}$ and X as above, and let τ_p denote the product topology.*

(i) Each $\pi_\alpha : (X, \tau_p) \to (X_\alpha, \tau_\alpha)$ is continuous.

(ii) If η is a topology on X such that each $\pi_\alpha : (X, \eta) \to (X_\alpha, \tau_\alpha)$ is continuous, then $\tau_p \subset \eta$.

Proof.

(i) If $U_\alpha \in \tau_\alpha$, then $\pi_\alpha^{-1}(U_\alpha) \in \tau_p$ by definition.

(ii) If η is a topology as above, then for any $\alpha \in J$, and $U_\alpha \in \tau_\alpha$, $\pi_\alpha^{-1}(U_\alpha) \in \eta$. By taking finite intersections, any basic open set in τ_p is in η. Hence, $\tau_p \subset \eta$.

□

Theorem 5.4. *Let $f : Z \to X$ be a function. Then f is continuous iff $\pi_\alpha \circ f$ is continuous for each $\alpha \in J$*

Proof. If f is continuous, then for each $\alpha \in J, \pi_\alpha \circ f$ is continuous by Lemma 5.3. For the other direction, suppose $\pi_\alpha \circ f$ is continuous for each $\alpha \in J$ and we WTS: f is continuous. Then by Proposition 3.3, it suffices to show that

$$f^{-1}(U) \in \tau_Z$$

for any basic open set $U \subset X$. Hence, we write $U = \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_i)$, whence

$$f^{-1}(U) = \bigcap_{i=1}^{n} (\pi_{\alpha_i} \circ f)^{-1}(U_i) \in \tau_Z$$

□

Theorem 5.5. *Let $(X_\alpha), \tau_\alpha)$ be a family of topological spaces, and let $X = \prod X_\alpha$. Let $\mathcal{B}$ be the collection of sets of the form*

$$\prod U_\alpha$$

where $U_\alpha \in \tau_\alpha$ for each $\alpha \in J$. Then there is a unique topology τ_B on X which has $\mathcal{B}$ as a basis. This is called the box topology on X.

Proof. Identical to Theorem 5.2. □

(End of Week 3)

Example 5.6.

(i) If J is finite, then the product and box topologies on X coincide.

(ii) The basic open sets of τ_B are of the form

$$\prod U_\alpha$$

where $U_\alpha \in \tau_\alpha$ are any open sets. However, the basic open sets in τ_p are of the form

$$\prod U_\alpha$$

where $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in J$

(iii) In general, $\tau_p \subset \tau_B$.

(iv) If J is infinite, they may not coincide. Example: Let $\mathbb{R}^\omega$ denote the countable product of $\mathbb{R}$ with itself. In other words,

$$\mathbb{R}^\omega = \prod_{n=1}^{\infty} X_n,$$

where $X_n = \mathbb{R}$ (with the usual topology) for each $n \in \mathbb{N}$. In $\mathbb{R}^\omega$,

$$U := \prod_{n=1}^{\infty} (-1/n, 1/n)$$

is open in the box topology, but not in the product topology.

Proof. Consider $0 \in U$. If $U \in \tau_p$, then there must be a basic open set B such that $0 \in B$ and $B \subset U$. But if B is a basic open set, then $\exists n_1, n_2, \ldots, n_k \in \mathbb{N}$ and open sets $U_i \subset \mathbb{R}$ such that

$$B = \bigcap_{i=1}^{k} \pi_i^{-1}(U_i) = U_{n_1} \times U_{n_2} \times \ldots U_{n_k} \times \mathbb{R} \times \mathbb{R} \times \ldots$$

Let $n = \max\{n_i : 1 \leq i \leq k\} + 1$, and $y = (0, 0, 0, \ldots, 1, 0, 0, \ldots)$, where 1 occurs in the n^{th} stage, then $y \in B$, but $y \notin U$. Hence, B is not a subset of U, so $U \notin \tau_p$. □

Theorem 5.7. *Let X be a set, (Y, τ_Y) be a topological space, and let $\mathcal{F}$ be a family of functions from $X \to Y$. Define $\mathcal{B}$ to be the collection of sets of the form*

$$f_1^{-1}(U_1) \cap f_2^{-1}(U_2) \cap \ldots f_n^{-1}(U_n) \qquad (*)$$

where $\{f_1, f_2, \ldots, f_n\} \subset \mathcal{F}$ and $U_i \in \tau_Y$. Then $\mathcal{B}$ satisfies the conditions of Theorem 3.5. Hence, there is a unique topology for which $\mathcal{B}$ is a basis. This is called the weak topology generated by $\mathcal{F}$s, and is denoted by $\tau_{\mathcal{F}}$.

Proof. We need to check two things from Theorem 3.5.

(i) For each $x \in X, \exists B \in \mathcal{B}$ such that $x \in B$

(ii) If $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$

Now,

(i) If $f \in \mathcal{F}$ is any function, then $X = f^{-1}(Y) \in \mathcal{B}$, so (i) holds.

(ii) If $B_1, B_2 \in \mathcal{B}$, then by definition, $B_1 \cap B_2 \in \mathcal{B}$.

□

Note that each $f \in \mathcal{F}$ is continuous if X is equipped with $\tau_{\mathcal{F}}$.

Theorem 5.8. *Let (X, τ_X) and (Y, τ_Y) be two topological spaces, and let $\mathcal{F}$ be a family of functions from X to Y. Suppose that each $f \in \mathcal{F}$ is continuous, then*

$$\tau_{\mathcal{F}} \subset \tau_X$$

ie. $\tau_{\mathcal{F}}$ is the smallest topology that makes all the elements of $\mathcal{F}$ continuous.

Proof. Suppose each $f \in \mathcal{F}$ is continuous. Then for any $f_1, f_2, \ldots, f_n \in \mathcal{F}$ and any $U_1, U_2, \ldots, U_n$ in τ_Y, the sets $f_i^{-1}(U_i) \in \tau_X$. Hence,

$$f_1^{-1}(U_1) \cap f_2^{-1}(U_2) \cap \ldots \cap f_n^{-1}(U_n) \in \tau_X$$

Since every member of the basis of $\tau_{\mathcal{F}}$ is in τ_X, it follows by Theorem 3.5 that $\tau_{\mathcal{F}} \subset \tau_X$. □

6. Closed Sets

Definition 6.1. Let (X, τ) be a topological space. A subset $A \subset X$ is said to be closed if $X \setminus A$ is open.

Example 6.2. (i) $[a, b]$ is closed in $\mathbb{R}$

(ii) $A = \{(x, y) \in \mathbb{R}^2 : x \geq 0, \text{ and } y \geq 0\}$ is closed in $\mathbb{R}^2$ because $\mathbb{R}^2 \setminus A = \mathbb{R} \times (-\infty, 0) \cup (-\infty, 0) \times \mathbb{R}$

(iii) If τ is the discrete topology, then every subset of X is closed.

(iv) If τ is the co-finite (or finite complement) topology on $\mathbb{R}$, then the only closed sets are finite sets and $\mathbb{R}$.

Lemma 6.3. *Let X be a topological space. Then*

(i) $\emptyset$ and X are closed in X

(ii) If $\{F_\alpha\}$ are closed in X, then so is $\bigcap F_\alpha$

(iii) If F_1, F_2 are closed in X, then so is $F_1 \cup F_2$

Theorem 6.4. *Let $Y \subset X$. A set $A \subset Y$ is closed in Y (wrt the subspace topology) if and only if $\exists F \subset X$ closed in X such that $A = F \cap Y$*

Proof. HW □

Corollary 6.5. *Let $Y \subset X$. If $A \subset Y$ is closed in Y, and Y is closed in X, then A is closed in X.*

Definition 6.6. Let $A \subset X$

(i) The interior of A, $\text{int}(A)$ is the union of all open sets contained in A.

(ii) The closure of A, $\overline{A}$, is the intersection of all open sets containing A.

Remark 6.7.

(i) $\text{int}(A) \subset A \subset \overline{A}$

(ii) A is open iff $\text{int}(A) = A$ and A is closed iff $A = \overline{A}$

(iii) $\text{int}(A)$ is the largest open set contained in A. ie. If $U \subset A$ is open in X, then $U \subset \text{int}(A)$.

(iv) Similarly, $\overline{A}$ is the smallest closed set containing A. If $F \subset X$ is closed and $A \subset F$, then $\overline{A} \subset F$.

(v) If $A \subset Y \subset X$, we write $\text{cl}_X(A)$ and $\text{cl}_Y(A)$ to denote the closures of A with respect to X and Y respectively.

Lemma 6.8. *Let $A \subset Y \subset X$. Then $cl_Y(A) = cl_X(A) \cap Y$*

Proof. Note that

$$cl_Y(A) = \bigcap\{F \subset Y : F \text{ closed, and } A \subset F\}$$

By Theorem 6.4,

$$cl_Y(A) = \bigcap\{G \cap Y : G \subset X \text{ closed in X, and } A \subset G\}$$

which is clearly $\mathrm{cl}_X(A) \cap Y$. □

Theorem 6.9. *Let $A \subset X$ and $x \in X$.*

(i) $x \in \overline{A}$ iff, for every open set U containing x, $U \cap A \neq \emptyset$.

(ii) If the topology on X has a basis $\mathcal{B}$, then $x \in \overline{A}$ iff, for every basic open set $B \in \mathcal{B}, B \cap A \neq \emptyset$.

Note: An open set U containing a point x is called a neighbourhood of x.

Proof. We only prove (i): If $x \in \overline{A}$, let U be an open set containing x. If $x \in A$, then $U \cap A \neq \emptyset$ so there is nothing to prove. If $x \notin A$, suppose $U \cap A = \emptyset$. Then $F := X \setminus U$ is closed, and $A \subset F$. By Remark 6.7, $\overline{A} \subset F$, so that $\overline{A} \cap U = \emptyset$, whence $x \notin \overline{A}$. This is a contradiction.

Conversely, suppose every open set U containing x has the property that $U \cap A \neq \emptyset$. WTS: $x \in \overline{A}$. By definition,

$$\overline{A} = \bigcap\{F : F \subset X \text{ closed, and } A \subset F\}$$

So choose $F \subset X$ closed such that $A \subset F$. WTS: $x \in F$. Suppose $x \notin F$, then $x \in U := X \setminus F$, which is open. Hence, $U \cap A \neq \emptyset$. However, $A \subset F$, so this is impossible. Hence, $x \in F$ as required. □

Corollary 6.10. *Let (X, d) be a metric space and $A \subset X$. Then $x \in \overline{A}$ if and only if there is a sequence $(x_n) \subset A$ such that $x_n \to x$.*

Proof. (i) Suppose there is a sequence $(x_n) \subset A$ such that $x_n \to x$, then, for any open set U containing $x, \exists \epsilon > 0$ such that $B(x, \epsilon) \subset U$. Then $\exists N \in \mathbb{N}$ such that $x_n \in B(x, \epsilon)$ for all $n \geq N$. Hence, $U \cap A \neq \emptyset$, and so $x \in \overline{A}$

(ii) Conversely, suppose $x \in \overline{A}$. Fix $n \in \mathbb{N}$ and $U_n := B(x, 1/n)$. Then $U_n \cap A \neq \emptyset$ so $\exists x_n \in A$ such that $d(x, x_n) < 1/n$. It follows that $x_n \to x$.

□

Definition 6.11. Let (X, τ) be a topological space and $A \subset X$. A point $x \in X$ is said to be a limit point of A if, for every open set U containing x, $U \cap A$ contains a point of A other than x. Equivalently,

$$x \in \overline{(A \setminus \{x\})}$$

Write A' for the set of limit points of A.

Example 6.12.

(i) If $A \subset \mathbb{R}$ is a finite set, then A has no limit points. Similarly, $\mathbb{Z} \subset \mathbb{R}$ has no limit points.

(ii) Let τ be the co-finite topology on $\mathbb{R}$, and $A = \mathbb{Z}$, and let $x \in \mathbb{R}$ be any point. If U is an open neighbourhood of x, then $U \cap (\mathbb{Z} \setminus \{x\}) \neq \emptyset$ because U contains all but finitely many points of $\mathbb{R}$. Hence, every point of $\mathbb{R}$ is a limit point of $\mathbb{Z}$

(iii) If $A = [0, 1]$, then every point of A is a limit point of A.

(iv) If $A = \{1/n : n \in \mathbb{N}\}$, then 0 is the only limit point of A.

Proof. If $x \in A'$, then

(i) If $x < 0$, then $U := (x - |x|/2, x + |x|/2)$ is a neighbourhood of x, and $U \cap A = \emptyset$. Hence, $x \notin A'$.

(ii) If $x > 1$, then a similar argument shows that $x \notin A'$.

(iii) If $1 \geq x > 0$, and $x \notin A$, then $\exists N \in \mathbb{N}$ such that

$$\frac{1}{N+1} < x < \frac{1}{N}$$

So if $\delta = \min\{1/N - x, x - \frac{1}{N+1}\}$, then $U := (x - \delta/2, x + \delta/2)$ is an open neighbourhood of x such that $U \cap A = \emptyset$

(iv) If $1 \geq x > 0$ and $x \in A$, then $x = 1/N$ ofr some $N \in \mathbb{N}$. Once again,

$$\frac{1}{N+1} < x < \frac{1}{N-1}$$

so a similar argument shows that $x \notin A'$

(v) If $x = 0$, and U is an open set containing 0, then $\exists \delta > 0$ such that $(-\delta, +\delta) \subset U$. Choose $N \in \mathbb{N}$ such that $1/N < \delta$, so that $1/N \in U$, so that $U \cap (A \setminus \{0\}) \neq \emptyset$. Hence, $0 \in A'$.

□

Theorem 6.13. $\overline{A} = A \cup A'$

Proof. (i) $\overline{A} \subset A \cup A'$: Let $F := A \cup A'$ and $U := X \setminus F$. We claim that U is open. To see this, fix $x \in X \setminus F$. Then by definition, $\exists$ a neighbourhood V of x such that $V \cap (A \setminus \{x\}) = \emptyset$. Furthermore, $x \notin A$, so that $V \cap A = \emptyset$. Hence, $V \subset U$, so that U is open. Hence, F is closed, and since $A \subset F$, it follows that $\overline{A} \subset F$.

(ii) $A \cup A' \subset \overline{A}$: If $x \in A$, then $x \in \overline{A}$. Also, if $x \in A'$, then $x \in \overline{A}$ by definition. Hence, $A \cup A' \subset \overline{A}$.

□

Corollary 6.14. *A set A is closed iff it contains all its limit points.*

Example 6.15. Let $X = \mathbb{R}^\omega$ with the box topology, and

$$A := \{(x_n) \in X : x_n > 0 \quad \forall n \in \mathbb{N}\}$$

and let $0 = (0, 0, \ldots)$. Then

(i) $0 \in \overline{A}$: If U is any basic open set containing 0, then

$$U = \prod U_n$$

where $U_n \subset \mathbb{R}$ is open and contains 0. Hence, $\exists x_n \in U_n$ such that $x_n > 0$, so that $x := (x_n) \in A \cap U$. Hence, $A \cap U \neq \emptyset$.

(ii) Let $x^m = (x_n^m)$ be a sequence in A. Then consider the diagonal $a_n := x_n^n > 0$, and the open set $U_n = (-a_n, a_n) \subset \mathbb{R}$. Define $U := \prod U_n$, so that $0 \in U$. However, $x^m \notin U$ for all $m \in \mathbb{N}$. Hence, there is no sequence in A that converges to 0.

(iii) Hence, the box topology on $\mathbb{R}^\omega$ is not induced by a metric.

Definition 6.16. Let $A \subset X$

(i) A is said to be <u>dense</u> in X if $\overline{A} = X$. Equivalently, $U \cap A \neq \emptyset$ for any open set $U \subset X$

(ii) X is said to be <u>separable</u> if it has a countable dense subset.

(End of Week 4)

Example 6.17.

(i) $\mathbb{Q}$ is dense in $\mathbb{R}$, so $\mathbb{R}$ is separable.

Proof. If $x \in \mathbb{R}, \delta > 0$, then $(x - \delta, x + \delta) \cap \mathbb{Q} \neq \emptyset$. By Theorem 6.9, $\overline{\mathbb{Q}} = \mathbb{R}$ $\square$

(ii) If X, Y are topological spaces and A, B are dense in X and Y respectively. Then $A \times B$ is dense in $X \times Y$

Proof. If $U \subset X$ and $V \subset Y$ are open, then $U \cap A \neq \emptyset, V \cap B \neq \emptyset$. Hence, $(U \times V) \cap (A \times B) \neq \emptyset$ as required. $\square$

(iii) Hence, $\mathbb{R}^n$ is separable because $\mathbb{Q}^n$ is dense in it.

(iv) $\mathbb{R}^\omega$ is separable with respect to the product topology because

$$A = \{(x_n) \in \mathbb{R}^\omega : \exists N \in \mathbb{N} \text{ such that } x_n = 0 \forall n \geq N, x_n \in \mathbb{Q}\}$$

is dense in $\mathbb{R}^\omega$

Proof. Let
$$A_N = \{(x_n) : x_n \in \mathbb{Q}, x_n = 0 \quad \forall n \geq N\}$$
Then $A_N \cong \mathbb{Q}^{N-1}$, so A_N is countable. Hence, $A = \bigcup A_N$ is also countable. Now if U is a basic open set in $\mathbb{R}^\omega$, then write $U = \prod U_n$, where $U_n = \mathbb{R}$ for all $n \geq N$. Then $U_i \cap \mathbb{Q} \neq \emptyset$ for all $1 \leq i \leq N$, so choose $x_i \in U_i \cap \mathbb{Q}$. Then

$$x = (x_1, x_2, \ldots, x_N, 0, 0, \ldots)$$

is in $U \cap A$. Hence, $U \cap A \neq \emptyset$, so $\overline{A} = \mathbb{R}^\omega$ □

(v) $\mathbb{R}^\omega$ with the box topology is not separable.

Proof. Suppose $A = \{y^n\}$ is a countable subset of $\mathbb{R}^\omega$, we show that A is not dense. For each $n \in \mathbb{N}$, write
$$y^n = (y_1^n, y_2^n, \ldots, y_m^n, \ldots)$$
Now, $y_n^n \in \mathbb{R}$, so choose an open set $U_n \subset \mathbb{R}$ such that $y_n^n \notin U_n$. Then $U := \prod U_n$ is open in $\mathbb{R}^\omega$ and has the property that $y^n \notin U$ for all $n \in \mathbb{N}$. Hence, $A \cap U = \emptyset$ as required. □

Theorem 6.18. *Let $f : X \to Y$ be a function. Then TFAE:*

(i) f is continuous.

(ii) For every $A \subset X, f(\overline{A}) \subset \overline{f(A)}$

(iii) $f^{-1}(B)$ is closed in X whenever B is closed in Y.

Proof. (i) (i) $\Rightarrow$ (ii): Suppose f is continuous and $y \in f(\overline{A})$, then WTS: $y \in \overline{f(A)}$. Write $y = f(x)$ for some $x \in \overline{A}$, and choose an open set U such that $y \in U$. Then $f^{-1}(U)$ is an open neighbourhood of x. Hence, $f^{-1}(U) \cap A \neq \emptyset$, so choose $z \in f^{-1}(U) \cap A$. Then $f(z) \in U \cap f(A)$. Hence $U \cap f(A) \neq \emptyset$ so that $y \in \overline{f(A)}$.

(ii) (ii) $\Rightarrow$ (iii): Suppose B is closed, WTS: $A := f^{-1}(B)$ is closed. We have $f(A) = f(f^{-1}(B)) \subset B$ so if $x \in \overline{A}$, then

$$f(x) \in f(\overline{A}) \subset \overline{f(A)} \subset \overline{B} = B$$

Hence, $x \in f^{-1}(B) = A$. Hence, $\overline{A} \subset A$ whence $A = \overline{A}$ is closed.

(iii) (iii) $\Rightarrow$ (i): Take complements and apply the hypothesis.

□

7. Continuous Functions

Definition 7.1. A function $f : X \to Y$ is called a

(i) open map if $f(U)$ is open whenever $U \subset X$ is open.

(ii) homeomorphism if f is bijective, continuous, and $f^{-1} : Y \to X$ is also continuous. Equivalently, f is bijective, continuous and an open map. If such a homeomorphism exists, we say that X and Y are homeomorphic, and write $X \cong Y$.

Example 7.2.

(i) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x + 3$ is a homeomorphism because $g(y) := \frac{1}{2}(y-3)$ is the inverse.

(ii) Let $f : (-1, 1) \to \mathbb{R}$ given by $f(x) = x/(1-x^2)$. Then f is a homeomorphism with inverse

$$g(y) := \frac{2y}{1 + (1 + 4y^2)^{1/2}}$$

Hence, $(-1, 1) \cong \mathbb{R}$.

(iii) Let $Q = [-1, 1]^2 \subset \mathbb{R}^2$ and $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be the square and the disc in $\mathbb{R}^2$. Define $f : D \to Q$ by $f(0, 0) = (0, 0)$ and if $(x, y) \neq (0, 0)$, then

$$f(x, y) = \frac{\sqrt{x^2 + y^2}}{\max\{|x|, |y|\}}(x, y)$$

and $g : Q \to D$ by $g(0, 0) = (0, 0)$ and if $(x, y) \neq (0, 0)$, then

$$g(x, y) = \frac{\max\{|x|, |y|\}}{\sqrt{x^2 + y^2}}(x, y)$$

Hence, $Q \cong D$.

(iv) Let $f : [0, 1) \to S^1$ be $f(t) = (\cos(t), \sin(t))$. Then f is bijective and continuous, but not a homeomorphism, because if $U = [0, 1/4)$, then $p := f(0) \in f(U)$ is not an interior point of $f(U)$.

Note: This does not mean that $[0, 1) \ncong S^1$, but merely that *this* function is not a homeomorphism.

Theorem 7.3 (Rules for constructing Continuous functions)**.** *Let X, Y, Z be topological spaces.*

(i) (Constant function): If $f : X \to Y$ maps X to a single point $y_0 \in Y$, then f is continuous.

(ii) (Inclusion): If $Y \subset X$ has the subspace topology, then the inclusion map $\iota : Y \to X$ is continuous.

(iii) (Composition): If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

(iv) (Restricting the domain): If $f : X \to Y$ is continuous and $A \subset X$ has the subspace topology, then $f|_A : A \to Y$ is continuous.

(v) *(Restricting the range): Suppose $f : X \to Y$ is continuous, and $A \subset Y$ has the subspace topology. If $f(X) \subset A$, then the function $g : X \to A$ given by f is continuous.*

(vi) *(Expanding the range): Suppose $f : X \to Y$ is continuous, and $Y \subset Z$ has the subspace topology, then $f : X \to Z$ is continuous.*

Proof.

(i) If U is an open set, then $f^{-1}(U) = X$ if $y_0 \in U$ and $f^{-1}(U) = \emptyset$ if $y_0 \notin Y$. In either case, $f^{-1}(U)$ is open.

(ii) If $U \subset X$ is open, then $\iota^{-1}(U) = U \cap Y$, which is open in Y by definition.

(iii) HW1.

(iv) $f|_A = f \circ \iota$ where $\iota : A \to X$ is the inclusion map. So apply (iii).

(v) If $U \subset A$ is open, then $U = V \cap A$ for some open set $V \subset X$. Then $g^{-1}(U) = f^{-1}(V) \cap f^{-1}(A) = f^{-1}(V) \cap X = f^{-1}(V)$, which is open in X.

(vi) If $U \subset Z$ is open, then $f^{-1}(U) = f^{-1}(U \cap Y)$, which is open in X.

□

Theorem 7.4 (Pasting Lemma)**.**

(i) *Let $X = \bigcup_{\alpha \in J} U_\alpha$ where U_α is open, and let $f : X \to Y$ such that $f|_{U_\alpha} : U_\alpha \to Y$ is continuous for each $\alpha \in J$. Then $f : X \to Y$ is continuous.*

(ii) *Let $X = A \cup B$ where A and B are closed. Let $f : A \to Y$ and $g : B \to Y$ be continuous functions such that $f(x) = g(x)$ for all $x \in A \cap B$. Then $h : X \to Y$ given by*

$$h(x) = \begin{cases} f(x) & : x \in A \\ g(x) & : x \in B \end{cases}$$

is a well-defined continuous function from X to Y.

Proof.

(i) If $V \subset Y$ is open, then

$$f^{-1}(V) = \bigcup_{\alpha \in J} f^{-1}(V) \cap U_\alpha = \bigcup_{\alpha \in J} f|_{U_\alpha}^{-1}(V)$$

(ii) If $C \subset Y$ is a closed set, then [Check!]

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

which is closed.

□

Example 7.5.

(i) Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} 0 & : x \leq 0 \\ x & : x \geq 0 \end{cases}$$

defines a continuous function.

(ii) Let $f, g : X \to \mathbb{R}$ be continuous functions. Then

$$h_1(x) := \min\{f(x), g(x)\} \text{ and } h_2(x) := \max\{f(x), g(x)\}$$

are continuous functions [HW]

(iii) (Part (ii) of the Pasting Lemma fails for infinitely many closed sets). Let $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$, and $A_0 = \{0\}, A_i = \{1/i\}$ for $i \in \mathbb{N}$. Define $f_i : A_i \to \mathbb{R}$ by

$$f_i = \begin{cases} 0 & : i = 0 \\ 1 & : i \neq 0 \end{cases}$$

Then each f_i is continuous, and $A_i \cap A_j = \emptyset$ so they agree on the intersections. However, the function $f : X \to \mathbb{R}$ obtained by pasting them is not continuous.

Example 7.6 (Stereographic Projection)**.** Consider $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, and fix the north pole $N = (0, 0, 1)$. We claim

$$S^2 \setminus \{N\} \cong \mathbb{R}^2$$

Consider the plane passing through the equatorial circle. Fix $P = (x, y, z) \in S^2$. Draw a line from N through P, and let it meet the plane at the point $Q := (u, v, 0)$. Now taking ratios, we get

$$\begin{aligned} \frac{x}{y} &= \frac{u}{v} \\ \frac{y}{1-z} &= v \\ x^2 + y^2 + z^2 &= 1 \end{aligned}$$

Solving, we get

$$\begin{aligned} x &= \frac{2u}{1 + u^2 + v^2}, u = \frac{x}{1-z} \\ y &= \frac{2v}{1 + u^2 + v^2}, v = \frac{y}{1-z} \\ z &= \frac{1 - u^2 - v^2}{1 + u^2 + v^2} \end{aligned}$$

This gives a function $F : S^2 \setminus \{N\} \to \mathbb{R}^2$ given by

$$F(x, y, z) := (u, v, 0) = \left(\frac{x}{1-z}, \frac{y, 1-z}{,} 0\right),$$

and $G : \mathbb{R}^2 \to S^2 \setminus \{N\}$ given by

$$G(u,v) := (x,y,z) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2}\right).$$

Note that the map

$$(u,v) \mapsto 1+u^2+v^2$$

is continuous from $\mathbb{R}^2 \to \mathbb{R} \setminus \{0\}$ and

$$t \mapsto 1/t$$

is continuous from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$. Hence, by composition, both F and G are continuous, and inverses of each other. Hence, $S^2 \setminus \{N\} \cong \mathbb{R}^2$.

Note: The stereographic projection has the property that it preserves angles (such a map is called a conformal map). This is the same property that the mercator projection also has.

8. The Quotient Topology

Remark 8.1. Many spaces are constructed from other spaces by gluing, ie. by identifying parts of the space to obtain another space.

(i) A cylinder is obtained from a rectangle by identifying one pair of opposite edges.

(ii) The torus is obtained from a rectangle in $\mathbb{R}^2$ by identifying both pairs of opposite edges.

(iii) Consider X to be the union of two discs in $\mathbb{R}^2$. If we identify the boundary of one with the bounday of the other, we obtain the sphere S^2.

Definition 8.2. Let X be a set.

(i) An equivalence relation on X is a subset $R \subset X \times X$ such that, for all $x, y, z \in X$,

   (i) (Reflexive): $(x,x) \in R$

   (ii) (Symmetric): $(x,y) \in R \Rightarrow (y,x) \in R$

   (iii) (Transitive): $\{(x,y),(y,z)\} \subset R \Rightarrow (x,z) \in R$

   We write $x \sim y$ iff $(x,y) \in R$.

(ii) For $x \in X$, write

$$[x] := \{y \in X : y \sim x\}$$

for the equivalence class of x. Note that $[x] \cap [y] = \emptyset$ or $[x] = [y]$. Hence the equivalence classes partition X.

(iii) Write $X/\sim = \{[x] : x \in X\}$ to be the set of equivalence classes of $(X, \sim)$, and let $p : X \to X^*$ be the map $x \mapsto [x]$.

Example 8.3.

(i) If $X = \bigsqcup_{\alpha \in J} A_\alpha$ is a partition of X. Write $x \sim y$ iff $\exists \alpha \in J$ such that $\{x, y\} \subset A_\alpha$. Then this is an equivalence relation whose equivalence classes are precisely the A_α.

(ii) Let $A \subset X$. Define $x \sim y$ iff $\{x, y\} \subset A$. Then $\sim$ is an equivalence relation whose equivalence classes are either A or singleton sets. In this case, we write

$$X/A := X/\sim$$

(iii) If $X = [0, 1]$, then define $0 \sim 1$ and $x \nsim y$ if $\{x, y\} \neq \{0, 1\}$. Then $X/\sim$ can be thought of as gluing the end-points of X.

(iv) If $X = \mathbb{R}$, write $x \sim y$ iff $x - y \in \mathbb{Z}$.

(v) If $X = [0, 1]^2$, write

$$\begin{aligned}
(x, 0) &\sim (x, 1), \text{ for } 0 \leq x \leq 1 \\
(0, y) &\sim (1, y), \text{ for } 0 \leq y \leq 1
\end{aligned}$$

This gives equivalence classes

$$\begin{aligned}
[(x, y)] &= \{(x, y)\} : 0 < x, y < 1 \\
[(x, 0)] &= \{(x, 0), (x, 1)\} : 0 < x < 1 \\
[(0, y)] &= \{(0, y), (1, y)\} : 0 < y < 1 \\
[(0, 0)] &= \{(0, 0), (1, 0), (0, 1), (1, 1)\}
\end{aligned}$$

ie. Opposite edges of the square are identified, and the vertices collapse to a single point.

Lemma 8.4. *Let X be a topological space, and Y any set. Suppose $p : X \to Y$ is a function. Define*

$$\tau_Y := \{U \subset Y : p^{-1}(U) \in \tau_X\}$$

Then

(i) τ_Y is a topology on Y,

(ii) $p : X \to Y$ is a continuous function.

(iii) If η is any topology on Y such that $p : X \to (Y, \eta)$ is continuous, then $\eta \subset \tau_Y$. ie. τ_Y is the largest topology that makes p continuous.

Proof.

(i) To see that τ_Y is a topology.

  (i) $\emptyset = p^{-1}(\emptyset)$ and $X = p^{-1}(Y)$, so $\emptyset, Y \in \tau_Y$

(ii) If $\{U_\alpha : \alpha \in J\} \subset \tau_Y$, then

$$p^{-1}(\bigcup U_\alpha) = \bigcup p^{-1}(U_\alpha) \in \tau_X$$

so $\bigcup U_\alpha \in \tau_Y$.

(iii) Similarly, τ_Y is closed under finite intersection.

(ii) Obvious.

(iii) Suppose η is as above, then for any $U \in \eta, p^{-1}(U) \in \tau_X$, so $U \in \tau_Y$ by definition. Hence, $\eta \subset \tau_Y$.

□

(End of Week 5)

Definition 8.5. Let X be a set and $\sim$ an equivalence relation of X. Let $p : X \to X/\sim$ be the map $x \mapsto [x]$. The quotient topology on $X/\sim$ is the topology induced by p as in the above lemma. ie. A set $U \subset X/\sim$ is open iff

$$\bigcup_{[x]\in U} [x]$$

is open in X.

Example 8.6.

(i) If $X = [0, 1]$ with $0 \sim 1$. Then $U = \{[x] : 0 \leq x < 1/4\}$ is not an open set because

$$\bigcup_{[x]\in U} [x] = [0, 1/4) \cup \{1\}$$

whereas $U = \{[x] : 0 \leq x < 1/4, \text{ or } 3/4 < x \leq 1\}$ is an open set.

(ii) Similarly, if $X = [0, 1]^2$ with the relation in Example 8.3, then (draw picture of open set bounded by an edge, and not having a counterpart on the opposite edge)

Theorem 8.7 (Universal Property of Quotient Spaces)**.** *Let X be a set with an equivalence relation $\sim$, let $X/\sim$ be given the quotient topology, and let $p : X \to X/\sim$ be the natural map. Let Y be a topological space, and $f : X \to Y$ be a function such that*

$$x \sim x' \Rightarrow f(x) = f(x')$$

Then $\exists$ a unique function $\overline{f} : X/\sim \to Y$ such that

$$f = \overline{f} \circ p$$

Furthermore, f is continuous iff $\overline{f}$ is continuous.

Proof.

(i) Given $f : X \to Y$ as above, define $\overline{f} : X^* \to Y$ by

$$\overline{f}([x]) := f(x)$$

This is well-defined and satisfies $\overline{f} \circ p = f$. Furthermore, if $g : X/\sim \to Y$ is any other function such that $g \circ p = f$. Then $g \circ p = \overline{f} \circ p$. But p is surjective, so $g = \overline{f}$, so $\overline{f}$ is unique.

(ii) Suppose $\overline{f}$ is continuous, then $f = \overline{f} \circ p$ is continuous by Lemma 8.4. Conversely, suppose f is continuous. WTS: $\overline{f}$ is continuous. So choose an open set $U \subset Y$, then WTS: $\overline{f}^{-1}(U) \subset X/\sim$ is open. By definition, this is equivalent to asking if $p^{-1}(\overline{f}^{-1}(U) = (\overline{f} \circ p)^{-1}(U)$ is open in X, which is true.

□

Example 8.8.

(i) Let $X = [0,1]$ with $0 \sim 1$, then $X^* \cong S^1$

Proof. Define $f : X \to S^1$ by $f(x) = e^{2\pi i x}$, then f is continuous, and $f(0) = f(1)$. Hence, we get a continuous function $\overline{f} : X/\sim \to S^1$ as above. We want to construct an inverse $g : S^1 \to X/\sim$. Write

$$A_1 = \{z \in S^1 : \operatorname{Im}(z) \geq 0\}, \text{ and } A_2 = \{z \in S^1 : \operatorname{Im}(z) \leq 0\}$$

Then A_1 and A_2 are closed sets and $A_1 \cap A_2 = \{\pm 1\}$. We now use the pasting lemma. Given $z \in A_1, \exists$ unique $t \in [0, 1/2]$ such that $z = e^{2\pi i t}$. Define $h_1 : A_1 \to [0,1]$ by $h_1(z) = t$. Similarly, if $z \in A_2, \exists$ unique $t' \in [1/2, 1]$ such that $z = e^{2\pi i t'}$, so define $h_2(z) = t'$. Note that h_1 and h_2 are continuous, but do not agree on $A_1 \cap A_2$ because

$$h_1(1) = 0, \text{ but } h_2(1) = 1$$

Now define $g_i : A_i \to X/\sim$ by $g_i = p \circ h_i$. Then g_i are continuous (because the h_i are continuous), and they agree on $A_1 \cap A_2$. Hence by pasting lemma, they define a continuous function $g : S^1 \to X/\sim$. Now note that

$$g \circ \overline{f}([t]) = g(f(t)) = g(e^{2\pi i t}) = [t]$$

and similarly,

$$(\overline{f} \circ g)(z) = z \quad \forall z \in S^1$$

Hence, $\overline{f}$ is a homeomorphism. □

(ii) If $X = \mathbb{R}$ and $x \sim y$ iff $x - y \in \mathbb{Z}$, then define $f : \mathbb{R} \to S^1$ by $f(x) = e^{2\pi i x}$. As above, we get a homeomorphism $\mathbb{R}/\sim \cong S^1$.

(iii) Similarly, if $X = [0,1]^2$ with the equivalence relation in Part (v) of Example 8.3, then $X/\sim \cong S^1 \times S^1$. This is the torus.

(iv) Let $D^2 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$. Then $S^1 \subset D^2$. We claim

$$D^2/S^1 \cong S^2$$

Proof. (i) Write $D^2 = int(D^2) \sqcup S^1$. Now define $f_1 : \mathbb{R}^2 \to int(D^2)$ by

$$f_1(x,y) = \frac{1}{\sqrt{x^2+y^2+1}}(x,y)$$

Then f_1 is a homeomorphism. Let $f_2 : \mathbb{R}^2 \to S^2 \setminus \{N\}$ be the inverse of the stereographic projection, so $\widehat{f} = f_2 \circ f_1 : int(D^2) \to S^2 \setminus \{N\}$ is a homeomorphism.

(ii) Define $f : D^2 \to S^2$ by

$$f(x) = \begin{cases} \widehat{f}(x) & : x \in int(D^2) \\ N & : x \in S^1 \end{cases}$$

We claim that f is continuous. It suffices to check continuity on S^1, so fix $x_0 \in S^1$ and an open set $U \subset S^2$ containing $N = f(x_0)$. Then $\exists \delta > 0$ such that $B_{\mathbb{R}^3}(N,\delta) \cap S^2 \subset U$. By definition of the stereographic projection, $\exists R > 0$ such that

$$\sqrt{x^2+y^2} > R \Rightarrow f_2(x,y) \in U$$

Hence, $\exists 0 < r < 1$ such that

$$\sqrt{x^2+y^2} > r \Rightarrow \widehat{f}(x,y) \in U$$

Hence, $f^{-1}(U)$ contains the set

$$V = \{(x,y) \in D^2 : x^2+y^2 > r^2\}$$

which is open in D^2 and contains x_0

(iii) Thus, f is continuous. Clearly, $x \sim y$ if and only if $f(x) = f(y)$, so by Theorem 8.7, f induces a map

$$\overline{f} : D^2/S^1 \to S^2$$

This map is both continuous and bijective. We will show later this is enough to conclude that $\overline{f}$ is a homeomorphism.

□

Definition 8.9.

(i) Consider

$$S^n = \{(x_1,x_2,\ldots,x_{n+1}) \in \mathbb{R}^{n+1} : \sum x_i^2 = 1\}$$

Define $\overline{x} \sim \overline{y}$ iff $\overline{y} = -\overline{x}$ (antipodal points are identified). Then we define

$$\mathbb{R}P^n := S^n/\sim$$

This is called the <u>real projective space</u>.

(ii) Consider $X = [0,1]^2$, and define $\sim$ by $(0,y) \sim (1, 1-y)$. The quotient space $X/\sim$ is called the Möbius strip.

(iii) Let $X = [0,1]^2$ and define $\sim$ by $(0,y) \sim (1, 1-y)$ and $(x,0) \sim (x,1)$. The quotient space $X/\sim$ is called the Klein bottle.

III. Properties of Topological Spaces

1. The Hausdorff property

Definition 1.1. A topological space X is said to be Hausdorff (T_2) if, for each $x, y \in X$ and distinct point, then $\exists$ open sets U, V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Example 1.2.

(i) Every metric space is Hausdorff.

Proof. If $x, y \in X$ such that $x \neq y$, then $\delta := d(x, y) > 0$, so let $U = B(x, \delta/2)$ and $V = B(y, \delta/2)$ □

(ii) If X is Hausdorff, and $Y \subset X$, then Y is Hausdorff.

Proof. If $x, y \in Y$ are distinct, then $\exists U, V \subset X$ open such that $x \in U, y \in V$ and $U \cap V = \emptyset$. So let $U' = U \cap Y$ and $V' = V \cap Y$. □

(iii) If X and Y are Hausdorff, then so is $X \times Y$.

Proof. If $(x_1, y_1) \neq (x_2, y_2)$, then assume without loss of generality that $x_1 \neq x_2$, so $\exists U, V \subset X$ open such that $U \cap V = \emptyset$ and $x_1 \in U, x_2 \in V$. Now consider $U' = U \times Y, V' = V \times Y$. Then $U' \cap V' = \emptyset$ and $(x_1, y_1) \in U', (x_2, y_2) \in V'$. □

(iv) Similarly if each X_α is Hausdorff, then so is $\prod X_\alpha$ in either the product or the box topology.

(v) If X has the indiscrete topology, then it is not Hausdorff.

(vi) If $\mathbb{R}$ has the co-finite topology, then it is not Hausdorff.

Proof. Any two open sets must intersect non-trivially. □

Definition 1.3. A topological space X is said to be T_1 is singleton sets are closed in X. Equivalently, if $x \neq y$ are distinct points, then $\exists$ an open set U such that $x \in U$ and $y \notin U$.

Example 1.4. (i) If X is T_2, then it is T_1

Proof. If $x \in X$, then WTS: $X \setminus \{x\}$ is open. But if $y \in X \setminus \{x\}$, then by the Hausdorff property, $\exists V$ open such that $y \in V$ and $V \subset X \setminus \{x\}$. Hence, $X \setminus \{x\}$ is open as required. □

(ii) $\mathbb{R}$ with the co-finite topology is T_1 but not T_2

Proof. If $x \in \mathbb{R}$, then by definition, $\mathbb{R} \setminus \{x\}$ is an open set, so $\{x\}$ is closed. $\square$

(iii) If X has the indiscrete topology and $|X| \geq 2$, then X is not T_1

Theorem 1.5. *Let X be Hausdorff, and $(x_n) \subset X$. Then (x_n) can converge to atmost one point in X.*

Proof. If $x_n \to x$, and $x \neq y$, then choose neighbourhoods U, V such that $x \in U, y \in V$ and $U \cap V = \emptyset$. Then $\exists N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. Hence, at most finitely many x_j may lie in V. Hence, (x_n) does not converge to y. $\square$

Example 1.6. Recall that if $\mathbb{R}$ has the co-finite topology, and $x_n = n$, then for any open set $U \subset \mathbb{R}, \exists N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq \mathbb{N}$. Hence, $x_n \to a$ for all $a \in \mathbb{R}$.

Remark 1.7.

(i) Let X be a topological space and X^* be a quotient space of X. Then a set $A \subset X^*$ is closed iff
$$\bigcup_{[x] \in A} [x]$$
is closed in X. Hence, X^* is T_1 if and only each $[x]$ is closed in X.

(ii) For example, all the spaces constructed in the previous section are T_1. However, if $A = \mathbb{Q} \subset \mathbb{R}$, then $\mathbb{R}/\mathbb{Q}$ (the topological space) is not T_1 because $\mathbb{Q}$ is not closed in $\mathbb{R}$. Hence, it is not true that if X is Hausdorff, then X^* is Hausdorff.

(End of Week 6)

2. Connectedness

Definition 2.1. Let X be a topological space.

(i) A <u>separation</u> of X is a pair $\{U, V\}$ of non-empty open sets such that $X = U \cup V$ and $U \cap V = \emptyset$.

(ii) A space X is said to be <u>connected</u> if it does not have a separation.

(iii) A set $A \subset X$ is called <u>cl-open</u> if it is both closed an open.

Lemma 2.2. *X is connected iff the only sets in X that are both open and closed are $\emptyset$ and X (ie. X has no non-trivial cl-open sets)*

Proof. If X has a non-trivial cl-open set U, then $V := X \setminus U$ is cl-open, and $\{U, V\}$ is a separation of X. Conversely, if X is not connected, then it has a separation $\{U, V\}$ of disjoint non-empty sets. Then U is a non-trivial cl-open set. $\square$

Example 2.3.

(i) If X has the indiscrete topology, then X is connected.

(ii) If X has the discrete topology and $|X| \geq 2$, then X is disconnected.

(iii) $\mathbb{R}$ is connected.

(iv) $\mathbb{Q} \subset \mathbb{R}$ is not connected.

Lemma 2.4. *If $A \subset X$ is connected, and $A \subset B \subset \overline{A}$, then B is connected. In particular, $\overline{A}$ is connected.*

Proof. If B has a separation $\{U, V\}$, then $U_1 := U \cap A, V_1 := V \cap A$ are disjoint open subsets of A. Furthermore, $U_1 \neq \emptyset$ because $U \subset \overline{A}$ is open (by Theorem 6.9). Similarly, $V_1 \neq \emptyset$, so $\{U_1, V_1\}$ is a separation of A. This is a contradiction. □

Theorem 2.5. *Any interval in $\mathbb{R}$ is connected. In particular, $\mathbb{R}$ is connected.*

Proof. By the previous lemma, it suffices to consider closed intervals $Y = [a, b]$.

(i) Suppose $\{U, V\}$ is a separation of Y, then $U = U_1 \cap Y, V = V_1 \cap Y$ for some open sets $U_1, V_1 \subset \mathbb{R}$. Assume WLOG that $a \in U$. Since U is open in $Y, \exists \delta > 0$ such that $[a, a + \delta) \subset U$. Define

$$c := \sup A, \text{ where } A := \{x \in [a, b] : [a, x] \subset U\}$$

Note that $c > a$ by the above argument.

(ii) Claim: $c \in U$.

Proof. For each $\epsilon > 0, c - \epsilon$ is not an upper bound for the set A, so $\exists x \in A$ such that

$$c - \epsilon < x < c$$

Now $[a, x] \subset U$, so Hence, $(c - \epsilon, c + \epsilon) \cap U \neq \emptyset$. Hence, $c \in cl_{\mathbb{R}}(U)$ by Theorem 6.9. But Y is closed in $\mathbb{R}$, so $c \in cl_Y(U)$ (Lemma 6.8). But U is closed in Y, so $c \in U$. □

(iii) Claim: $c = b$.

Proof. Suppose $c < b$, then since $c \in U$ and U is open in $Y, \exists \delta > 0$ such that $[c, c + \delta) \subset U \cap Y$. Hence, $[a, c + \delta/2] \subset U$, which contradicts the fact that $c = \sup A$. Hence, $c = b$. □

Thus, $[a, b] \subset U$, so that V is empty. □

Proposition 2.6. *The only connected subsets of $\mathbb{R}$ are intervals.*

Proof. Suppose $Y \subset \mathbb{R}$ is connected is not an interval. Then $\exists a < c < b$ such that $\{a, b\} \subset Y$ and $c \notin Y$. Hence, $U := (-\infty, c) \cap Y$ and $V := (c, \infty) \cap Y$ form a separation of Y. □

Theorem 2.7. *Let X be a topological space and $\{A_\alpha : \alpha \in J\}$ be a collection of connected sets such that*

$$\bigcap A_\alpha \neq \emptyset$$

Then $A := \bigcup A_\alpha$ is connected.

Proof. Let $\{U, V\}$ be a separation of A, then for any $\beta \in J$, $\{U \cap A_\beta, V \cap A_\beta\}$ are two disjoint cl-open sets in A_β. By Lemma 2.2, either $U \cap A_\beta = A_\beta$ or $V \cap A_\beta = A_\beta$. i.e. either $A_\beta \subset U$ or $A_\beta \subset V$. Let

$$J_1 := \{\alpha \in J : A_\alpha \subset U\} \text{ and } J_2 = \{\alpha \in J : A_\alpha \subset V\}$$

Since $\{U, V\}$ is a separation of A, it follows that J_1, J_2 are both non-empty. However, if $x \in \cap A_\alpha$, then $x \in U \cap V$. This contradicts the fact that $U \cap V = \emptyset$. $\square$

Theorem 2.8. *Let X, Y be connected, then $X \times Y$ is connected.*

Proof. Fix $a \in X, b \in Y$, then $Y_a := \{a\} \times Y \cong Y$ is connected, and $X_b := X \times \{b\}$ is connected. Furthermore, $X_a \cap Y_b = \{(a, b)\} \neq \emptyset$. Hence, $X_b \cup Y_a$ is connected by the previous lemma. Now consider $A_b := X_b \cup Y_a, b \in Y$. Then A_b is connected, and

$$X \times Y = \bigcap A_b = Y_a \neq \emptyset$$

So by the previous theorem, $X \times Y$ is connected. $\square$

Example 2.9.

(i) Let $X = \mathbb{R}^\omega$ with the product topology, then X is connected.

Proof. Write

$$X_n = \{(x_1, x_2, \ldots, x_n, 0, 0, \ldots) : x_i \in \mathbb{R}\} \subset X$$

Then $X_n \cong \mathbb{R}^n$, so X_n is connected by the previous theorems and induction. Furthermore, $\bigcap X_n = \{\overline{0}\} \neq \emptyset$. Hence,

$$A := \bigcup_{n=1}^{\infty} X_n$$

is connected. We claim: $X = \overline{A}$. Fix $x = (x_n) \in X$ and an open set U containing x. Then we may assume that

$$U := \prod_{n=1}^{\infty} U_n$$

where $U_n = \mathbb{R}$ for all $n \geq N$. Then for

$$y := (x_1, x_2, \ldots, x_N, 0, 0, \ldots)$$

we have $y \in A$ and $y \in U$, so $U \cap A \neq \emptyset$. Hence, $\overline{A} = X$, so X is connected by Lemma 2.4. $\square$

(ii) Let $X = \mathbb{R}^\omega$ with the box topology, then X is disconnected.

Proof. Let

$$A := \{(x_n) \in \mathbb{R}^\omega : \exists M \in \mathbb{N} \text{ such that } |x_n| \leq M \quad \forall n \in \mathbb{N}\}$$

be the set of all bounded sequences. Then $A \neq \emptyset$ and $A \neq X$. We claim that A is cl-open, which would prove that $\mathbb{R}^\omega$ is disconnected.

- To see that A is open, fix $x = (x_n) \in A$, and consider

$$V := \prod_{n=1}^{\infty} (x_n - 1, x_n + 1)$$

  Then V is open, and if $y = (y_n) \in V$, then

$$|y_n| < |x_n| + 1$$

  so $(y_n) \in A$.

- To see that A is closed, fix $x = (x_n) \notin A$, and

$$V := \prod_{n=1}^{\infty} (x_n - 1, x_n + 1)$$

  If $y = (y_n) \in V$ is bounded, then $|x_n| \leq |y_n| + 1$ would imply that $x \in A$. This is a contradiction, so $V \subset X \setminus A$. Hence, $X \setminus A$ is open, so A is closed.

□

Theorem 2.10. *Let $f : X \to Y$ be a continuous function. If X is connected, then so is $f(X)$ (ie. the continuous image of a connected set is connected).*

Proof. If $f(X)$ has a separation $\{U, V\}$, then $\{f^{-1}(U), f^{-1}(V)\}$ would be open sets, and

$$X = f^{-1}(f(X)) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

and

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$$

so $\{f^{-1}(U), f^{-1}(V)\}$ would be a separation of X. Since X is connected, this cannot happen. □

Corollary 2.11. *If X is connected, and $\sim$ and equivalence relation on X, then $X/\sim$ is connected.*

Theorem 2.12 (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $d \in \mathbb{R}$ such that $f(a) < d < f(b)$. Then $\exists c \in [a, b]$ such that $f(c) = d$.*

Proof. By the previous theorems, $f([a, b])$ is a connected subset of $\mathbb{R}$, and is hence an interval. In particular, $f(a), f(b) \in f([a, b])$, so $d \in f([a, b])$. This implies the result. □

Corollary 2.13. $\mathbb{R}^n \cong \mathbb{R}$ *iff* $n = 1$

(In fact, it is true that $\mathbb{R}^n \cong \mathbb{R}^m$ implies that $n = m$, but that is much harder to prove.)

Proof. Assume $n > 1$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a homeomorphism. We will show (in the next section) that $\mathbb{R}^n \setminus \{0\}$ is connected, so $f(\mathbb{R}^n \setminus \{0\}) = f(\mathbb{R}^n) \setminus \{f(0)\}$ must be connected. But

$$f(\mathbb{R}^n \setminus \{0\}) = f(\mathbb{R}^n) \setminus \{f(0)\} = \mathbb{R} \setminus \{c\} = (-\infty, c) \bigsqcup (c, \infty)$$

which is disconnected. This is a contradiction □

3. Path Connectedness

Definition 3.1. Let X be a topological space.

(i) A path between two points $x, y \in X$ is a continuous function $f : [0,1] \to X$ such that $f(0) = x, f(1) = y$.

(ii) A space X is said to be path connected if any two points in X are connected by a path.

Remark 3.2. Every interval $[a,b]$ is homeomorphic to $[0,1]$ (via the map $t \mapsto at + (1-t)b$), so we may as well write $f : [a,b] \to X$ is the above definition.

Proposition 3.3. *A path connected space is connected.*

Proof. If $\{U, V\}$ is a separation for X, then choose $x \in U, y \in V$. By hypothesis, there is path $f : [0,1] \to X$ such that $f(0) = x, f(1) = y$. Consider $U' := f^{-1}(U)$ and $V' := f^{-1}(V)$. Then these are non-empty open sets and $[0,1] = f^{-1}(X) = f^{-1}(U) \cup f^{-1}(V)$, so $[0,1]$ must be disconnected. This contradicts 2.5. $\square$

Theorem 3.4. *If $f : X \to Y$ is continuous, and X is path connected, then $f(X)$ is path connected.*

Proof. Given $u, v \in f(X)$, write $u = f(x), v = f(y)$ for some $x, y \in X$. Let $g : [0,1] \to X$ be a path from x to y, then $f \circ g$ is path from u to v. $\square$

Corollary 3.5. *If X is path connected, then any quotient space of X is path connected.*

Definition 3.6. A set $X \subset \mathbb{R}^n$ is said to be convex if, for any $x, y \in X$ and $0 \leq t \leq 1$, the point $z := tx + (1-t)y \in X$.

Lemma 3.7. *Any convex subset of $\mathbb{R}^n$ is path connected. In particular, $\mathbb{R}^n$, and every (closed or open) ball in $\mathbb{R}^n$ is path connected.*

Proof. Consider the straight line path $f : [0,1] \to X$ by $f(t) := tx + (1-t)y$ and check that this is continuous. $\square$

Lemma 3.8. *Let X be a topological space and $\{A_\alpha : \alpha \in J\}$ be a collection of path connected sets such that, for any two $\alpha, \beta \in J$, $\exists \gamma \in J$ such that*

$$A_\alpha \cap A_\gamma \neq \emptyset \text{ and } A_\beta \cap A_\gamma \neq \emptyset$$

Then $A := \bigcup A_\alpha$ is path connected.

Proof. Fix $x, y \in A$, then $\exists \alpha, \beta \in J$ such that $x \in A_\alpha, y \in A_\beta$. Let $\gamma \in J$ as in the hypothesis, and $z_1 \in A_\alpha \cap A_\gamma, z_2 \in A_\beta \cap A_\gamma$. Since A_α is path connected, $\exists f_1 : [0,1] \to A_\alpha$ continuous such that $f_1(0) = x, f_1(1) = z_1$. Similarly, $\exists f_2 : [1,2] \to A_\gamma$ such that

$f_2(1) = z_1, f_2(2) = z_2$, and $\exists f_3 : [2,3] \to A_\beta$ such that $f_3(2) = z_2$ and $f_3(3) = y$. Define $h : [0,3] \to A$ by

$$h(x) = \begin{cases} f_1(x) & : 0 \leq x \leq 1 \\ f_2(x) & : 1 \leq x \leq 2 \\ f_3(x) & : 2 \leq x \leq 3 \end{cases}$$

Then h is continuous by pasting lemma and Theorem 7.3, and $h(0) = x, h(3) = y$. So by Remark 3.2, A is path connected. □

Example 3.9.

(i) If $n > 1$, then $\mathbb{R}^n \setminus \{0\}$ is path connected.

Proof. For each $1 \leq i \leq n$, let

$$A_i := \{\overline{x} \in \mathbb{R}^n : x_i > 0\}, \text{ and } B_i := \{\overline{x} \in \mathbb{R}^n : x_i < 0\}$$

Then A_i and B_i are convex (check!) and satisfy the hypotheses of Lemma 3.8. Hence,

$$\mathbb{R}^n \setminus \{0\} = \bigcup A_i \cup B_i$$

is path connected. □

(ii) $S^n \subset \mathbb{R}^{n+1}$ is path connected.

Proof. The map $g : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ given by $x \mapsto x/d(x,0)$ is a continuous surjective map. So apply Theorem 3.4. □

(iii) The following quotient spaces are all path connected: The Torus, The Mobius strip, the Klein bottle, the real projective space.

Theorem 3.10. *If each X_α is path connected, then $\prod X_\alpha$ is path connected with the product topology.*

Proof. Given $x = (x_\alpha), y = (y_\alpha) \in X := \prod X_\alpha$, for each $\beta \in J$, there is a path $f_\beta : [0,1] \to X_\beta$ such that $f(0) = x_\beta$ and $f(1) = y_\beta$. Define $f : [0,1] \to X$ by $f(t) = (f_\alpha(t))$, then f is continuous because each component of f is continuous. And clearly $f(0) = x, f(1) = y$, so X is path connected. □

Remark 3.11. Note that the above result is not true with the box topology: $\mathbb{R}^\omega$ is not connected with the box topology, so cannot be path connected. (See Example 2.9)

Example 3.12 (The Topologists' Sine Curve)**.** Define

$$S := \{(x, \sin(1/x)) : 0 < x \leq 1\} \subset \mathbb{R}^2$$

and let $X = \overline{S}$. Then note that

$$X = S \cup \{0\} \times [-1,1]$$

Then X is connected, but not path connected.

Proof. The map $f : (0,1] \to S$ given by $x \mapsto (x, \sin(1/x))$ is continuous, and $(0,1]$ is connected. Hence, S is connected (Note: In fact, S is path connected). By Lemma 2.4, X is connected. We claim there is no path from $(0,0)$ to any point of S. Suppose $f : [0,1] \to X$ is such a path, consider

$$A = \{t \in [0,1] : f(t) \in \{0\} \times [-1,1]\}$$

and let $a := \sup(A)$. By hypothesis, $a < 1$. Consider $f|_{[a,1]}\colon [a,1] \to X$ and write $f(t) = (x(t), y(t))$. Then $x(0) = 0$ and $x(t) > 0$ for all $t > a$, so that $y(t) = \sin(1/x(t))$ for all $t > a$. We claim: $\exists (t_n) \subset [a,1]$ such that $t_n \to a$ and $y(t_n) = (-1)^n$.

For $n \in \mathbb{N}$ fixed, choose $0 < u < x(a + 1/n)$ such that $\sin(1/u) = (-1)^n$. By the intermediate value theorem, $\exists a < t_n < a + 1/n$ such that $f(t_n) = (t_n, (-1)^n)$. This proves the claim.
Hence, $t_n \to 0$ and $f(t_n) = (t_n, (-1)^n)$ does not converge. Hence, f is not continuous. $\square$

Remark 3.13.

(i) The above example also shows that even if A is path connected, then $\overline{A}$ may not be path connected (compare with Lemma 2.4)

(ii) There are two other examples similar to the topologists' sine curve:

  (i) The deleted infinite broom: For $n \in \mathbb{N}$, let L_n denote the line segment in $\mathbb{R}^2$ connecting $(0,0)$ to $(1, 1/n)$. Let

  $$S := \bigcup_{n=1}^{\infty} L_n, \text{ and } X := S \setminus \{(0,1)\}$$

  Then S is called the infinite broom, and X the deleted infinite broom. Once again, X is connected, but not path connected.

  (ii) The deleted comb space: Define

  $$D := ([0,1] \times \{0\}) \cup \bigcup_{n=1}^{\infty} (\{1/n\} \times [0,1]) \cup [0,1]$$

  and $X := D \setminus \{(0,1)\}$. Then D is called the comb space, and X the deleted comb space. Once again, X is connected, but not path connected.

4. Local Connectedness

Definition 4.1. Let X be a topological space. Write $x \sim y$ if there is a connected subspace $A \subset X$ such that $\{x, y\} \subset A$.

Lemma 4.2. *The above relation is an equivalence relation, and the equivalence classes are the maximal connected subsets of X (ie. if C is an equivalence class, and B is a connected set such that $C \subset B$, then $C = B$). These equivalence classes are called the connected components of X.*

Proof. That this is an equivalence class is easy to see. For any $x \in X$,

$$\begin{aligned}[x] &= \{y \in X : x \sim y\} \\ &= \{y \in X : \exists A_y \text{ connected, such that } \{x, y\} \subset A_y\} \\ &= \bigcup_{y \in [x]} A_y\end{aligned}$$

Each A_y is connected, and $\bigcap A_y \supset \{x\} \neq \emptyset$, so by 2.7, $[x]$ is connected. Furthermore, if B is a connected set such that $[x] \subset B$, and $y \in B$, then $\{x, y\} \subset B$, so by definition, $y \in [x]$. Hence, $[x]$ is maximal as well. □

Definition 4.3. Let X be a topological space. Write $x \sim_h y$ if there is a path $f : [0,1] \to X$ such that $f(0) = x, f(1) = y$.

Lemma 4.4. *The above relation is an equivalence relation, and the equivalence classes are the maximal path connected subsets of X. These are called the path components of X.*

Proof. To show that $\sim_h$ is an equivalence relation:

(i) $x \sim x$: Consider the constant path

(ii) $x \sim y \Rightarrow y \sim x$: If $f : [0,1] \to X$ is such that $f(0) = x, f(1) = y$, take $g(s) := f(1-s)$, then g is continuous, $g(0) = y, g(1) = y$.

(iii) If $x \sim y, y \sim z$: To show that $x \sim z$, simply use the pasting lemma as in 3.8 to join the two paths.

That the equivalence classes are path connected, and maximal is exactly as in 4.2. □

Example 4.5.

(i) If X is connected, it has only one component.

(ii) If $X = \mathbb{Q}$, then the connected components are singletons.

Proof. If $A \subset X$ has at least two points, then $\exists a, b \in A$ and $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x < b$. Hence, $U := (-\infty, x) \cap A$ and $V := (x, \infty) \cap A$ forms a separation of A, so A is disconnected. Hence, the only connected sets are singletons. □

Definition 4.6.

(i) A topological space X is said to be locally connected if, for each $x \in X$ and each open set $U \ni x, \exists$ an open neighbourhood $V \subset U$ of x that is connected.

(ii) We define locally path connected similarly.

Example 4.7.

(i) Locally path connected implies locally connected.

(ii) $A = (0,1) \sqcup (2,3)$ is locally (path) connected, but not connected.

(iii) If $A = \{0\} \cup \{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$, then A is not locally connected because, for any $1 > \delta > 0$, $B(0,\delta) \cap A$ is a finite set, and hence disconnected.

(iv) However, connected does not imply local connectedness: Consider the topologists' sine curve X from 3.12, and $x = (0,1) \in X$. Fix $\delta < 1$ and consider $U = B(x,\delta) \cap X$. Then U is a disjoint union of infinitely many line segments $U = \sqcup L_n$. Each such L_n is a cl-open set in U, so U is disconnected.

(v) Similarly, path connectedness does not imply local path connectedness: Define

$$X = \bigcup_{n=1}^{\infty} \left\{ \left(\frac{1}{n}, y) \right) : y \in \mathbb{R} \right\} \cup \{(0,y) : y \in \mathbb{R}\} \cup \{(x,0) : x \in \mathbb{R}\}$$

Then X is clearly path connected, but if $x = (0,1) \in X$, and $\delta < 1$, then $U = B(x,\delta) \cap X$ is once again a disjoint union of line segments. Hence, U is not path connected either.

Lemma 4.8. *(i) If X is locally connected, then components are open sets. Hence each component is cl-open.*

(ii) If X is locally path connected, then each path component is open in X. Hence, each path component is cl-open.

Proof. We prove (i), because (ii) is identical: If C is a component of x and $x \in C$, then $\exists$ a connected neighbourhood U of x. It follows that $U \subset C$, so C is open. Now if each component is open, and X is a disjoint union of components, then each component must also be closed. □

Theorem 4.9. *Let X be a topological space.*

(i) Every path component is contained in a connected component of X.

(ii) If X is locally path connected, then the components and path components coincide.

Proof. (i) is obvious, so we prove (ii): Let P be a path component, and $x \in P$, then $P \subset C_x$, the connected component of x. Also, P is a cl-open set in X, so P is cl-open in C_x. Since C_x is connected, it follows that $P = C_x$. □

(End of Week 7)

Example 4.10.

(i) If $X \subset \mathbb{R}^n$ is open, then it is locally path connected.

Proof. Let $x \in X$, then $\exists$ a n-cell $V := \prod_{i=1}^n (a_i, b_i) \subset X$ such that $x \in V$. But each $(a_i, b_i) \subset \mathbb{R}$ is path connected by Lemma 3.7, so V is path connected by Theorem 3.10. □

(ii) More generally, if X is locally connected, and $Y \subset X$ is open, then Y is locally connected.

5. Compactness

Remark 5.1. Consider some nice properties of the interval $[0,1]$:

(i) If $f : [0,1] \to \mathbb{R}$ is continuous, then f is bounded.

(ii) If $f : [0,1] \to \mathbb{R}$ is continuous, then it is uniformly continuous. ie. For all $\epsilon > 0, \exists \delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

(iii) Every sequence in $[0,1]$ has a convergent subsequence.

Note that these properties are also shared by other sets, for instance, finite sets. Compactness is a generalization of finiteness in the context of topological spaces.

(iv) Example: If $f : (0,1) \to \mathbb{R}$ is given by $f(x) = 1/x$, then f is not uniformly continuous, and is not bounded. ie. $[0,1]$ should be compact, but $(0,1)$ should not.

Definition 5.2. Let X be a topological space.

(i) A collection $\mathcal{U}$ of subsets of X is called an open cover for X if every member of $\mathcal{U}$ is open, and, for each $x \in X, \exists U \in \mathcal{U}$ such that $x \in U$.

(ii) Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of X. We say $\mathcal{V}$ is a subcover of $\mathcal{U}$ if $\mathcal{V} \subset \mathcal{U}$.

Example 5.3.

(i) $\{X\}$ is an open cover for X. Similarly, the topology τ (or any basis of τ) is an open cover for X.

(ii) If $\mathcal{U}$ is an open cover for X, and $\mathcal{W} \subset \tau$ is any collection of open sets, then $\mathcal{U} \cup \mathcal{W}$ is an open cover, and $\mathcal{U}$ is a subcover of $\mathcal{U} \cup \mathcal{W}$.

(iii) If X is a metric space. For each $x \in X$, choose $\delta_x > 0$. Then $\mathcal{U} := \{B(x, \delta_x) : x \in X\}$ is an open cover for X.

(iv) If $\mathcal{U}$ is an open cover for X, and $\mathcal{V}$ is an open cover for Y, then $\mathcal{W} := \{U \times V : U \in \mathcal{U}, V \in \mathcal{V}\}$ is an open cover for $X \times Y$.

(v) If $\mathcal{U}$ is an open cover for X, and X^* is any quotient space of X, then $\mathcal{V} := \{\pi(U) : U \in \mathcal{U}\}$ is an open cover for X^* (where $\pi : X \to X^*$ denotes the quotient map).

Definition 5.4. A topological space X is said to be <u>compact</u> if whenever $\mathcal{U}$ is an open cover for $X, \exists$ finitely many elements $\mathcal{V} := \overline{\{U_1, U_2, \ldots, U_n\}} \subset \mathcal{U}$ such that $\mathcal{V}$ is an open cover for X. i.e. Every open cover of X has a finite subcover.

Example 5.5.

(i) Any finite set is compact.

Proof. If $\mathcal{U}$ is an open cover for X, then $\mathcal{U} \subset \mathcal{P}(X)$, which is itself finite. Hence, $\mathcal{U}$ is finite. $\square$

(ii) $(0,1)$ is not compact.

Proof. Let $U_n := (1/n, 1)$, then $\{U_n\}$ is an open cover without a finite subcover. $\square$

Theorem 5.6. *$[0,1] \subset \mathbb{R}$ is compact.*

Proof. Let $\mathcal{U}$ be an open cover for $[0,1]$. Since $0 \in [0,1], \exists U \in \mathcal{U}$ such that $0 \in U$. Hence, $\exists \delta > 0$ such that $[0,\delta) \subset U$. Now define

$$A := \{x \in [0,1] : [0,x] \text{ is contained in finitely many elements of } \mathcal{U}\}$$

Then, by the above argument, $\delta/2 \in A$. So define

$$c := \sup(A)$$

We claim that $c = 1$. If $c < 1$, then $c \in [0,1]$, so $\exists V \in \mathcal{U}$ such that $c \in V$. Hence, $\exists \delta > 0$ such that $(c-\delta, c+\delta) \subset V$. Since $c = \sup(A), c - \delta$ is not an upper bound for A. Hence, $\exists x \in A$ such that

$$c - \delta < x \leq c$$

Now, $[a,x]$ is covered by finitely many members of $\mathcal{U}$, say $\{U_1, U_2, \ldots, U_k\}$. Also, $[x, c+\delta/2] \subset (c-\delta, c+\delta) \subset V$. Hence, $[a, c+\delta/2]$ is covered by $\{U_1, U_2, \ldots, U_k, V\}$. In particular,

$$c + \delta/2 \in A$$

contradicting the fact that $c = \sup(A)$. Thus, $c = 1$, and the proof is complete. $\square$

Proposition 5.7. *A closed subspace of a compact space is compact.*

Proof. Let $Y \subset X$ be a closed and X compact. Let $\mathcal{U}$ be an open cover for Y. Then for each $V \in \mathcal{U}, \exists V' \subset X$ open such that $V = V' \cap Y$. Consider

$$\mathcal{U}' := \{V' : V \in \mathcal{U}\} \bigcup \{X \setminus Y\}$$

This is an open cover for X, so has a finite subcover $\mathcal{V} \subset \mathcal{U}'$. Consider

$$\{W \cap Y : W \in \mathcal{V}\}$$

then this is a cover of Y that is finite, and a subcover of $\mathcal{U}$ [Check!] $\square$

Lemma 5.8 (The Tube Lemma)**.** *Let X, Y be topological spaces with Y compact. Let $x_0 \in X$, and suppose $N \subset X \times Y$ is open such that*

$$x_0 \times Y \subset N$$

Then $\exists W \subset X$ open such that $x_0 \in W$ and

$$W \times Y \subset N$$

Note: A set of the form $W \times Y$ is called a tube about $x_0 \times Y$.

Proof. For each $(x_0, y) \in x_0 \times Y$, choose a basic open set $U_y \times V_y$ such that $(x_0, y) \in U_y \times V_y$ and
$$U_y \times V_y \subset N$$
The collection $\{U_y \times V_y : y \in Y\}$ forms an open cover for $x_0 \times Y \cong Y$. Hence, it has a finite subcover
$$\{U_1 \times V_1, U_2 \times V_2, \ldots, U_n \times V_n\}$$
Consider $W := U_1 \cap U_2 \cap \ldots \cap U_n$, then if $x \in W$ and $y \in Y$, then $\exists 1 \leq i \leq n$ such that $(x_0, y) \in U_i \times V_i \subset N$. Hence, $(x, y) \in U_i \times V_i$, so
$$(x, y) \in N$$
So $W \times Y \subset N$ □

(End of Week 8)

Theorem 5.9. *The finite product of compact spaces is compact.*

Proof. By induction, we prove it for two spaces, so let X, Y be compact, and let $\mathcal{U} = \{U_\alpha\}$ be an open cover for $X \times Y$. Fix $x_0 \in X$, then $\mathcal{U}$ is an open cover for $x_0 \times Y$. Since $x_0 \times Y \cong Y$ is compact, it has a finite subcover $\{U_1, U_2, \ldots, U_n\}$. Let
$$N := U_1 \cup U_2 \cup \ldots \cup U_n$$
then N is an open set containing $x_0 \times Y$. Let $W \subset X$ be an open set such that
$$W \times Y \subset N$$
as in the previous lemma. Then $W \times Y$ is covered by finitely many sets of $\mathcal{U}$, namely $\{U_1, U_2, \ldots, U_n\}$.

Hence, for each $x \in X$, there is an open neighbourhood W_x of x such that $W_x \times Y$ is covered by finitely many elements of $\mathcal{U}$. Now the collection $\{W_x : x \in X\}$ forms an open cover for X, so has a finite subcover $\{W_1, W_2, \ldots, W_n\}$. Now each $W_i \times Y$ is covered by finitely many elements of $\mathcal{U}$, so
$$\bigcup_{i=1}^{n} W_i \times Y$$
is covered by finitely many elements of $\mathcal{U}$. But
$$X \times Y \subset \bigcup_{i=1}^{n} W_i \times Y$$
so this completes the proof. □

Definition 5.10. A collection $\mathcal{C}$ of subsets of X is said to have the finite intersection property if, for each finite subcollection $\{C_1, C_2, \ldots, C_n\} \subset \mathcal{C}$, the intersection

$$C_1 \cap C_2 \cap \ldots \cap C_n$$

is non-empty.

Theorem 5.11. *Let X be a topological space, then X is compact iff, for every collection $\mathcal{C}$ of closed sets with the finite intersection property,*

$$\bigcap_{C \in \mathcal{C}} C \neq \emptyset$$

Proof. Define $\mathcal{U}$ by

$$\mathcal{U} := \{X \setminus C : C \in \mathcal{C}\}$$

Then

(i) $\mathcal{U}$ is a collection of open sets.

(ii) $\mathcal{U}$ is an open cover for X if and only if

$$\bigcap_{C \in \mathcal{C}} C = \emptyset$$

(iii) A finite subcollection $\{U_1, U_2, \ldots, U_n\}$ of $\mathcal{U}$ covers X if and only if, the corresponding subcollection $C_i := X \setminus U_i$ has the property that

$$C_1 \cap C_2 \cap \ldots \cap C_n = \emptyset$$

Now suppose X is compact: If $\mathcal{C}$ has the finite intersection property and

$$\bigcap_{C \in \mathcal{C}} C = \emptyset$$

then $\mathcal{U}$ is a cover for X. By compactness, it must have a finite subcover. By (iii), this would violate the finite intersection property.

The converse is similar. $\square$

Corollary 5.12. *Let X be a compact topological space. Let $\{C_i\}$ be a sequence of non-empty closed subsets of X such that*

$$C_1 \supset C_2 \supset \ldots \supset C_i \supset C_{i+1} \supset \ldots$$

(Such a sequence is called a nested sequence of closed sets.) Then

$$\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$$

6. Compact Subsets of $\mathbb{R}^n$

Example 6.1. Fix real numbers $a_i < b_i$ for $1 \leq i \leq n$, then

$$X := \prod_{i=1}^{n} [a_i, b_i]$$

is compact in $\mathbb{R}^n$. Such a set is called a n-cell.

Proof. Any set of the form $[a, b] \subset \mathbb{R}$ is homeomorphic to $[0, 1]$ via the map

$$t \mapsto tb + (1 - t)a$$

Hence, $[a, b]$ is compact. Hence, X is compact by Theorem 5.9. □

Definition 6.2. Let (X, d) be a metric space and $Y \subset X$. Y is said to be <u>bounded</u> if $\exists M > 0$ such that

$$d(x, y) \leq M \quad \forall x, y \in Y$$

By the triangle inequality, this is equivalent to: $\exists x_0 \in X$ and $M' > 0$ such that

$$d(x_0, y) \leq M' \quad \forall y \in Y$$

Lemma 6.3. *Let X be a metric space and $Y \subset X$ be a compact set, then Y is bounded.*

Proof. Fix $x_0 \in Y$. Then consider

$$\mathcal{U} := \{B(x_0, r) \cap Y : r > 0\}$$

If $y \in Y$, then $\exists r > 0$ such that $d(x_0, y) < r$, so $\mathcal{U}$ is an open cover for Y. Hence it has a finite subcover $\{B(x_0, r_1) \cap Y, \ldots, B(x_0, r_n) \cap Y\}$. Let

$$M := \max\{r_i : 1 \leq i \leq n\} > 0$$

Then for any $y \in Y, \exists 1 \leq i \leq n$ such that $y \in B(x_0, r_i) \cap Y$, so $d(x_0, y) < r_i \leq M$. Hence, Y is bounded. □

Recall: Let X be a set. Two metrics d_1 and d_2 on X are said to be equivalent if $\exists K, M > 0$ such that

$$Kd_1(x, y) \leq d_2(x, y) \leq Md_1(x, y) \quad \forall x, y \in X$$

Note: If a set $Y \subset X$ is bounded with respect to d_1, then it is bounded with respect to d_2 and vice versa.

Lemma 6.4. *Let X be a Hausdorff space and $Y \subset X$ compact, then Y is closed.*

Proof. If $x \notin Y$, then for each $y \in Y, \exists$ open sets U_y and V_y such that $x \in U_y, y \in V_y$ and $U_y \cap V_y = \emptyset$. Now $\{V_y : y \in Y\}$ is an open cover of Y, which must have a finite subcover $\{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}$. Set

$$U := \bigcap_{i=1}^{n} U_{y_i}$$

Then U is open, $x \in U$, and $U \cap V_{y_i} = \emptyset$ for all i. Hence, $U \cap Y = \emptyset$, so $U \subset Y^c$, whence Y^c is open. □

Theorem 6.5 (Heine-Borel). *Let $X \subset \mathbb{R}^n$, then X is compact if and only if X is both closed and bounded (with respect to the Euclidean metric).*

Proof. If X is compact, X is closed and bounded by the previous two lemmas. If X is closed and bounded, and is non-empty, fix $x_0 \in X$, then

$$X - x_0 := \{a - x_0 : a \in X\}$$

is homeomorphic to X and contains 0. To show that X is compact, it suffices to show that $X - x_0$ is compact, so we may assume WLOG that $0 \in X$. Since X is bounded with respect to the Euclidean metric, it is bounded with respect to the sup-metric because they are equivalent (Example 2.14). Hence, $\exists M > 0$ such that

$$\max\{|y_i| : 1 \leq i \leq n\} = d_\infty(0, y) \leq M \quad \forall y \in X$$

Hence, if $y \in X$, then $|y_i| \leq M$ for all $1 \leq i \leq n$. ie. X is contained in the set

$$Z := \prod_{i=1}^{n} [-M, M]$$

Now Z is compact because it is an n-cell. Since $X \subset Z$ and X is closed in $\mathbb{R}^n$, X is closed in Z (Why?). Hence X is compact by Proposition 5.7. □

Example 6.6. Let $X = \mathbb{Z}$ with the discrete metric

$$d(x, y) = \begin{cases} 1 & : x \neq y \\ 0 & : x = y \end{cases}$$

Then X is closed and bounded, but not compact. Hence, the above theorem does not hold for all metric spaces.

Definition 6.7. Let X be a topological space. A point $x \in X$ is said to be <u>isolated</u> if $\{x\}$ is an open set in X.

Theorem 6.8. *Let X be a non-empty compact, Hausdorff space. If X has no isolated points, then X is uncountable.*

Proof. (i) We claim that: If $x \in X$ and U an open set of X, then $\exists$ a non-empty open set $V \subset U$ such that $x \notin \overline{V}$.: Since U is non-empty, and $U \neq \{x\}$ (since x is not isolated), $\exists y \in U, y \neq x$. Choose open sets W_1, W_2 such that $y \in W_1, x \in W_2$ and $W_1 \cap W_2 = \emptyset$. Then $V := W_1 \cap U$ is open, $V \subset U$ and $W_2 \subset V^c$, so $x \notin \overline{V}$.

(ii) Now we show that X is uncountable. Suppose $A = \{x_n\}$ is a countable subset of X, we WTS: $A \neq X$.

  (i) For x_1, take $U = X$, then $\exists V_1$ open such that $x_1 \notin \overline{V_1}$.

  (ii) For $x_2 \in X$, take $U = V_1$, then $\exists V_2$ open such that $V_2 \subset V_1$ and $x_2 \notin \overline{V_2}$.

  (iii) Thus proceeding, we get a sequence of open sets

$$V_1 \supset V_2 \supset \dots$$

  such that $x_n \notin \overline{V_n}$. Now consider the nested sequence of closed sets

$$\overline{V_1} \supset \overline{V_2} \supset \dots$$

  and note that each set is non-empty. By Corollary 5.12, $\exists x \in X$ such that

$$x \in \bigcap_{n \in \mathbb{N}} \overline{V_n}$$

  Since $x_n \notin \overline{V_n}$, it follows that $x \notin A$. Hence, $A \neq X$, so X is uncountable.

□

Corollary 6.9. *Any closed, bounded interval in $\mathbb{R}$ is uncountable.*

7. Continuous Functions on Compact Sets

Theorem 7.1. *Let $f : X \to Y$ be a continuous function, and X compact. Then $f(X)$ is compact.*

Proof. If $\mathcal{U}$ is an open cover for $f(X)$, then

$$\mathcal{V} := \{f^{-1}(U) : U \in \mathcal{U}\}$$

is an open cover for X [Check!]. Let $\{f^{-1}(U_1), f^{-1}(U_2), \dots, f^{-1}(U_n)\}$ be a finite subcover of $\mathcal{V}$, then $\{U_1, U_2, \dots, U_n\}$ is a finite subcover of $f(X)$ [Check!]. □

Corollary 7.2. *The quotient of a compact space is compact.*

Proof. The quotient map $\pi : X \to X^*$ is surjective and continuous, so the previous theorem applies. □

Definition 7.3. Let $f : X \to \mathbb{R}$ be a function.

(i) We say that f is <u>bounded below</u> if $\exists m \in \mathbb{R}$ such that $f(x) \geq m$ for all $x \in X$.

(ii) Similarly, we define f to be <u>bounded above</u>.

(iii) If f is bounded below, we say that f <u>attains its infimum</u> at a point $x_0 \in X$ if

$$f(x_0) \leq f(x) \quad \forall x \in X$$

(iv) We say that f <u>attains its supremum</u> at x_1 if

$$f(x) \leq f(x_1) \quad \forall x \in X$$

The points x_0 and x_1 (if they exist, and they need not be unique) are called <u>extreme points</u> of f.

Example 7.4.

(i) Let $f : (0, 1) \to \mathbb{R}$ be given by $f(x) = 1/x$, then f is not bounded above.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = e^{-x}$, then f is bounded below, but it does not attain its infimum 0.

Theorem 7.5 (Extreme Value Theorem)**.** *Let X be compact and $f : X \to \mathbb{R}$ continuous, then $\exists x_0, x_1 \in X$ such that*

$$f(x_0) \leq f(x) \leq f(x_1) \quad \forall x \in X$$

Proof. Since $f(X)$ is compact, by the Heine-Borel theorem, it is closed and bounded. In particular,

$$m := \inf\{f(x) : x \in X\}$$

exists and is finite. m is a limit point of $f(X)$ and $f(X)$ is closed, so $m \in f(X)$. Hence, $\exists x_0 \in X$ such that $f(x_0) = m$. The proof for the upper bound is analogous. □

Theorem 7.6. *Let $f : X \to Y$ be a continuous, bijective function. If X is compact, and Y is Hausdorff, then f is a homeomorphism.*

Proof. We want to show that f is an open map. It suffices to show that f is a closed map. If $F \subset X$ is closed, then F is compact. Hence, $f(F)$ is compact in Y, so $f(F)$ is closed in Y. □

Example 7.7.

(i) This completes the proof from Example 8.8,

$$D^2/S^1 \cong S^2$$

(ii) In the Mid-Sem Exam Q.6, we had

$$A := \{(x, y) : 1 \leq \sqrt{x^2 + y^2} \leq 2\}$$

and we had constructed a continuous bijective function $f : S^1 \times [1, 2] \to A$. Note that $S^1 \times [1, 2]$ is compact and A is Hausdorff, so f is a homeomorphism.

Definition 7.8. Let (X, d) be a metric space and $A \subset X$. Given $x \in X$, define the distance of x from A as

$$d(x, A) := \inf\{d(x, y) : y \in A\}$$

Lemma 7.9. *The function $p : X \to \mathbb{R}$ given by $p(x) := d(x, A)$ is a continuous function. Furthermore, $p(x) = 0$ if and only if $x \in \overline{A}$*

Proof. (i) If $x_1, x_2 \in X, y \in A$

$$d(x_1, A) \leq d(x_1, y) \leq d(x_1, x_2) + d(x_2, y)$$

This is true for all $y \in A$, so

$$d(x_1, A) \leq d(x_1, x_2) + d(x_2, A)$$

so

$$d(x_1, A) - d(x_2, A) \leq d(x_1, x_2)$$

By symmetry, $d(x_2, A) - d(x_1, A) \leq d(x_1, x_2)$ so

$$|d(x_1, A) - d(x_2, A)| \leq d(x_1, x_2)$$

From this continuity follows [Why?]

(ii) Suppose $x \in \overline{A}$, then $\exists y_n \in A$ such that $d(x, y_n) \to 0$. Hence, $d(x, A) = 0$. Conversely, if $d(x, A) = 0$, then for each $n \in \mathbb{N}$, $1/n$ is not a lower bound for the set

$$\{d(x, y) : y \in A\}$$

So $\exists y_n \in A$ such that $d(x, y_n) < 1/n$. Clearly, $y_n \to x$, so $x \in \overline{A}$

□

Definition 7.10. Let (X, d) be a metric space and $A \subset X$. The <u>diameter</u> of A is defined as

$$\text{diam}(A) := \sup\{d(x, y) : x, y \in A\}$$

Theorem 7.11 (Lebesgue Number Lemma)**.** *Let $\mathcal{U}$ be an open cover of a metric space (X, d). If X is compact, $\exists \delta > 0$ such that if $A \subset X$ such that diam$(A) < \delta$, then $\exists U \in \mathcal{U}$ such that $A \subset U$.*

Note: Any number δ as above is called a <u>Lebesgue number</u> for the cover $\mathcal{U}$. Note if δ is a Lebesgue number for $\mathcal{U}$ and $\delta' < \delta$, then δ' is also a Lebesgue number for $\mathcal{U}$.

Proof. Let $\{U_1, U_2, \ldots, U_n\}$ be a finite subcover of $\mathcal{U}$ and define $A_i := X \setminus U_i$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x, A_i)$$

Then f is continuous by the previous lemma, so it must attain its minimum at some point $x \in X$. Now, $\exists U_i$ such that $x \in U_i$, so $x \notin A_i$ so by the previous lemma, $d(x, A_i) > 0$, whence $f(x) > 0$, so if $\delta := f(x)$, then

$$f(y) \geq \delta \quad \forall y \in X$$

Now if A is a set of diameter less than δ, then fix $x_0 \in A$, then

$$A \subset B(x_0, \delta)$$

Now, assume that $d(x_0, A_m)$ is the maximum of $\{d(x_0, A_1), d(x_0, A_2), \ldots, d(x_0, A_n)\}$. Then

$$\delta \leq f(x_0) \leq d(x_0, C_m)$$

Hence, for each $y \in C_m, d(x_0, y) \geq \delta$, whence

$$B(x_0, \delta) \subset X \setminus C_m = U_m \Rightarrow A \subset U_m$$

□

Definition 7.12. Let $f : X \to Y$ be a continuous function between two metric spaces. We say that f is uniformly continuous if, for each $\epsilon > 0, \exists \delta > 0$ such that

$$d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon$$

Example 7.13. Let $f : (0, 1) \to \mathbb{R}$ given by $f(x) = 1/x$, then f is not uniformly continuous.

Theorem 7.14. *Let $f : X \to Y$ be a continuous function between metric spaces. If X is compact, then f is uniformly continuous.*

Proof. Consider $\epsilon > 0$ and set

$$\mathcal{V} := \{B(y, \epsilon/2) : y \in Y\}$$

Then $\mathcal{V}$ is an open cover for Y, so

$$\mathcal{U} := \{f^{-1}(B(y, \epsilon/2)) : y \in Y\}$$

is an open cover for X. Let $\delta > 0$ be a Lebesgue number for $\mathcal{U}$. Then if $x_1, x_2 \in X$ such that $d_X(x_1, x_2) < \delta$, then $A := \{x_1, x_2\}$ has diameter $< \delta$, so $\exists y \in Y$ such that

$$A \subset f^{-1}(B(y, \epsilon/2))$$

Hence, $\{f(x_1), f(x_2)\} \subset B(y, \epsilon/2)$ so by the triangle inequality,

$$d_Y(f(x_1), f(x_2)) < \epsilon$$

□

(End of Week 9)

8. Compactness in Metric Spaces

Definition 8.1. Let X be a topological space.

(i) X is said to be sequentially compact if, for any sequence $(x_n) \subset X$, there is a subsequence (x_{n_k}) of (x_n) that converges to a point in X.

(ii) Recall: If $A \subset X$. A point $x \in X$ is called a limit point of A if, for each open set U containing x, $U \cap (A \setminus \{x\}) \neq \emptyset$

(iii) X is said to be limit point compact if every infinite subset of X has a limit point in X.

Lemma 8.2. *If X is compact, then it is limit point compact.*

Proof. Let $A \subset X$ be an infinite set, and suppose A has no limit point. Then, for each $x \in X$, there is an open set U_x containing x such that $U_x \cap (A \setminus \{x\}) = \emptyset$. Then, $\mathcal{U} := \{U_x : x \in X\}$ is an open cover for X which has a finite subcover $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$. Then each U_{x_1} contains atmost one point of A (possibly x_i). Hence A is finite. $\square$

Example 8.3. Let $Y = \{1, 2\}$ with the indiscrete topology $\tau_Y = \{\emptyset, Y\}$, and let

$$X := \mathbb{N} \times Y$$

with the product topology, where $\mathbb{N}$ is given the usual discrete topology. Then X is limit point compact but not compact.

Proof. If $A \subset X$ is any non-empty set, and assume that $(n, 1) \in A$. If U is an open set containing $(n, 2)$, then U contains a basic open neighbourhood $W = \{n\} \times Y$, so

$$(n, 1) \in W \cap (A \setminus \{(n, 2)\})$$

whence $U \cap (A \setminus \{(n, 1)\}) \neq \emptyset$. Thus, X is limit point compact.

However, the open cover $\{\{n\} \times Y : n \in \mathbb{N}\}$ does not have a finite subcover, so X is not compact. $\square$

Lemma 8.4. *Let X be Hausdorff, $A \subset X$ and $x \in X$ a limit point of A. Then for any open neighbourhood U of x, $U \cap (A \setminus \{x\})$ is infinite.*

Proof. Suppose $U \cap (A \setminus \{x\})$ is finite, then write $U \cap A = \{a_1, a_2, \ldots, a_n\}$. For each i, there are open sets V_i, W_i such that $x \in V_i$ and $a_i \in W_i$ such that $V_i \cap W_i = \emptyset$. If

$$V := \bigcap_{i=1}^{n} V_i$$

Then V is an open set containing x and $V \cap (A \setminus \{x\}) = \emptyset$, so x cannot be a limit point of A. $\square$

Definition 8.5. A metric space X is said to be totally bounded if, for each $\epsilon > 0$, there are finitely many points $\{x_1, x_2, \ldots, x_n\} \subset X$ such that

$$\{B(x_i, \epsilon) : 1 \leq i \leq n\}$$

covers X. Such a collection of open set is called an ϵ-net of X.

Lemma 8.6. *If X is sequentially compact, then it is totally bounded.*

Proof. Suppose X is not totally bounded, then $\exists \epsilon > 0$ for which there is no finite epsilon net. In particular, if $x_1 \in X$, then $X \neq B(x_1, \epsilon)$, so $\exists x_2 \in X$ such that

$$d(x_1, x_2) \geq \epsilon$$

Now, $\{B(x_1, \epsilon), B(x_2, \epsilon)\}$ is not an open cover for X, so $\exists x_3 \in X$ such that

$$d(x_3, x_1)\epsilon \text{ and } d(x_3, x_2) \geq \epsilon$$

Thus proceeding, we obtain a sequence $(x_n) \subset X$ such that if $m > n$, then

$$d(x_m, x_n) \geq \epsilon$$

Such a sequence cannot have a convergent subsequence [Why?] contradicting the fact that X is sequentially compact. □

Lemma 8.7 (Lebesgue Number Lemma - II)**.** *If X is a sequentially compact metric space and $\mathcal{U}$ is an open cover for X, then $\exists \delta > 0$ such that, for any $y \in X, \exists U \in \mathcal{U}$ such that $B(y, \epsilon) \subset U$.*

Proof. Suppose $\mathcal{U}$ does not have a Lebesgue number, then $\delta = 1/n$ does not work. So $\exists x_n \in X$ such that $B(x_n, 1/n)$ is not contained in any single member of $\mathcal{U}$. Then (x_n) has a convergent subsequence $x_{n_k} \to x$. Now $x \in X$, so $\exists U \in \mathcal{U}$ such that $x \in U$. Choose $\delta > 0$ such that $B(x, \delta) \subset U$, then $\exists n_k \in \mathbb{N}$ such that

$$d(x_{n_k}, x) < \delta/2 \text{ and } 1/n_k < \delta/2$$

Then by the triangle inequality

$$B(x_{n_k}, 1/n_k) \subset B(x, \delta) \subset U$$

This contradicts the assumption on the x_n. □

Theorem 8.8. *If X is a metric space, then TFAE:*

(i) *X is compact*

(ii) *X is limit point compact.*

(iii) *X is sequentially compact.*

Proof. (i) $\Rightarrow$ (ii): Lemma 8.2.

(ii) $\Rightarrow$ (iii): If $(x_n) \subset X$ is a sequence, then let $A := \{x_n\}$. If A is finite, then there is a subsequence $(n_k) \subset \mathbb{N}$ such that x_{n_k} is constant, and hence convergent. Suppose A is infinite, then it has a limit point x. In particular,

$$B(x,1) \cap (A \setminus \{x\}) \neq \emptyset$$

so choose $n_1 \in \mathbb{N}$ such that $x_{n_1} \in B(x,1)$. Now,

$$B(x,1/2) \cap (A \setminus \{x\}) \neq \emptyset$$

By the previous lemma, $B(x,1/2) \cap (A \setminus \{x\})$ is infinite. In particular,

$$B(x,1/2) \cap (A \setminus \{x, x_1, x_2, \ldots, x_{n_1}\}) \neq \emptyset$$

So $\exists n_2 > n_1$ such that

$$x_{n_2} \in B(x,1/2) \cap (A \setminus \{x\})$$

Thus proceeding, for each $k \in \mathbb{N}$, we choose $n_k > n_{k-1}$ such that

$$x_{n_k} \in B(x,1/k) \cap (A \setminus \{x\})$$

Now $d(x, x_{n_k}) < 1/k$, so $x_{n_k} \to x$.

(iii) $\Rightarrow$ (i): If X is sequentially compact, choose an open cover $\mathcal{U}$ of X. By the Lebesgue Number Lemma II, $\exists \delta > 0$ such that any ball of radius δ is contained in a single member of $\mathcal{U}$. However, X is totally bounded by Lemma 8.6, so finitely many balls $\{B(x_1,\delta), B(x_2,\delta), \ldots, B(x_n,\delta)\}$ cover X. Hence, finitely many members of $\mathcal{U}$ cover X.

□

Theorem 8.9 (Bolzano-Weierstrass)**.** *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Proof. If $(x_m) \subset \mathbb{R}^n$ is bounded, then $\exists M \geq 0$ such that

$$(x_m) \subset \prod_{i=1}^{n} [-M, M] =: Z$$

Z is compact, so it is sequentially compact. □

Example 8.10. A metric space without the Bolzano-Weierstrass property: Let

$$X := \{(x_n) \in \mathbb{R}^\omega : (x_n) \text{ is bounded}\}$$

Define a metric on X by

$$d(\overline{x}, \overline{y}) := \sup\{|x_n - y_n| : n \in \mathbb{N}\}$$

This is a well-defined metric on X. Now consider e^n to be the standard basis vector in X. Then $d(e^n, 0) = 1$, so $\{e^n\}$ is a bounded sequence in X. However, e^n does not have a convergent subsequence because $d(e^n, e^m) = 1$ if $n \neq m$.

9. Local Compactness

Definition 9.1. A topological space X is said to be locally compact if, for each $x \in X$, there is an open neighbourhood V of x such that $\overline{V}$ is compact.

Example 9.2.

(i) Every compact space is locally compact.

(ii) $\mathbb{R}$ is locally compact because every closed interval $[a,b] = \overline{(a,b)}$ is compact.

(iii) $\mathbb{Q}$ is not locally compact because if $V \subset \mathbb{Q}$ is open, then $\exists a < b$ in $\mathbb{R}$ such that $(a,b) \cap \mathbb{Q} \subset V$. If $s \in \mathbb{R} \setminus \mathbb{Q}$ is an irrational such that $a < s < b$, then there is a sequence $(x_n) \subset V$ that converges to s in $\mathbb{R}$, so (x_n) cannot have a convergent subsequence. Hence, $\overline{V}$ cannot be compact.

(iv) $\mathbb{R}^\omega$ with the product topology is not locally compact, because if V is a non-empty open set, then V contains an open set of the form

$$(a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \ldots$$

If $\overline{V}$ were compact, then

$$[a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] \times \mathbb{R} \times \mathbb{R} \times \ldots$$

would be compact, but it is not [Check! Use the fact that $\mathbb{R}$ is not compact].

Theorem 9.3. *Let X be a topological space, then $\exists$ a compact space Y such that*

(i) $X \subset Y$

(ii) $Y \setminus X$ *is a singleton.*

Proof. Define $Y := X \sqcup \{\infty\}$ as a new set, and define τ_Y as the collection of sets U satisfying one of the two following properties:

(i) $U \subset X$ is open in X

(ii) $\infty \in U$ and $Y \setminus U$ is compact in X

We show that τ_Y is a topology on Y, and that Y is compact.

(i) $\emptyset \in \tau_Y$ because $\emptyset \in \tau_X$

(ii) $Y \in \tau_Y$ because $Y \setminus Y = \emptyset$ is compact in X

(iii) If $\{U_\alpha\}$ is a collection of members of τ_Y, we set $U := \bigcup U_\alpha$ consider two cases:

    (i) If $\infty \notin U$, then $U \in \tau_X$ so $U \in \tau_Y$

(ii) If $\infty \in U$, then choose $I \subset J$ such that $\infty \in U_\beta$ iff $\beta \in I$, so $U_\beta = Y \setminus C_\beta$ for all $\beta \in J$, where $C_\beta \subset X$ is compact, then

$$\bigcup_{\alpha \in J} U_\alpha = \left(\bigcup_{\beta \in I} (Y \setminus C_\beta) \right) \cup \left(\bigcup_{\gamma \in I^c} U_\gamma \right)$$

Now $\bigcap_{\beta \in I} C_\beta$ is compact, so

$$\bigcup_{\beta \in I} Y \setminus C_\beta$$

is in τ_Y, so $U \in \tau_Y$.

(iv) If $U_1, U_2 \in \tau_Y$, we WTS: $U_1 \cap U_2 \in \tau_Y$. Consider cases again:

(i) If $\infty \notin U_1 \cup U_2$, then $U_1 \cap U_2 \in \tau_X \subset \tau_Y$

(ii) If $\infty \in U_1, \infty \notin U_2$, then $U_1 = Y \setminus C$ for $C \subset X$ compact, so

$$U_1 \cap U_2 = (Y \setminus C) \cap U_2 = (X \setminus C) \cap U_2 \in \tau_X \subset \tau_Y$$

(iii) Similarly if $\infty \in U_2 \setminus U_1$

(iv) If $\infty \in U_1 \cap U_2$, then $U_i = (Y \setminus C_i)$ as above, so

$$U_1 \cap U_2 = Y \setminus (C_1 \cup C_2)$$

but $C_1 \cup C_2$ is compact in X.

We now show that Y is compact: Suppose $\mathcal{U}$ is an open cover for Y, then $\exists U \in \mathcal{U}$ such that $\infty \in U$, so $U = Y \setminus C$ for some compact $C \subset X$. There are finitely many elements $\{U_1, U_2, \dots, U_n\}$ of $\mathcal{U}$ that cover C, so

$$\{U_1, U_2, \dots, U_n\} \cup \{U\}$$

covers Y.

□

Lemma 9.4. *If X is a locally compact and Hausdorff, then the space constructed above is Hausdorff.*

Proof. If $x, y \in Y$ with $x \neq y$. If $x, y \in X$, then we use the fact that X is Hausdorff to produce open sets as required. So assume $y = \infty$, then choose a neighbourhood V of x such that $\overline{V}$ is compact. Then $U := X \setminus \overline{V}$ is an open neighbourhood of y and $U \cap V = \emptyset$. So Y is Hausdorff. □

Theorem 9.5. *If X is locally compact and Hausdorff, and suppose Y_1 and Y_2 are two spaces such that*

(i) Both Y_1 and Y_2 are compact.

(ii) $X \subset Y_1$ *and* $X \subset Y_2$

(iii) $Y_1 \setminus X$ *is a singleton and* $Y_2 \setminus X$ *is a singleton.*

Then there is a homeomorphism $p : Y_1 \to Y_2$ *such that* $p|_X = id_X$.

Proof. Suppose $Y_1 \setminus X = \{y_1\}$ and $Y_2 \setminus X = \{y_2\}$, then define $p : Y_1 \to Y_2$ by

$$p(z) = \begin{cases} z & : z \in X \\ y_2 & : z = y_1 \end{cases}$$

Then p is clearly a well-defined bijection. Also, if $U \subset Y_2$ is an open set such that $U \subset X$, then $p^{-1}(U) = U \subset Y_1$ is open. If $U \subset Y_2$ is open and $\infty \in Y_2$, then $F := Y_2 \setminus U = X \setminus U$ is closed in Y_2. But Y_2 is compact, so F is compact in Y_2. Since $F \subset X$, F is compact in X. But $X \subset Y_1$, so F is compact in Y_1. But Y_1 is Hausdorff, so F is closed in Y_1. Hence, $Y_1 \setminus F = p^{-1}(U)$ is open in Y_1. Hence, p is continuous. But $p : Y_1 \to Y_2$ is a continuous bijection from a compact space to a Hausdorff space, so it is a homeomorphism. □

Definition 9.6. Given a locally compact Hausdorff space, we have shown that $\exists$ a compact space Y such that $X \subset Y$ and $Y \setminus X$ is a singleton. Furthermore, Y is unique in the sense of Theorem 9.5. This space Y is called the one-point compactification of X, and is denoted by X^+.

Example 9.7. If $X = \mathbb{R}^n$, then $X^+ \cong S^n$

Proof. The stereographic projection gives a continuous injective map $p : X \to S^n$, and is a homeomorphism onto its range $p(X) = S^n \setminus \{N\}$. Identifying X with $p(X)$, we see that S^n satisfies the conditions of Theorem 9.3. By Theorem 9.5, $S^n \cong X^+$. □

Note: For $n = 2$, $S^2 \cong (\mathbb{R}^2)^+$ is referred to as the Riemann sphere.

IV. Separation Axioms

1. Regular Spaces

Assume that all spaces are T_1: Singleton sets are closed.

Definition 1.1. A topological space X is said to be regular (or T_3) if, for any closed set $A \subset X$ and any $x \notin A$, there are open sets $U, V \subset X$ such that $A \subset U, x \in V$ and $U \cap V = \emptyset$.

Example 1.2.

(i) Every regular space is Hausdorff.

(ii) Let $K = \{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$ and define a topology on $\mathbb{R}$ as follows: Define

$$\mathcal{B}_1 := \{ \text{ open intervals in } \mathbb{R}\}$$
$$\mathcal{B}_2 := \{(a,b) \setminus K : a < b \text{ in } \mathbb{R}\}$$

Then $\mathcal{B} := \mathcal{B}_1 \cup \mathcal{B}_2$ forms a basis for a topology on $\mathbb{R}$ (HW), which we denote by τ_K. Then $\mathbb{R}_K := (\mathbb{R}, \tau_K)$ is Hausdorff but not regular.

Proof. $\mathbb{R}_K$ is Hausdorff because distinct points can be separated by open intervals. To see that $\mathbb{R}_K$ is not regular, note that K is closed in $\mathbb{R}_K$ and $0 \notin K$. However, if U is an open set containing 0, then U must contain a basic open set around 0. It cannot contain sets of the form $(-r, r)$ because they intersect K. So suppose $(-r, r) \setminus K \subset U$. Let $n \in \mathbb{N}$ such that $1/n < r$. Let V be an open set containing K and choose a basic open set (a, b) around $1/n$ contained in V. Then

$$1/n \in (a,b) \text{ and } 1/n < r \Rightarrow ((a,b) \setminus K) \cap (-r,r) \neq \emptyset$$

Hence, $U \cap V \neq \emptyset$, so K and 0 cannot be separated. $\square$

(End of Week 10)

Proposition 1.3. *Every compact Hausdorff space is regular.*

Proof. If X is compact and $A \subset X$ closed, $x \notin A$, then A is compact. For each $y \in A$, there are open sets U_y, V_y such that $x \in U_y, y \in V_y$ and $U_y \cap V_y = \emptyset$. Now $\{V_y \cap A : y \in A\}$ forms an open cover for A. Choose a finite subcover $\{V_{y_i} \cap A : 1 \leq i \leq n\}$ and consider

$$U := \bigcap_{i=1}^n U_{y_i} \text{ and } V := \bigcup_{i=1}^n V_{y_i}$$

Then U and V are open, $A \subset V, x \in U$ and $U \cap V = \emptyset$. $\square$

Theorem 1.4. *X is regular iff, for each $x \in X$ and an open neighbourhood U of x, there is an open neighbourhood V of x such that $\overline{V} \subset U$.*

Proof. Suppose X is regular, and $x \in X, U$ an open neighbourhood of X. Then, $X \setminus U$ is closed and does not contain x, so there are open sets V, W such that $x \in V, X \setminus U \subset W$ and $V \cap W = \emptyset$. We claim that $\overline{V} \subset U$. If $y \notin U$, then $y \in W$ and $W \cap V = \emptyset$, so $y \notin \overline{V}$. Hence, $\overline{V} \subset U$.

Conversely, suppose the given condition holds and $x \in X, A \subset X$ closed and $x \notin A$. Then $U := X \setminus A$ is an open set containing x, so there is an open set V such that $\overline{V} \subset U$. Then $W := X \setminus \overline{V}$ is open, contains A and $V \cap W = \emptyset$. □

Corollary 1.5. *Every subspace of a regular space is regular.*

Proof. If $Y \subset X$, where X is regular, suppose U is an open neighbourhood of x in Y, then $U = U' \cap Y$ for some open set $U' \subset X$. Choose $V' \subset X$ open such that $\overline{V'} \subset U'$. Now take $V := V' \cap Y$, which is open in Y, contains x and by Lemma 6.8,

$$cl_Y(V) = cl_X(V) \cap Y \subset cl_X(V') \cap Y \subset U' \cap Y = U$$

□

Corollary 1.6. *Every locally compact Hausdorff space is regular.*

Proof. Let X be locally compact and Hausdorff, and $X \subset X^+$ its one point compactification. X^+ is regular, so X must also be regular. □

Corollary 1.7. *Any product of regular spaces is regular.*

Proof. Suppose X_α is regular for all $\alpha \in J$, and $X := \prod_{\alpha \in J} X_\alpha$. Let $x := (x_\alpha \in X$ and $U \subset X$ an open neighbourhood of x. Then we may assume that U is a basic open set of the form

$$U_{\alpha_1} \times U_{\alpha_2} \times \ldots \times U_{\alpha_n} \times \prod_\beta X_\beta$$

Now $x_{\alpha_i} \in U_{\alpha_i}$, so there are open sets V_{α_i} such that $\overline{V_{\alpha_i}} \subset U_{\alpha_i}$. Then

$$V := V_{\alpha_1} \times V_{\alpha_2} \times \ldots \times V_{\alpha_n} \times \prod_\beta X_\beta$$

is an open neighbourhood of x such that $\overline{V} \subset U$ [Why?] □

2. Normal Spaces

Definition 2.1. A topological space X is said to be <u>normal</u> if, whenever A and B are disjoint closed sets, there are open sets U, V such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

Lemma 2.2. *X is normal iff, given a closed set $A \subset X$ and an open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$.*

Proof. HW. $\square$

Proposition 2.3. *Every metric space is normal.*

Proof. If $A, B \subset X$ are disjoint closed sets. For each $a \in A$, $a \notin B$, so $\exists \epsilon_a > 0$ such that $B(a, \epsilon_a) \subset X \setminus B$. Define

$$U := \bigcup_{a \in A} B(a, \epsilon_a/2)$$

Then U is open and it contains A. Similarly, define

$$V := \bigcup_{b \in B} B(b, \epsilon_b/2)$$

where ϵ_b is chosen as above. Then, if $z \in U \cap V$, then $\exists a \in A, b \in B$ such that

$$z \in B(a, \epsilon_a/2) \cap B(b, \epsilon_b/2)$$

Assume WLOG that $\epsilon_a \leq \epsilon_b$, then by triangle inequality,

$$d(a, b) \leq d(a, z) + d(z, b) < \frac{\epsilon_a}{2} + \frac{\epsilon_b}{2} \leq \epsilon_a$$

Hence, $B(a, \epsilon_a) \cap B \neq \emptyset$ contradicting the choice of ϵ_a. $\square$

Proposition 2.4. *Every compact Hausdorff space is normal.*

Proof. Let X be a compact Hausdorff space and $A, B \subset X$ disjoint closed sets. By Proposition 1.3, X is regular, so for each $a \in A$, there are open sets U_a and V_a such that

$$a \in U_a, B \subset V_a \text{ and } U_a \cap V_a = \emptyset$$

So $\{U_a : a \in A\}$ is an open cover for A. But A is compact, so there is a finite subcover $\{U_{a_1}, U_{a_2}, \ldots, U_{a_k}\}$. Define

$$U := \bigcup_{i=1}^{k} U_{a_i} \text{ and } V := \bigcap_{i=1}^{n} V_{a_i}$$

Then U, V are open, $A \subset U, B \subset V$ and $U \cap V = \emptyset$ [Check!]. $\square$

Proposition 2.5. *A closed subspace of a normal space is normal.*

Proof. If $Y \subset X$ is closed and X is normal. We use Lemma 2.2. Suppose $A \subset Y$ is closed and $U \subset Y$ an open set such that $A \subset U$. Then write $U = U' \cap Y$ for some open set $U' \subset X$. Since A is closed in Y and Y is closed in X, A is closed in X. Hence, there is an open set $V' \subset X$ such that $A \subset V'$ and $\overline{V'} \subset U'$. Now set

$$V := V' \cap Y$$

Then $A \subset V$ and by Lemma 6.8,

$$cl_Y(V) = cl_X(V) \cap Y \subset cl_X(V') \cap Y \subset U' \cap Y = U$$

$\square$

Example 2.6.

(i) Every normal space is regular. Hence, every normal space is Hausdorff.

(ii) Let $X = \mathbb{R}$ with the topology whose basis are sets of the form

$$[a, b)$$

where $-\infty < a < b \leq \infty$. This topology is denoted by τ_ℓ and it contains the usual topology. It follows that $\mathbb{R}_\ell := (\mathbb{R}, \tau_\ell)$ is normal.

(iii) $X := \mathbb{R}_\ell \times \mathbb{R}_\ell$ is thus a product of regular spaces, so it is regular. However, it is not normal [without proof]. Hence,

 (i) The product of normal spaces is not necessarily normal.

 (ii) This is an example of a space that is regular but not normal.

Theorem 2.7 (Urysohn's Lemma for metric spaces)**.** *Let (X, d) be a metric space and $A, B \subset X$ disjoint closed sets. Then $\exists f : X \to [0, 1]$ continuous such that*

$$f(x) = 0 \quad \forall x \in A \text{ and } f(y) = 1 \quad \forall y \in B$$

Proof. Recall that $x \mapsto d(x, A)$ is continuous and $d(x, A) = 0$ iff $x \in \overline{A}$. Define $f : X \to [0, 1]$ by

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}$$

Note that the denominator is non-zero because $A \cap B = \emptyset$. Now check that f satisfies the required properties. ☐

Lemma 2.8. *Let X be a normal space and $A, B \subset X$ disjoint closed sets. Let $P := \mathbb{Q} \cap [0, 1]$, then there is a sequence of open sets $\{U_p : p \in P\}$ such that*

(i) $A \subset U_0$ and $U_1 = X \setminus B$

(ii) For all $p, q \in P, p < q \Rightarrow \overline{U_p} \subset U_q$

Proof. Define $U_1 := X \setminus B$. Since $A \subset U_1$, define U_0 by Lemma 2.2 such that

$$A \subset U_0 \text{ and } \overline{U_0} \subset U_1$$

Now arrange P in a sequence $\{0, 1, p_1, p_2, \ldots\}$. We wish to define U_{p_1}: Note that $0 < p_1 < 1$ and $\overline{U_0} \subset U_1$, so by Lemma 2.2, there is an open set U_{p_1} such that

$$\overline{U_0} \subset U_{p_1} \text{ and } \overline{U_{p_1}} \subset U_1$$

Now we proceed by induction. Having define $\{U_0, U_1, U_{p_1}, \ldots, U_{p_n}\}$, we wish to define $U_{p_{n+1}}$. Since $0 < p_{n+1} < 1$, choose an immediate predecession p_i and an immediate successor p_j among $\{0, 1, p_1, p_2, \ldots, p_n\}$. Note that $\overline{U_{p_i}} \subset U_{p_j}$. So by Lemma 2.2, there is an open set $U_{p_{n+1}}$ such that

$$\overline{U_{p_i}} \subset U_{p_{n+1}} \text{ and } \overline{U_{p_{n+1}}} \subset U_{p_j}$$

By induction, we define U_p for all $p \in P$ satisfying (i) and (ii). ☐

Lemma 2.9. *Let X be a normal space and $A, B \subset X$ disjoint closed sets. Let $\{U_p : p \in \mathbb{Q} \cap [0,1]\}$ be a sequence of open sets as in the previous lemma. Define $U_p = \emptyset$ if $p < 0$ and $U_q = X$ if $q > 1$. Now define $f : X \to \mathbb{R}$ by*

$$f(x) := \inf \mathbb{Q}(x)$$

where $\mathbb{Q}(x) := \{p \in \mathbb{Q} \cap [0,1] : x \in U_p\}$.

(i) $f(x) \in [0,1]$ *for all* $x \in X$.

(ii) *For any* $r \in \mathbb{Q}, x \in \overline{U_r} \Rightarrow f(x) \leq r$, *and*

(iii) $x \notin U_r \Rightarrow f(x) \geq r$

Proof. Note that f is well-defined because, for any $x \in X$, $x \in U_p$ for all $p > 1$, so $(1, \infty) \cap \mathbb{Q} \subset \mathbb{Q}(x)$. Hence, $f(x) \leq 1$. Similarly, $x \notin U_p$ for all $p < 0$. Hence, $f(x) \geq 0$.

If $x \in \overline{U_r}$, then for any $p > r$, $x \in U_p$. Hence,

$$(r, \infty) \cap \mathbb{Q} \subset \mathbb{Q}(x)$$

Since the infimum of a subset is greater than the infimum of a super set, $f(x) \leq r$. Similarly, if $x \notin U_r$, then $x \notin U_s$ for all $s < r$. Hence,

$$\mathbb{Q}(x) \subset (r, \infty) \cap \mathbb{Q}$$

As before, this implies $f(x) \geq r$ □

Theorem 2.10 (Urysohn's Lemma)**.** *Let X be a normal space and $A, B \subset X$ disjoint closed sets. Then $\exists f : X \to [0,1]$ continuous such that*

$$f(x) = 0 \quad \forall x \in A \text{ and } f(y) = 1 \quad \forall y \in B$$

Proof. Let $\{U_p : p \in \mathbb{Q}\}$ and $f : X \to \mathbb{R}$ defined as above. For any $x \in X$, and $r < 0$, $x \notin U_r$, so $f(x) \geq 0$. Similarly, $f(x) \leq 1$. Furthermore, if $x \in A$, then $x \in U_0$, so $f(x) = 0$. Similarly, $f(y) = 1$ for all $y \in B$. It suffices to show that f is continuous.

Fix $x_0 \in X$ and U an open set containing $f(x_0)$. WTS: $\exists$ an open set $V \subset X$ containing x_0 such that $f(V) \subset U$. Choose $c, d \in \mathbb{R}$ such that $(c, d) \subset U$. Now there exists $p, r \in \mathbb{Q}$ such that $[p, r] \subset (c, d) \subset U$, and let

$$V := U_r \setminus \overline{U_p}$$

Note that V is open, and if $z \in V$, then $z \in U_r$ and $z \notin \overline{U_p}$. So by the previous lemma,

$$p \leq f(x) \leq r$$

Hence, $f(V) \subset U$ as required. □

Corollary 2.11. *Let X be a normal space and $A, B \subset X$ disjoint closed sets. Given $a, b \in \mathbb{R}$ with $a < b$, $\exists f : X \to [a, b]$ continuous such that*

$$f|_A = a \text{ and } f|_B = b$$

Proof. Simply compose the function $g : X \to [0, 1]$ produced by Urysohn's lemma with the map $[0, 1] \to [a, b]$ given by

$$t \mapsto (1 - t)a + tb$$

$\square$

3. Tietze's extension Theorem

Definition 3.1. Let (X, d) be a metric space.

(i) A sequence $(x_n) \subset X$ is said to be Cauchy if, for each $\epsilon > 0, \exists N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

(ii) X is said to be complete if every Cauchy sequence in X converges to a point in X.

Example 3.2.

(i) Every convergent sequence is Cauchy.

(ii) Let $X = \mathbb{Q}^c$, and $x_n := \sqrt{2}/n$, then (x_n) is Cauchy, but does not converge to a point in X. Hence X is not complete.

(iii) $X = (0, 1)$ is not complete because $(1/n)$ is Cauchy but not convergent.

Lemma 3.3. *Let (X, d) be a metric space and $(x_n) \subset X$ Cauchy. Then (x_n) is bounded. i.e. $\exists x_0 \in X$ and $M \geq 0$ such that $d(x_n, x_0) \leq M$ for all $n \in \mathbb{N}$.*

Proof. Fix $\epsilon = 1$, then $\exists N \in \mathbb{N}$ such that

$$d(x_n, x_m) < 1 \quad \forall n, m \geq 1$$

For $x_0 \in X$ fixed, let

$$M := \max\{d(x_0, x_i) : 1 \leq i \leq N\} + 1$$

Then for any $n \in \mathbb{N}$, if $n \leq N$, then $d(x_n, x_0) \leq M$. And if $n \geq N$, then

$$d(x_n, x_0) \leq d(x_n, x_N) + d(x_N, x_0) \leq M$$

$\square$

Lemma 3.4. *Let (X, d) be a metric space and (x_n) a Cauchy sequence. If (x_n) has a convergent subsequence, then (x_n) converges.*

Proof. Suppose $x_{n_k} \to x$ is a convergent subsequence. For any $\epsilon > 0$, choose $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \epsilon/2 \quad \forall n, m \geq N$$

Now choose $K \in \mathbb{N}$ such that

$$d(x_{n_i}, x) < \epsilon/2 \quad \forall i \geq K$$

Hence, $N_0 := \max\{N, n_K\}$ has the property that

$$d(x_n, x) < \epsilon \quad \forall n \geq N_0$$

□

Lemma 3.5. *Every sequence in $\mathbb{R}$ has a monotone subsequence.*

Proof. Let $(x_n) \subset \mathbb{R}$ and suppose (x_n) has no monotone increasing subsequence. We show that (x_n) has a monotone decreasing subsequence. We claim: $\exists n_1 \in \mathbb{N}$ such that $x_n < x_{n_1}$ for all $n > n_1$.

Proof. Suppose not, then set $n_1 = 1$. Then $\exists n_2 > n_1$ and x_{n_2} such that $x_{n_2} > x_{n_1}$. Similarly, $\exists n_3 > n_2$ such that $x_{n_3} > x_{n_2}$ and so on. Thus, we produce a subsequence (x_{n_k}) that is strictly increasing. This contradicts the assumption that (x_n) has no increasing subsequence. □

Now choose $n_1 \in \mathbb{N}$ such that $x_n < x_{n_1}$ for all $n > n_1$. Now consider the subsequence $\{x_{n_1}, x_{n_1+1}, x_{n_1+2}, \ldots\}$. By the same argument as above, $\exists n_2 > n_1$ such that $x_n < x_{n_2}$ for all $n > n_2$. In particular,

$$x_{n_2} < x_{n_1}$$

and

$$x_n < x_{n_2} \quad \forall n > n_2$$

Thus proceeding (by induction) there is a subsequence (x_{n_k}) that is strictly decreasing.

□

Theorem 3.6. *$\mathbb{R}$ is complete.*

Proof. Let $(x_n) \subset \mathbb{R}$ be Cauchy, then by the previous lemmas, (x_n) is bounded and has a monotone subsequence. But every monotone bounded subsequence in $\mathbb{R}$ is convergent (to its supremum or infimum). Some the previous lemma applies. □

Definition 3.7. Let X be a topological space and (Y, d) a metric space.

(i) A function $f : X \to Y$ is said to be <u>bounded</u> if $f(X)$ is a bounded subset of Y (i.e. $\exists y_0 \in X$ and $M \geq 0$ such that $d(f(x), y_0) \leq M$ for all $x \in X$.

(ii) Let $C_b(X, Y)$ denote the set of all continuous, bounded functions $f : X \to Y$

Theorem 3.8. *Define* $d_\infty : C_b(X,Y) \times C_b(X,Y) \to \mathbb{R}$ *by*

$$d_\infty(f,g) := sup\{d(f(x),g(x)) : x \in X\}$$

Then this defines a metric on $C_b(X,Y)$.

Proof. HW $\square$

(End of Week 11)

Theorem 3.9. *If* (Y,d) *is a complete metric space, and* $(C_b(X,Y), d_\infty)$ *is complete.*

Proof. Let $(f_n) \subset C_b(X,Y)$ be a Cauchy sequence. For any $x \in X$,

$$d(f_n(x), f_m(x)) \leq d_\infty(f_n, f_m)$$

Hence, $(f_n(x))$ is Cauchy in Y. Hence, $\exists z_x \in Y$ such that $f_n(x) \to z_x$. Define $f : X \to Y$ by $f(x) = z_x$. We claim that f is continuous and bounded.

(i) Since (f_n) is Cauchy, it is bounded. Hence, $\exists M \geq 0$ such that

$$\sup_{x \in X} d(f_n(x), 0) \leq M \quad \forall n \in \mathbb{N}$$

For any $x \in X$ fixed, $f_n(x) \to f(x)$. Hence, $d(f(x),0) \leq M$ [Why?]. Hence, f is bounded.

(ii) To see that $f_n \to f$ wrt d_∞: Fix $\epsilon > 0$, then $\exists N \in \mathbb{N}$ such that

$$d_\infty(f_n, f_m) < \epsilon/2 \quad \forall n,m \geq M$$

Hence for $x \in X$ fixed,

$$d(f_n(x), f_m(x)) < \epsilon/2 \quad \forall n,m \geq N$$

Let $m \to \infty$, then

$$d(f_n(x), f(x)) \leq \epsilon/2 \quad \forall n \geq N$$

Hence, $d_\infty(f_n, f) < \epsilon \quad \forall n \geq N$. Hence, $f_n \to f$ in d_∞

(iii) To see that f is continuous: Let $x_0 \in X$ and $\epsilon > 0$, then $\exists N \in \mathbb{N}$ such that

$$d_\infty(f_n, f) < \epsilon/3 \quad \forall n \geq N$$

Since f_N is continuous, $\exists U \subset X$ open such that $x_0 \in U$ and

$$d(f_N(y), f_N(x_0)) < \epsilon/3 \quad \forall y \in U$$

Hence, for all $y \in U$,

$$d(f(y), f(x_0)) < \epsilon$$

□

Corollary 3.10. *Let X be any topological space. The set $C_b(X) := C_b(X, \mathbb{R})$ is a complete metric space with respect to the metric*

$$d_\infty(f, g) := \sup_{x \in X} |f(x) - g(x)|$$

Theorem 3.11 (Tietze's Extension Theorem)**.** *Let X be a normal topological space and $Y \subset X$ closed. Let $f : Y \to \mathbb{R}$ be a continuous function, then $\exists h : X \to \mathbb{R}$ continuous such that*

$$h(y) = f(y) \quad \forall y \in Y$$

(h is called a continuous extension of f)

Proof. Assume first that f is bounded and

$$c := \sup\{|f(y)| : y \in Y\}$$

Define

$$E_0 := \{x \in X : f(x) \leq -c/3\} = f^{-1}(-\infty, -c/3]$$
$$F_0 := \{x \in X : f(x) \geq c/3\} = f^{-1}[c/3, \infty)$$

Then E_0 and F_0 are disjoint closed sets. By Corollary 2.11, $\exists g_0 : X \to \mathbb{R}$ such that

$$-c/3 \leq g_0(x) \leq c/3 \quad \forall x \in X$$

and

$$g_0|_{E_0} = -c/3 \text{ and } g_0|_{F_0} = c/3$$

Hence,

$$|g_0(x)| \leq c/3 \quad \forall x \in X$$
$$|f(y) - g_0(y)| \leq 2c/3 \quad \forall y \in Y$$

Let $f_1 := f - g_0$. Then by the above argument, $\exists g_1 : X \to \mathbb{R}$ continuous such that

$$|g_1(x)| \leq 2c/9 \quad \forall x \in X$$
$$|f(y) - g_0(y) - g_1(y)| \leq 4c/9 \quad \forall y \in Y$$

Thus proceeding, we obtain a sequence (g_n) of continuous functions such that

$$|g_n(x)| \leq 2^n c/3^{n+1} \quad \forall x \in X$$
$$|f(y) - h_n(y)| \leq 2^{n+1} c/3^{n+1} \quad \forall y \in Y$$

where $h_n := g_0 + g_1 + \ldots + g_n$. Now note that if $m > n$,

$$\begin{aligned}|h_n(x) - h_m(x)| &= \left|\sum_{i=m+1}^{n} g_i(x)\right| \\ &\leq \sum_{i=m+1}^{n} |g_i(x)| \\ &\leq \sum_{i=m+1}^{n} \frac{2^i c}{3^{i+1}} \leq \frac{2^{m+1}c}{3^{m+1}}\end{aligned}$$

Hence,

$$d_\infty(h_n, h_m) \leq \frac{2^{m+1}c}{3^{m+1}}$$

Since the RHS goes to zero, (h_n) form a Cauchy sequence in $C_b(X, \mathbb{R})$. By the previous lemma, $\exists h \in C_b(X, \mathbb{R})$ such that $h_n \to h$. Now if $y \in Y$, then

$$|f(y) - h_n(y)| \leq \frac{2^{n+1}c}{3^{n+1}}$$

Letting $n \to \infty$, we see that $h = f$ on Y.

Now suppose f is not bounded. Let $g : \mathbb{R} \to (-1, 1)$ be a homeomorphism (is there one?). Now define $\widetilde{f} := g \circ f$. Now $\widetilde{f}$ is bounded, so $\exists \widetilde{h} : X \to \mathbb{R}$ continuous such that $\widetilde{h}|_Y = \widetilde{f}$. Now define $h := g^{-1} \circ \widetilde{h}$, and check that h satisfies the required conditions. □

4. Urysohn Metrization Theorem

Definition 4.1. A topological space (X, τ) is said to be <u>metrizable</u> if there exists a metric d on X such that $\tau = \tau_d$.

Proposition 4.2. $\mathbb{R}^\omega$ *with the product topology is metrizable.*

Proof. Let $\overline{d} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the metric given by

$$\overline{d}(a, b) = \min\{|a - b|, 1\}$$

Define $D : \mathbb{R}^\omega \times \mathbb{R}^\omega \to \mathbb{R}$ by

$$D(x, y) := \sup\left\{\frac{\overline{d}(x_i, y_i)}{i}\right\}$$

Then [Check!] that D is a metric on $\mathbb{R}^\omega$. We claim that the product topology τ_p on $\mathbb{R}^\omega$ coincides with τ_D

(i) $\tau_p \subset \tau_D$: Let U be a basic open set in τ_p of the form

$$U := U_1 \times U_2 \times \ldots \times U_n \times \mathbb{R} \times \mathbb{R} \times \ldots$$

Let $x = (x_i) \in U$, so for $1 \leq i \leq n, x_i \in U_i$, so $\exists \epsilon_i > 0$ such that

$$(x_i - \epsilon_i, x_i + \epsilon_i) \subset U_i$$

Assume $\epsilon_i < 1$ for all i, and let $\epsilon := \min\{\epsilon_i/i : 1 \leq i \leq n\}$, then we claim that

$$B_D(x, \epsilon) \subset U$$

To see this, suppose $y = (y_i) \in B_D(x, \epsilon)$, then for $1 \leq i \leq n$,

$$\frac{\overline{d}(x_i, y_i)}{i} \leq D(x, y) < \epsilon$$

Hence, $\overline{d}(x_i, y_i) \leq \epsilon_i < 1$, so $|x_i - y_i| < \epsilon_i$. Hence, $y_i \in U_i$ for all $1 \leq i \leq n$. Hence, $y \in U$, so

$$B_D(x, \epsilon) \subset U$$

Thus, U is a union of sets of the form $B_D(x, \epsilon)$, and so $U \in \tau_D$. Since U is a generic basic open set, it follows that $\tau_p \subset \tau_D$.

(ii) $\tau_D \subset \tau_p$: Let $U \in \tau_D$ be open, and $x \in U$. Then $\exists \epsilon > 0$ such that $B_D(x, \epsilon) \subset U$. Choose $N \in \mathbb{N}$ such that $1/N < \epsilon$, and consider

$$V := (x_1 - \epsilon, x_1 + \epsilon) \times \ldots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \ldots$$

We claim that $V \subset B_D(x, \epsilon)$. To see this, suppose $y = (y_i) \in V$, then for $i \geq N$,

$$\frac{\overline{d}(x_i, y_i)}{i} \leq \frac{1}{N}$$

because $\overline{d}(x_i, y_i) \leq 1$. Furthermore, if $1 \leq i \leq N$, then

$$\frac{\overline{d}(x_i, y_i)}{i} \leq \frac{d(x_i, y_i)}{i} \leq \frac{1}{Ni} < \epsilon$$

Hence, $D(x, y) < \epsilon$. This is true for any $y \in V$, so $V \subset B_D(x, \epsilon) \subset U$. Hence, U is a union of open sets in τ_p, and so $U \in \tau_p$. Thus, $\tau_D \subset \tau_p$ as well.

$\square$

(End of Week 12)

Definition 4.3. A topological space is called <u>second countable</u> if it has a countable basis.

Example 4.4.

(i) $\mathbb{R}^n$ is second countable.

(ii) If $\mathbb{R}$ is given the discrete metric, then it is not second countable.

(iii) Every second countable space is separable.

Proof. Let $\{B_n : n \in \mathbb{N}\}$ be a countable basis for X. For each $n \in \mathbb{N}$, choose $x_n \in B_n$ and let $D := \{x_n : n \in \mathbb{N}\}$. Then D is dense in X, because if U is any non-empty open set, then $\exists n \in \mathbb{N}$ such that $B_n \subset U$, so $x_n \in U$ which implies $D \cap U \neq \emptyset$. □

(iv) Any separable metric space is second countable.

Proof. Let (X, d) be a separable metric space and $A := \{x_n\}$ be a countable dense subset of X. Let $B_{m,n} := B(x_m, 1/n)$, then we claim that $\mathcal{B} := \{B_{m,n}\}$ forms a basis for τ_d.

(i) If $x \in X$, then $\exists x_m \in A$ such that $d(x_m, x) < 1$. Hence, $x \in B_{m,1}$. So $\mathcal{B}$ covers X.

(ii) Furthermore, if $x \in B_{m_1,n_1} \cap B_{m_2,n_2}$ then let $\alpha := \min\{1/2n_1, 1/2n_2\}$. Choose $m_3 \in \mathbb{N}$ such that $d(x, x_{m_3}) < \alpha$ and let $n_3 \in \mathbb{N}$ such that $1/n_3 < \alpha$, then [Check!]

$$B_{m_3,n_3} \subset B_{m_1,n_1} \cap B_{m_2,n_2}$$

and $x \in B_{m_3,n_3}$.

(iii) Thus, $\mathcal{B}$ forms a basis for some topology τ on X. Since $\mathcal{B} \subset \tau_d$, it follows that $\tau \subset \tau_d$.

(iv) However, if $U \in \tau_d$ and $x \in U$, then $\exists \epsilon > 0$ such that $B_d(x, \epsilon) \subset U$. Now choose $m \in \mathbb{N}$ such that $d(x, x_m) < \epsilon/2$, and let $n \in \mathbb{N}$ such that $1/n < \epsilon/2$, then $x \in B_{m,n}$ and $B_{m,n} \subset B_d(x, \epsilon) \subset U$. Hence, every $U \in \tau_d$ is obtained as a union of elements of $\mathcal{B}$.

Hence, $\mathcal{B}$ is a basis for τ_d. □

Lemma 4.5. *Every regular, second countable space is normal.*

Proof. Let X be a regular space with a countable basis $\mathcal{B}$, and let $A, B \subset X$ be two closed disjoint sets. WTS: $\exists$ open sets U and V such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

(i) For each $x \in A$, $x \notin B$, so there is an open sets U, V such that $x \in U, B \subset V$ and $U \cap V = \emptyset$. Since X is regular, there is an open set W such that $x \in W$ and $\overline{W} \subset U$. Choose a basic open set $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \subset W$. Thus,

$$\overline{B_x} \cap B = \emptyset$$

Thus, we obtain an open cover $\{B_x : x \in A\}$ for A which is countable, so we denote it by $\{U_n : n \in \mathbb{N}\}$. Note that

$$\overline{U_n} \cap B = \emptyset \quad \forall n \in \mathbb{N}$$

Similarly, we obtain an open cover $\{V_n : n \in \mathbb{N}\}$ of B which is countable such that

$$\overline{V_n} \cap A = \emptyset \quad \forall n \in \mathbb{N}$$

(ii) If $U := \bigcup U_n$ and $V := \bigcup V_n$, then $A \subset U, B \subset V$, but U and V need not be disjoint. So define

$$U_n' := U_n \setminus \left[\bigcup_{i=1}^{n} \overline{V_n}\right] \text{ and } V_n' := V_n \setminus \left[\bigcup_{i=1}^{n} \overline{U_n}\right]$$

Then each U_n' and V_n' is open.

(iii) If $x \in A$, then $\exists n \in \mathbb{N}$ such that $x \in U_n$. But $\overline{V_i} \cap A = \emptyset$ for all i. Hence, $x \in U_n'$. Thus, $\{U_n' : n \in \mathbb{N}\}$ forms an open cover for A. Define

$$U' := \bigcup_{n=1}^{\infty} U_n'$$

Then $A \subset U'$. Similarly, if

$$V' := \bigcup_{n=1}^{\infty} V_n'$$

Then $B \subset V'$.

(iv) We claim that $U' \cap V' = \emptyset$. Suppose $x \in U' \cap V'$, then $\exists n, m \in \mathbb{N}$ such that $x \in U_n'$ and $x \in V_m'$. Assume $n > m$, then $x \notin V_m$ by definition of U_n'. This is a contradiction, so $U' \cap V' = \emptyset$.

□

Lemma 4.6. *Let X be a regular space with a countable basis. Then there is a sequence of functions $f_n : X \to [0,1]$ such that, for any $x_0 \in X$ and open set U containing $x_0, \exists n \in \mathbb{N}$ such that $f_n(x_0) = 1$ and $f_n = 0$ on $X \setminus U$.*

Proof. Note that X is normal so Urysohn's lemma applies. Let $\{B_n : n \in \mathbb{N}\}$ be a countable basis for X. Define

$$D := \{(n,m) \in \mathbb{N} \times \mathbb{N} : \overline{B_n} \subset B_m\}$$

For each $(n,m) \in D$, Urysohn's lemma implies that there is a function $g_{n,m} : X \to [0,1]$ such that

$$g_{n,m}|_{\overline{B_n}} = 1 \text{ and } g_{n,m}|_{X \setminus B_m} = 0$$

This collection $\{g_{n,m}\} = \{f_n\}$ is countable, and it satisfies the required condition: If $x_0 \in X$ and U is an open set such that $x_0 \in U$, then $\exists$ a basic open set B_m such that $x_0 \in B_m$ and $B_m \subset U$. Furthermore, by regularity, $\exists$ a basic open set B_n such that $x_0 \in B_n$ and $\overline{B_n} \subset B_m$. Now

$$g_{n,m}(x_0) = 1 \text{ and } g_{n,m}|_{X \setminus U} = 0$$

□

Theorem 4.7 (Urysohn's Metrization Theorem)**.** *Every regular, second countable space is metrizable.*

Proof.

(i) We construct a continuous function $F : X \to \mathbb{R}^\omega$ as follows: Let $\{f_n\}$ be a sequence as in the previous lemma, and define

$$F(x) := (f_n(x))$$

Then F is continuous because each coordinate function f_n is continuous.

(ii) F is injective: If $x \neq y$, then there is an open set U such that $x \in U$ and $y \notin U$. Choose $n \in \mathbb{N}$ such that $f_n(x) = 1$ and $f_n|_{X\setminus U} = 0$. In particular, $f_n(y) = 0$. Hence, $F(x) \neq F(y)$.

(iii) Let $Z := F(X)$. We claim that $F : X \to Z$ is a homeomorphism. F is clearly surjective, so it suffices to show that F is an open map. Let $U \subset X$ be an open set. WTS: $F(U) \subset Z$ is open. Fix $z \in F(U)$, then $\exists x \in U$ such that

$$F(x) = z$$

Choose $n \in \mathbb{N}$ such that $f_n(x) = 1$ and $f_n|_{X\setminus U} = 0$. Define

$$V := \pi_n^{-1}((0, \infty)) \subset \mathbb{R}^\omega$$

and set

$$W := V \cap Z$$

Then W is open in Z since V is open in $\mathbb{R}^\omega$. Furthermore, $f_n(x) > 0$, so $z \in W$. We claim: $W \subset F(U)$. To see this, fix $y \in W$, then $\exists x' \in X$ such that $F(x') = y$. Now, $\pi_n(y) > 0$, but

$$\pi_n(y) = \pi_n(F(x')) = f_n(x')$$

Since $f_n = 0$ on $X \setminus U$, it follows that $x' \in U$. Hence, $x' \in F(U)$. Thus, $W \subset F(U)$. Hence, every $z \in F(U)$ is an interior point of $F(U)$, so $F(U)$ is open.

(iv) Thus, $F : X \to Z$ is a homeomorphism. Since $Z \subset \mathbb{R}^\omega$ and $\mathbb{R}^\omega$ is metrizable, it follows that Z is metrizable, and so X is too.

□

Corollary 4.8. *Every compact, Hausdorff, second countable space is metrizable.*

Example 4.9.

(i) Every metric space is certainly regular, but need not have a countable basis (See Example 4.4).

(ii) Let $K = \{1/n : n \in \mathbb{N}\}$. Define

$$\mathcal{B}_1 := \{\text{ open intervals in } \mathbb{R} \text{ with rational end-points}\}$$
$$\mathcal{B}_2 := \{(a,b) \setminus K : a < b \text{ in } \mathbb{Q}\}$$

Then $\mathcal{B} := \mathcal{B}_1 \cup \mathcal{B}_2$ forms a basis for a topology on $\mathbb{R}$, which we denote by τ_K. Then $\mathbb{R}_K := (\mathbb{R}, \tau_K)$ is Hausdorff, has a countable basis, but is not metrizable because it is not regular. Thus, regularity is necessary for Urysohn's metrization theorem to hold.

5. Imbedding of Manifolds

Definition 5.1. An <u>m-manifold</u> is a Hausdorff topological space X with a countable basis such that for each $x \in X$, there is a neighbourhood U_x of x such that U_x is homeomorphic with an open subset of $\mathbb{R}^m$.

Example 5.2.

(i) $\mathbb{R}^m$ is an m-manifold. So is any open subset of $\mathbb{R}^m$.

(ii) $[0, 1]$ is not a 1-manifold, because any neighbourhood of 0 is of the form $[0, \delta)$, which is not homeomorphic to an open subset of $\mathbb{R}$.

(iii) S^1 is a 1-manifold. In general, S^m is an m-manifold (without proof)

(iv) A 1-manifold is called a curve, and a 2-manifold is called a surface.

(v) The torus $S^1 \times S^1$ is a surface. In general, if X and Y are manifolds, then so is $X \times Y$.

(End of Week 13)

Theorem 5.3. *Let X be an m-manifold. Then X is*

(i) Locally path connected.

(ii) Locally compact.

(iii) Regular

(iv) Metrizable.

Proof.

(i) Let $x \in X$ and U an open neighbourhood of x. WTS: $\exists V \subset U$ open such that $x \in V$ and V is path connected. To see this, choose a neighbourhood U_x of x and a homeomorphism

$$g : U_x \to U'_x \subset \mathbb{R}^m$$

where U'_x is open in $\mathbb{R}^m$. Then $U_x \cap U$ is open and

$$g|_{U_x \cap U} \colon U_x \cap U \to g(U'_x \cap U) \subset \mathbb{R}^m$$

is a homeomorphism. Since $g(U'_x \cap U)$ is an open subset of $\mathbb{R}^m$ containing $g(x)$, and $\mathbb{R}^m$ is locally path connected, there is an open set $V' \subset g(U'_x \cap U)$ that is path connected and containing $g(x)$. Then $V := g^{-1}(V')$ is open, path connected, contains x and $V \subset U$.

(ii) Local compactness is identical to part (i).

(iii) Let $x \in X$ and an open set U containing x. WTS: $\exists V$ open such that $x \in V$ and $\overline{V} \subset U$. Choose U_x open and a homeomorphism

$$g : U_x \to U'_x \subset \mathbb{R}^m$$

as before. Since $U \cap U_x$ is open in U_x,

$$g(U \cap U_x) \subset U'_x$$

is open and contains $g(x)$. Since $U'_x \subset \mathbb{R}^m$ and $\mathbb{R}^m$ is regular, U'_x is regular by Corollary 1.5. Hence, there is an open set V' such that $g(x) \in V'$ and

$$\overline{V'} \subset g(U \cap U_x)$$

Then $V := g^{-1}(V')$ is open, contains x and since g is a local homeomorphism

$$\overline{V} = \overline{g^{-1}(V')} = g^{-1}(\overline{V'}) \subset g^{-1}(g(U \cap U_x)) \subset U \cap U_x \subset U$$

Hence, X is regular.

(iv) X has a countable basis, so Urysohn's metrization theorem applies.

□

Definition 5.4. Let X be a topological space.

(i) Let $f : X \to \mathbb{R}$ be a function. The <u>support</u> of f is the set

$$\operatorname{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}$$

(ii) Let $\mathcal{U} := \{U_1, U_2, \ldots, U_n\}$ be an open cover for X. A partition of unity dominated by $\mathcal{U}$ is a family of continuous functions $f_i : X \to \mathbb{R}$ such that

  (i) $\operatorname{supp}(f_i) \subset U_i$ for all $1 \leq i \leq n$

  (ii) For each $x \in X$, $f_1(x) + f_2(x) + \ldots + f_n(x) = 1$

Lemma 5.5. *Let X be a normal space and $\mathcal{U} := \{U_1, U_2, \ldots, U_n\}$ be an open cover for X. Then there is an open cover $\mathcal{V} := \{V_1, V_2, \ldots, V_n\}$ such that*

$$\overline{V_i} \subset U_i$$

for all $1 \leq i \leq n$.

Proof. We induct on n: If $n = 1$, then $U_1 = X$ so take $V_1 = U_1$. If $n \geq 2$, note that

$$A := X \setminus \left[\bigcup_{i=2}^{n} U_i\right]$$

is closed and $A \subset U_1$. Since X is normal, there is an open set V_1 such that

$$A \subset V_1 \text{ and } \overline{V_1} \subset U_1$$

The collection $\{V_1, U_2, \ldots, U_n\}$ now covers X. Proceeding by induction, suppose that we have produced a cover

$$\{V_1, V_2, \ldots, V_{k-1}, U_k, U_{k+1}, \ldots, U_n\}$$

such that $\overline{V_i} \subset U_i$ for all $1 \leq i \leq k-1$. Let

$$A := X \setminus \left[\left(\bigcup_{i=1}^{k-1} V_i\right) \cup \left(\bigcup_{j=k+1}^{n} U_j\right)\right]$$

Then A is closed and contained in U_k. Choose V_k open such that $A \subset V_k$ and $\overline{V_k} \subset U_k$. Now $\{V_1, V_2, \ldots, V_k, U_{k+1}, \ldots, U_n\}$ forms an open cover. Proceeding thus, we exhaust all U_i's. $\square$

Theorem 5.6. *Let X be a normal space and $\mathcal{U}$ be a finite open cover for X. Then there is a partition of unity dominated by $\mathcal{U}$.*

Proof. Let $\mathcal{U} := \{U_1, U_2, \ldots, U_n\}$ be an open cover for X. Choose a cover $\mathcal{V} := \{V_1, V_2, \ldots, V_n\}$ such that $\overline{V_i} \subset U_i$ and an open cover $\mathcal{W} := \{W_1, W_2, \ldots, W_n\}$ such that $\overline{W_i} \subset V_i$ for all $1 \leq i \leq n$. By Urysohn's lemma, there exist function $\psi_i : X \to [0,1]$ such that

$$\psi_i|_{\overline{W_i}} = 1 \text{ and } \psi_i|_{X \setminus V_i} = 0$$

Then

$$\operatorname{supp}(\psi_i) \subset \overline{V_i} \subset U_i$$

For any $x \in X, \exists 1 \leq i \leq n$ such that $x \in W_i$, so $\psi_i(x) = 1$. Hence, define $f_i : X \to \mathbb{R}$ by

$$f_i(x) := \frac{\psi_i(x)}{\psi_1(x) + \psi_2(x) + \ldots + \psi_n(x)}$$

The denominator is never zero, so f_i is continuous, and is a partition of unity dominated by $\mathcal{U}$. $\square$

Theorem 5.7 (Imbedding Theorem)**.** *Let X be a compact m-manifold, then $\exists N \in \mathbb{N}$ and an injective map*

$$F : X \to \mathbb{R}^N$$

such that $F : X \to F(X)$ is a homeomorphism. (ie. F is an imbedding of X into $\mathbb{R}^n$)

Proof. For each $x \in X, \exists$ an open set U_x that is homeomorphic to an open subset of $\mathbb{R}^m$. Choose a finite subcover $\{U_1, U_2, \ldots, U_n\}$ and homeomorphisms

$$g_i : U_i \to V_i$$

where $V_i \subset \mathbb{R}^m$ is open. Let $\{f_1, f_2, \ldots, f_n\}$ be a partition of unity dominated by $\mathcal{U}$. Let $A_i := \text{supp}(f_i) \subset U_i$ and define $h_i : X \to \mathbb{R}^m$ by

$$h_i(x) := \begin{cases} f_i(x)g_i(x) & : x \in U_i \\ 0 & : x \in X \setminus A_i \end{cases}$$

If $x \in (X \setminus A_i) \cap U_i$, then $f_i(x) = 0$, so both definitions agree. So by pasting lemma, h_i is continuous. Define

$$F : X \to \underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \mathbb{R}}_{n \text{ times}} \times \underbrace{\mathbb{R}^m \times \mathbb{R}^m \times \ldots \times \mathbb{R}^m}_{n \text{ times}}$$

by

$$x \mapsto (f_1(x), f_2(x), \ldots, f_n(x), h_1(x), h_2(x), \ldots, h_n(x))$$

Then F is continuous. Suppose we show that F is injective, then since X is compact,

$$F : X \to F(X)$$

will be a homeomorphism. So suppose $x, y \in X$ such that $F(x) = F(y)$, then choose $1 \leq i \leq n$ such that $f_i(x) > 0$. Then $x \in U_i$ and $f_i(x) = f_i(y) > 0$ and $h_i(x) = h_i(y)$ implies that

$$g_i(x) = g_i(y)$$

But $g_i : U_i \to V_i$ is a homeomorphism, so $x = y$ as required. $\square$

(End of Week 14)

V. Instructor Notes

(i) As before, I was unable to cover Tychonoff's theorem and Lindeloff spaces, neither of which is a major loss. We did discuss Tychonoff's theorem though.

(ii) The students were coming out of COVID (the first half of the semester was online), so their learning losses were significant. I was surprised by their lack of enthusiasm though (less than half attended lectures, and no questions were forthcoming).

(iii) Barring a few students, most had very poor grades, and this is something that requires immediate attention.

Bibliography

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