## MTH 304: General Topology Semester 2, 2017-2018

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# I. Continuous Functions

## 1. First Definitions

Following [Crossley, Section 2.1,2.2]

Let  $S \subset \mathbb{R}$ . A function in this section will be a real-valued function whose domain is S.

- 1.1. Remark: Consider two graphs (one continuous and other discontinuous at x = 1). Continuity means that we can draw the graph of f without lifting our pencil. ie. If we approach a point on the x axis from either direction, the value of f(x) should be 'predicted' by the values of f(y) where y is near x. Furthermore, Continuity is a local property.
- 1.2. Definition: A function  $f: S \to \mathbb{R}$  is said to be sequentially continuous at  $a \in S$  if, for any sequence  $(x_n) \subset S$  such that  $x_n \to a$ , we have  $f(x_n) \to f(a)$ .
- 1.3. Example: f(x) = x/|x| for  $x \neq 0$  and f(0) = 1
  - (i) If we choose a = 0 and  $x_n = 1/n$ , then  $f(a) = \lim f(x_n)$
  - (ii) However, if we choose  $x_n = -1/n$ , then  $f(a) \neq \lim f(x_n)$ . So f is not sequentially continuous.
- 1.4. Definition: A function  $f: S \to \mathbb{R}$  is said to be continuous at a if, for every  $\epsilon > 0, \exists \delta > 0$  such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon \qquad (*)$$

1.5. Example:

(i)  $f(x) = x^2$  is continuous at 2

(a) If  $a = 0, \epsilon = 1$ , we want  $\delta > 0$  such that (\*) holds. ie. We want

 $|x| < \delta \Rightarrow |x^2| < 1$ 

Since  $|x^2| = |x|^2$ , we may choose  $\delta = 1$ .

(b) If  $a = 2, \epsilon = 1$ , we want  $\delta > 0$  such that (\*) holds. ie. We want

$$|x-2| < \delta \Rightarrow |x^2 - 2^2| < 1$$

Notice that  $\delta = 1$  does not work, because if x = 2.9 then  $x^2 \approx 9$ . However,

$$|x^2 - 2^2| = |x - 2||x + 2|$$

So  $\exists \delta > 0$  that works.

(ii)  $f(x) = x^2$  if  $x \neq 0$  and f(0) = 0.5 is discontinuous at 1.

(a) If  $\epsilon = 1$ , then  $\delta = 0.5$  works because if

 $|x| < 0.5 \Rightarrow |x^2| < 0.25 < 1$ , and |f(0)| = 0.5 < 1

(b) However, if  $\epsilon = 0.2$ , then no  $\delta > 0$  works because if  $|x| < \delta$ , then we may choose small enough x so that |x| < 0.5, so that  $|x^2| < 0.25$  and hence

$$|x^2 - 0.5| > 0.25$$

So f is discontinuous at 0.

- 1.6. Theorem: f is continuous at a if and only if it is sequentially continuous at a
  - *Proof.* (i) Suppose f is continuous at a and  $(x_n) \subset S$  is a sequence such that  $x_n \to a$ . WTS:  $f(x_n) \to f(a)$ , so choose  $\epsilon > 0$ , then  $\exists \delta > 0$  such that

$$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

For this  $\delta > 0, \exists N \in \mathbb{N}$  such that  $|x_n - a| < \delta$  for all  $n \ge N$ . Hence,

$$|f(x_n) - f(a)| < \epsilon \quad \forall n \ge N$$

This is true for any  $\epsilon > 0$  so  $f(x_n) \to f(a)$ 

(ii) Suppose f is sequentially continuous at a, but it is not continuous at a, then  $\exists \epsilon > 0$  for which no  $\delta$  works. Hence,  $\delta = 1/n$  does not work, so  $\exists x_n \in S$  such that

$$|x_n - a| < 1/n$$
, but  $|f(x_n) - f(a)| \ge \epsilon$ 

Clearly,  $x_n \to a$ , but  $f(x_n)$  does not converge to f(a). Hence, f is not sequentially continuous - a contradiction.

(End of Day 1)

### 2. Open Sets

[Crossley, Section 2.3]

2.1. Remark:  $\epsilon - \delta$  definition says that f is continuous at a if and only if, for any  $\epsilon > 0, \exists \delta > 0$  such that

$$x \in (a - \delta, a + \delta) \Rightarrow f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$$

#### 2.2. Definition:

(i) Open interval

- (ii) Open set: A set  $U \subset \mathbb{R}$  is open if and only if it can be written as a union of open intervals. (Note: We are not restricting ourselves to finite unions. ie. We are referring to 'arbitrary' unions)
- 2.3. Theorem: A set  $U \subset \mathbb{R}$  is open iff for all  $x \in U, \exists \delta_x > 0$  such that  $(x \delta_x, x + \delta_x) \subset U$ Note: The value of  $\delta_x$  depends on x.
  - *Proof.* (i) Suppose that, for any  $x \in U, \exists \delta_x > 0$  such that  $(x \delta_x, x + \delta_x) \subset U$ , then

$$U = \bigcup_{x \in U} (x - \delta_x, x + \delta_x)$$

so U is open.

(ii) Conversely, if U is open, then write  $U = \bigcup_{\alpha \in J} I_{\alpha}$ , where each  $I_{\alpha}$  is an open interval. If  $x \in U$ , then  $\exists \alpha \in J$  such that  $x \in I_{\alpha}$ . Write  $I_{\alpha} = (a, b)$ , then a < x < b, so

$$\delta_x = \min\{|x - a|/2, |b - x|/2\}$$

works.

2.4. Examples:

(i) (a, b)

- (ii) A closed interval (or even a half-open interval) is not open.
- (iii)  $\{0\}$  is not open. A finite set is not open.
- 2.5. Proposition:
  - (i) An arbitrary union of open sets is open.
  - (ii) A finite intersection of open sets is open.

*Proof.* (i) is obvious, so we prove (ii): By induction, it suffices to consider the case of two sets,  $U_1, U_2$  say. WTS:  $U_1 \cap U_2$  is open, so fix  $x \in U_1 \cap U_2$ , then  $\exists \delta_1, \delta_2 > 0$  such that  $(x - \delta_i, x + \delta_i) \subset U_i, i = 1, 2$ . Then if  $\delta = \min\{\delta_1, \delta_2\}$ , then  $(x - \delta, x + \delta) \subset U_1 \cap U_2$ , which verifies Theorem 2.3.

- 2.6. Example: A countable intersection of open sets may not be open.  $U_n = (-1/n, 1/n)$
- 2.7. Definition: A set  $F \subset \mathbb{R}$  is closed if  $F^c$  is open.

2.8. Examples:

- (i) Closed interval
- (ii)  $[2,\infty)$  is closed.
- (iii) Arbitrary intersection of closed sets is closed.
- (iv) Finite union of closed sets is closed.
- (v) [1,2) is neither open nor closed.

## 3. Continuity by Open Sets

[Crossley, Section 2.4]

3.1. Definition: Let  $f: X \to Y$  be a function between two sets and  $A \subset Y$ , then

$$f^{-1}(A) = \{ x \in X : f(x) \in A \}$$

Note: This definition does not imply that  $f^{-1}$  exists as a function. It is simply notation.

- 3.2. Example:  $f(x) = x^2 x = x(x 1)$ (i)  $f^{-1}(\mathbb{R}) = \mathbb{R}$ 
  - (ii)  $f^{-1}(\emptyset) = \emptyset$ (iii)  $f^{-1}[-1,\infty) = \mathbb{R}$ (iv)  $f^{-1}[0,\infty) = \mathbb{R} \setminus (0,1)$ (v)  $f^{-1}(\{0\}) = \{0,1\}$

3.3. Proposition: Let  $f: X \to Y$  and  $\{A_{\alpha} : \alpha \in J\}$  be a collection of subset of Y, then

(i)  $f^{-1}(\emptyset) = \emptyset$ (ii)  $f^{-1}(Y) = X$ (iii)  $f^{-1}(\bigcap A_{\alpha}) = \bigcap f^{-1}(A_{\alpha})$ (iv)  $f^{-1}(\bigcup A_{\alpha}) = \bigcup f^{-1}(A_{\alpha})$ 

- 3.4. Theorem: Let  $f : \mathbb{R} \to \mathbb{R}$ , then f is continuous if and only if  $f^{-1}(U)$  is open whenever U is open.
  - *Proof.* (i) Suppose f is continuous and U is open in  $\mathbb{R}$ . WTS:  $f^{-1}(U)$  is open, so fix  $x \in f^{-1}(U)$ . So that  $f(x) \in U$ , so  $\exists \epsilon > 0$  such that

$$(f(x) - \epsilon, f(x) + \epsilon) \subset U$$

By definition of continuity,  $\exists \delta > 0$  such that

$$|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon$$

So if  $y \in (x - \delta, x + \delta)$ , then  $f(y) \in (f(x) - \epsilon, f(x) + \epsilon) \subset U$ . Hence,

$$(x - \delta, x + \delta) \subset f^{-1}(U)$$

This is true for any  $x \in f^{-1}(U)$ . By Theorem 2.3,  $f^{-1}(U)$  is open.

(ii) Suppose  $f^{-1}(U)$  is open whenever U is open. Fix  $a \in \mathbb{R}, \epsilon > 0$ . Then

$$U = (f(a) - \epsilon, f(a) + \epsilon)$$

is open in  $\mathbb{R}$  so  $f^{-1}(U)$  is open. Since  $a \in f^{-1}(U), \exists \delta > 0$  such that

$$(a - \delta, a + \delta) \subset f^{-1}(U)$$

Hence, if  $x \in \mathbb{R}$  such that  $|x - a| < \delta$ , then  $|f(x) - f(a)| < \epsilon$ .

(End of Day 2)

# **II. Topological Spaces**

### 1. Definition and Examples

- 1.1. Definition: Let X be a set. A collection  $\tau$  of subsets of X is called a topology on X if
  - (i)  $\emptyset, X \in \tau$
  - (ii) If  $U_1, U_2 \in \tau$ , then  $U_1 \cap U_2 \in \tau$
  - (iii) If  $\{U_{\alpha} : \alpha \in J\}$  is an arbitrary collection of sets in  $\tau$ , then  $\bigcup_{\alpha \in J} U_{\alpha} \in \tau$

The pair  $(X, \tau)$  is called a topological space, and members of  $\tau$  are called open sets in X.

1.2. Examples:

- (i)  $X = \mathbb{R}$  and  $\tau$  = the collection of open sets in  $\mathbb{R}$  (as defined in the previous section) is a topological space. This is called the usual topology on  $\mathbb{R}$
- (ii) Let  $X = \mathbb{R}^2$ .
  - (a) Fix  $\overline{a} := (a_1, a_2) \in X, r > 0$ . An open disc in X centered at x of radius r is the set

$$B(\overline{a}, r) := \{ (x_1, x_2) \in \mathbb{R}^2 : \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} < r \}$$

- (b) A set  $U \subset \mathbb{R}^2$  is said to be open if it is a union of open discs. As in Theorem 2.3, a set  $U \subset \mathbb{R}^2$  is open if and only if, for any  $\overline{a} \in U, \exists r > 0$ such that  $B(\overline{a}, r) \subset U$ .
- (c) As in Proposition 2.5, an arbitrary union of open sets is open, and a finite intersection of open sets is open. Hence, this collection of open sets forms a topology on  $\mathbb{R}^2$ . This is called the Euclidean topology on  $\mathbb{R}^2$ .
- (iii) Let X be any set and  $\tau = \{\emptyset, X\}$ . This is called the indiscrete topology on X.
- (iv) Let X be any set and  $\tau = \mathcal{P}(X)$ . This is called the discrete topology on X.
- 1.3. Definition: Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be topological spaces. A function  $f : X \to Y$  is said to be continuous if  $f^{-1}(U) \in \tau_X$  whenever  $U \in \tau_Y$ . i.e. The inverse image of an open set is open.

Note: We think of continuity as a global property here, and don't care if a function is continuous at all but one point.

1.4. Example:

- (i) Let  $f : \mathbb{R} \to \mathbb{R}$  be  $f(x) = x^2$  is continuous, but f(x) = x/|x| if  $x \neq 0$  and f(0) = 1 is discontinuous.
- (ii) Let  $(X, \tau_d)$  be a discrete topological space, and  $(Y, \tau_Y)$  any topological space. If  $f: X \to Y$  is any function, then f is continuous.
- (iii) Similarly, if  $(X, \tau_X)$  is any topological space and  $(Y, \tau_i)$  is an indiscrete topological space, then any function  $f: X \to Y$  is continuous.
- (iv) Let  $f: X \to Y$  be a constant function, then f is continuous.

*Proof.* Suppose  $f(x) = y_0$  for all  $x \in X$ . Let U be an open set in Y, then

$$f^{-1}(U) = \begin{cases} \emptyset : \text{ if } y_0 \notin U \\ X & : \text{ if } y_0 \in U \end{cases}$$

In either case,  $f^{-1}(U)$  is open.

(v) Let  $A : \mathbb{R}^2 \to \mathbb{R}$  be the addition map A(x, y) = x + y. Then A is continuous.

*Proof.* Let  $U \subset \mathbb{R}$  be open. We WTS:  $A^{-1}(U)$  is open. As mentioned above, it suffices to show that, for any point  $(a, b) \in A^{-1}(U), \exists r > 0$  such that  $B((a, b), r) \subset A^{-1}(U)$ . So fix  $(a, b) \in A^{-1}(U)$ . Then  $a + b \in U$ , so  $\exists \epsilon > 0$ such that  $(a + b - \epsilon, a + b + \epsilon) \subset U$ . Note that  $A^{-1}((a + b - \epsilon, a + b + \epsilon))$ describes the region enclosed by (but not including) the two lines

$$x + y = a + b - \epsilon$$
 and  $x + y = a + b + \epsilon$ 

and (a, b) lies in this region. Now the distance of a point  $(x_0, y_0)$  from a line of the form  $\alpha x + \beta y + \gamma = 0$  is given by

$$d = \frac{|\alpha x_0 + \beta y_0 + \gamma|}{\sqrt{\alpha^2 + \beta^2}}$$

In this case, we get

$$d = \frac{|a+b+(-a-b-\epsilon)|}{\sqrt{2}} = \frac{\epsilon}{\sqrt{2}}$$

Hence, if  $(x, y) \in B((a, b), \epsilon/\sqrt{2})$ , then  $(x, y) \in A^{-1}((a + b - \epsilon, a + b + \epsilon))$ , and hence  $B((a, b), \epsilon/\sqrt{2}) \subset A^{-1}(U)$ , and so  $A^{-1}(U)$  is open. Hence, A is continuous.

- (vi) Similarly, the multiplication map  $M : \mathbb{R}^2 \to \mathbb{R}$  given by  $(x, y) \mapsto xy$  is also continuous [We will give a simpler proof later]
- (vii) Let  $d: \mathbb{R} \to \mathbb{R}^2$  be the diagonal map d(x) = (x, x). Then d is continuous.

*Proof.* Once again, fix an open set  $U \subset \mathbb{R}^2$  and a point  $x \in d^{-1}(U)$ . WTS:  $\exists \delta > 0$  such that  $(x - \delta, x + \delta) \subset d^{-1}(U)$ . Since  $(x, x) \in U$  and U is open,  $\exists \epsilon > 0$  such that  $B((x, x), \epsilon) \subset U$ . Consider the part of the line y = x inside this disc, and project it onto the X-axis. Note that if  $\delta = \epsilon/\sqrt{2}$ , then for any  $y \in (x - \delta, x + \delta)$ , we have

$$\sqrt{(x-y)^2 + (x-y)^2} < \epsilon \Rightarrow (y,y) \in B((x,x),\epsilon)$$
$$(x-\delta, x+\delta) \subset d^{-1}(U)$$

(End of Day 3)

1.5. Theorem: Let  $(X, \tau_X)$  be a topological space and  $Y \subset X$ . Define

$$\tau_Y := \{ U \cap Y : U \in \tau_X \}$$

Then  $\tau_Y$  is a topology on Y, and is called the subspace topology on Y. [HW]

1.6. Examples:

Hence,

(i)  $\mathbb{Z} \subset \mathbb{R}$ . We claim that every subset of  $\mathbb{Z}$  is open in the subspace topology (ie.  $\mathbb{Z}$  with the subspace topology is discrete). It suffices to show that every singleton is open. To do this, fix  $n \in \mathbb{N}$ , then (n - 1/2, n + 1/2) is open in  $\mathbb{R}$ and

$$(n-1/2, n+1/2) \cap \mathbb{Z} = \{n\}$$

- (ii)  $\mathbb{Q} \subset \mathbb{R}$ . Here the subspace topology is not discrete because if U is an open set in  $\mathbb{R}$ , then  $U \cap \mathbb{Q}$  contains infinitely many points. In particular, singelton sets are not open in  $\mathbb{Q}$ .
- (iii)  $S^1 \subset \mathbb{R}^2$ : An example of an open set is the intersection of any disc in  $\mathbb{R}^2$ with  $S^1$ . This will give arcs in  $S^1$ . Hence, every arc in  $S^1$  is an open set. Furthermore, since every open set in  $\mathbb{R}^2$  is a union of discs, every open set in  $S^1$  is a union of arcs.
- (iv)  $[0,1] \subset \mathbb{R}$ : Here, [0,1] is itself an open set since

$$[0,1] = \mathbb{R} \cap [0,1]$$

Furthermore, [0, 1/2) is also an open set in [0, 1].

(v) If  $Y = [0, 1] \cup [2, 3] \subset \mathbb{R}$ , then [0, 1] is an open set in Y because

$$[0,1] = (1/2,3/2) \cap Y$$

Similarly, [2,3] is also an open set. Hence, [0,1] is both open and closed in Y.

## 2. Metric Spaces

2.1. Definition: Let X be a set. A function  $d: X \times X \to \mathbb{R}$  is called a metric on X if

- (i)  $d(x,y) \ge 0$  for all  $(x,y) \in X \times X$
- (ii) d(x, y) = 0 if and only if x = y
- (iii) d(x,y) = d(y,x)

(iv) 
$$d(x,y) \le d(x,z) + d(z,y)$$
 for all  $x, y, z \in X$  (Triangle Inequality)

The pair (X, d) is called a metric space.

### 2.2. Examples:

- (i)  $\mathbb{R}$  with d(x, y) = |x y|
- (ii) Similarly,  $\mathbb{C}$  with d(z, w) = |z w|
- (iii)  $\mathbb{R}^n$  with

$$d(\overline{x}, \overline{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

*Proof.* Clearly, the first three axoims are satisfied, so it suffices to prove the triangle inequality. For this, note that

$$d(x,y)^{2} = \sum_{i=1}^{n} (x_{i} - y_{i})^{2}$$
  
= 
$$\sum_{i=1}^{n} (x_{i} - z_{i} + z_{i} - y_{i})^{2}$$
  
= 
$$\sum_{i=1}^{n} (x_{i} - z_{i})^{2} + (z_{i} - y_{i})^{2} + 2(x_{i} - z_{i})(z_{i} - y_{i})$$

But by Cauchy-Schwartz inequality,

$$\sum_{i=1}^{n} (x_i - z_i)(z_i - y_i) \le \sqrt{\sum_{i=1}^{n} (x_i - z_i)^2} \sqrt{\sum_{i=1}^{n} (z_i - y_i)^2} = d(x, z)d(z, y)$$

Hence,

$$d(x,y)^{2} \leq d(x,z)^{2} + d(y,z)^{2} + 2d(x,z)d(z,y) = [d(x,z) + d(y,z)]^{2}$$

which gives the triangle inequality.

(iv)  $\mathbb{R}^n$  with

$$d(\overline{x}, \overline{y}) = \max_{1 \le i \le n} |x_i - y_i|$$

This is called the uniform or supremum metric on  $\mathbb{R}^n$ , and the metric is written as  $d_{\infty}$ .

(v)  $\mathbb{R}^n$  with

$$d(\overline{x},\overline{y}) = \sum_{i=1}^{n} |x_i - y_i|$$

This is called the  $L^1$  metric on  $\mathbb{R}^n$ , and is written as  $d_1$ .

(vi) Let X be any set. Define  $d: X \times X \to \mathbb{R}$  by

$$d(x,y) = \begin{cases} 0 & : x = y \\ 1 & : x \neq y \end{cases}$$

This is called the discrete metric on X.

2.3. Definition: Let (X, d) be a metric space.

(i) An open ball of radius r > 0 centered at a point  $a \in X$  is the set

$$B(a, r) := \{ x \in X : d(x, a) < r \}$$

- (ii) A set  $U \subset \mathbb{R}$  is said to be open if it is a union of open balls. Equivalently, if, for each  $a \in U, \exists \delta_a > 0$  such that  $B(a, \delta_a) \subset U$
- 2.4. Theorem: Let (X, d) be a metric space, and  $\tau_d$  be the collection of open sets as defined above. Then  $\tau_d$  is a topology on X. This is called the metric topology on X induced by d.

*Proof.* (i) Clearly,  $\emptyset \in \tau_d$  and  $X \in \tau_d$ 

- (ii)  $\tau_d$  is closed under arbitrary union by definition.
- (iii) If  $U_1, U_2 \in \tau_d$ , WTS:  $U_1 \cap U_2 \in \tau_d$ , so fix  $a \in U_1 \cap U_2$ . Then  $\exists \delta_i > 0$ such that  $B(a, \delta_i) \subset U_i$ . Let  $\delta = \min\{\delta_1, \delta_2\}$ , then if  $x \in B(a, \delta)$ , then  $d(x, a) < \delta \le \delta_1 \Rightarrow x \in B(a, \delta_1) \subset U_1$ . Similarly,  $x \in U_2$ , so  $B(a, \delta) \subset U_1 \cap U_2$ .
- 2.5. Definition: Let (X, d) be a metric space. We say that a sequence  $(x_n) \subset X$  converges to a point  $a \in X$  if, for each  $\epsilon > 0, \exists N \in \mathbb{N}$  such that  $d(x_n, a) < \epsilon$  for all  $n \geq N$ . If this happens, we write  $x_n \to a$ .

#### (End of Day 4)

- 2.6. Theorem: Let  $(X, d_X)$  and  $(Y, d_Y)$  be two metric space,  $f : X \to Y$  a function. Then TFAE:
  - (i) For any  $a \in X$  and any sequence  $(x_n) \subset X$  such that  $x_n \to a$  implies  $f(x_n) \to f(a)$
  - (ii) For any  $a \in X$  and each  $\epsilon > 0, \exists \delta > 0$  such that

$$d_X(x,a) < \delta \Rightarrow d_Y(f(x), f(a)) < \epsilon$$

(iii)  $f^{-1}(U)$  is open in X whenever U is open in Y (with respect to the metric topologies on each).

Proof.

(i)  $\Rightarrow$  (ii): Suppose (i) holds and  $a \in X$  is fixed and  $\epsilon > 0$  given. Suppose no  $\delta > 0$  works, then for each  $n \in \mathbb{N}$ ,  $\delta = 1/n$  does not work. So  $\exists x_n \in X$  such that

$$d_X(x_n, a) < 1/n$$
, but  $d_Y(f(x_n), f(a)) \ge \epsilon$ 

So  $x_n \to a$  and  $f(x_n)$  does not converge to f(a) contradicting (i).

(ii)  $\Rightarrow$  (iii): Suppose U is open in X. WTS:  $f^{-1}(U)$  is open in Y, so choose  $a \in f^{-1}(U)$ . Then  $f(a) \in U$  and U is open, so  $\exists \epsilon > 0$  such that

$$B_Y(f(a),\epsilon) \subset U$$

Now by (ii), choose  $\delta > 0$  such that

$$d_X(x,a) < \delta \Rightarrow d_Y(f(x), f(a)) < \epsilon$$

Then clearly  $B_X(a, \delta) \subset f^{-1}(U)$ , so that  $f^{-1}(U)$  is open.

(iii)  $\Rightarrow$  (i) Suppose  $a \in X$  and  $x_n \to a$ . WTS:  $f(x_n) \to f(a)$ . So fix  $\epsilon > 0$ , then  $U = B_Y(f(a), \epsilon)$  is open so  $f^{-1}(U)$  is an open set containing a. Hence,  $\exists \delta > 0$  such that  $B_X(a, \delta) \subset f^{-1}(U)$ . Since  $x_n \to a, \exists N \in \mathbb{N}$  such that

$$d_X(x_n, a) < \delta \quad \forall n \ge N$$

Hence,  $x_n \in f^{-1}(U)$  so that  $f(x_n) \in U$ , whence

$$d_Y(f(x_n), f(a)) < \epsilon \quad \forall n \ge N$$

Hence,  $f(x_n) \to f(a)$ .

2.7. Example:

(i) Let  $M : \mathbb{R}^2 \to \mathbb{R}$  be the multiplication map  $(x, y) \mapsto xy$ . Then M is continuous.

*Proof.* Choose a sequence  $(x_n, y_n) \to (a, b)$ . Then

$$|x_n - a| \le \sqrt{|x_n - a|^2 + |y_n - b|^2} = d((x_n, y_n), (a, b)) \to 0$$

So  $x_n \to a$  in  $\mathbb{R}$ . Similarly,  $y_n \to b$  in  $\mathbb{R}$ . Hence,

$$|x_n y_n - ab| \le |x_n y_n - ay_n| + |ay_n - ab| = |x_n - a||y_n| + |a||y_n - b|$$

Since  $y_n \to b$ ,  $(y_n)$  is bounded, so  $\exists M > 0$  such that  $|y_n| \leq M$  for all  $n \in \mathbb{N}$ . Hence,

 $|x_n y_n - ab| \le M |x_n - a| + |a| |y_n - b| \to 0$ 

Hence, M is sequentially continuous, so it is continuous by the previous theorem.

(ii) Let  $P : \mathbb{R}^n \to \mathbb{R}$  be a polynomial function

$$P(x_1, x_2, \dots, x_n) = \sum a_{i_1, i_2, \dots, i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$$

Then P is continuous.

*Proof.* Similar to HW 1.4.

- 2.8. Theorem: Let  $(X, d_X)$  be a metric space and  $Y \subset X$ . Define  $d_Y : Y \times Y \to \mathbb{R}$  by  $d_Y(y_1, y_2) = d_X(y_1, y_2)$ . Then
  - (i)  $d_Y$  is a metric on Y, and
  - (ii) the metric topology induced on Y by  $d_Y$  coincides with the subspace topology induced on Y from  $(X, \tau_{d_X})$

*Proof.* Part (i) is trivial. To check part (ii), let  $\eta$  denote the subspace topology on Y and  $\tau$  denote the metric topology on Y induced by  $d_Y$ .

(i) To show  $\eta \subset \tau$ : So fix an open set  $V \in \eta$ , then  $\exists U$  open in  $(X, d_X)$  such that  $V = U \cap Y$ . To show that  $V \in \tau$ , we fix a point  $a \in V$ . WTS:  $\exists \delta > 0$  such that  $B_Y(a, \delta) \subset V$ . Since U is open,  $\exists \delta > 0$  such that

$$B_X(a,\delta) \subset U$$

Then note that  $B_Y(a, \delta) = B_X(a, \delta) \cap Y \subset U \cap Y = V$ .

(ii) To show  $\tau \subset \eta$ : It suffices to show that every open ball  $B_Y(a, r) \in \eta$ . But once again this follows from the fact that

$$B_Y(a,r) = B_X(a,r) \cap Y$$

- 2.9. Examples: Any subset of  $\mathbb{R}^n$  inherits a metric topology from  $\mathbb{R}^n$ , so is, in particular, a metric space. For instance, this applies to
  - (i) (The circle)  $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$
  - (ii) (The *n*-sphere)  $S^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$
  - (iii) (The cylinder)  $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 0 \le z \le 1\}$
  - (iv) (The Torus)  $T = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 4\sqrt{x^2 + y^2} + 3 = 0\}$
- 2.10. Theorem: Let  $f: X \to Y$  be an injective function and  $d_Y$  is a metric on Y. Define  $d_X: X \times X \to \mathbb{R}$  by

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$$

Then  $d_X$  is a metric on X, called the metric pulled back (or induced) by f. [HW] Note that f is automatically continuous in this situation.

2.11. Lemma: Let  $f: X \to Y$  be a bijective function and  $d_Y$  be a metric on Y. Let  $d_X$  be the metric on X induced by f. Then a function  $g: X \to Z$  (some other topological space) is continuous if and only if  $g \circ f^{-1}: Y \to Z$  is continuous.

*Proof.* Note that in the above situation,  $f^{-1}$  is automatically continuous from  $Y \to X$ . Hence, if g is continuous, so is  $g \circ f^{-1}$ . Conversely, if  $g \circ f^{-1}$  is continuous, then

$$g = g \circ f^{-1} \circ f$$

is also continuous.

2.12. Example:

(i) Let  $M_n(\mathbb{R})$  denote the set of all  $n \times n$  matrices with real entries. There is a map

$$f: M_n(\mathbb{R}) \to \mathbb{R}^n$$

that expands a matrix into a tuple. This map is clearly injective. Thus,  $M_n(\mathbb{R})$  is a metric space with the metric induced by f. i.e. we have

$$d((a_{i,j}), (b_{i,j})) = \sqrt{\sum_{i,j} (a_{i,j} - b_{i,j})^2}$$

- (ii) Consider the determinant map det :  $M_n(\mathbb{R}) \to \mathbb{R}$ . Note that det  $\circ f^{-1} : \mathbb{R}^{n^2} \to \mathbb{R}$  is a polynomial map which is continuous. Hence, by the previous Lemma, det is continuous.
- (iii) Note that  $GL_n(\mathbb{R})$ , the set of invertible  $n \times n$  matrices is the set

$$GL_n(\mathbb{R}) = det^{-1}(\mathbb{R} \setminus \{0\})$$

Hence,  $GL_n(\mathbb{R})$  is an open subset of  $M_n(\mathbb{R})$  and is a metric space in its own right.

### (End of Day 5)

2.13. Definition: Let X be a set and  $d_1, d_2$  be two metrics on X. We say that  $d_1$  and  $d_2$  are equivalent (In symbols,  $d_1 \sim d_2$ ) if  $\exists K, M > 0$  such that

$$Kd_1(x,y) \le d_2(x,y) \le Md_1(x,y) \quad \forall x, y \in X$$

2.14. Example: Let  $X = \mathbb{R}^n$  and  $d_1, d_2$  be the uniform and Euclidean metrics respectively. Then  $d_1 \sim d_2$ 

Proof.

$$d_1(\overline{x}, \overline{y}) = \max\{|x_i - y_i|\} \le d_2(\overline{x}, \overline{y})$$
$$d_2(\overline{x}, \overline{y}) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \le \sqrt{n} d_1(\overline{x}, \overline{y})$$

2.15. Theorem: Let  $d_1$  and  $d_2$  be equivalent metrics on a set X, then  $\tau_{d_1} = \tau_{d_2}$ 

*Proof.* By symmetry, it suffices to show that  $\tau_{d_1} \subset \tau_{d_2}$ . So let K, M > 0 such that

$$Kd_1(x,y) \le d_2(x,y) \le Md_1(x,y) \quad \forall x,y \in X$$

So fix  $U \in \tau_{d_1}$  and  $a \in U$ . Then  $\exists r > 0$  such that  $B_{d_1}(a, r) \subset U$ . Now if  $x \in B_{d_2}(a, rK)$ , then

$$d_1(x,a) \le \frac{d_2(x,a)}{K} < r$$

So  $B_{d_2}(a, rK) \subset B_{d_1}(a, r) \subset U$ . Hence,  $U \in \tau_{d_2}$  as required.

2.16. Example: (The converse of the previous theorem is not true) Let d be the usual metric on  $\mathbb{R}$  and

$$\rho(x, y) := \min\{|x - y|, 1\}$$

Then

(i)  $\tau_{\rho} = \tau_d$ 

*Proof.* Since  $\rho(x, y) \leq d(x, y)$ , it follows as above that

$$B_d(a,r) \subset B_\rho(a,r)$$

Hence,  $\tau_{\rho} \subset \tau_d$  [Check!]. Conversely, if  $U \in \tau_d$  and  $a \in U$ , then  $\exists r > 0$  such that  $B_d(a, r) \subset U$ . We may assume that r < 1, but in that case,

$$B_{\rho}(a,r) = B_d(a,r) \subset U$$

so that  $U \in \tau_{\rho}$  as well. Hence,  $\tau_d \subset \tau_{\rho}$  as required.

(ii)  $\rho$  is not equivalent to d

*Proof.* Note that  $\rho(x, y) \leq 1$  for all  $x, y \in \mathbb{R}$ . If  $\exists M > 0$  such that

$$d(x,y) \le M\rho(x,y)$$

Then this would imply that  $d(x, y) \leq M$  for all  $x, y \in \mathbb{R}$ . This is not true because d(n, 0) = n for all  $n \in \mathbb{N}$ .

### 3. Basis for a topology

3.1. Definition: Let  $(X, \tau)$  be a topological space. A collection  $\mathcal{B} \subset \tau$  of open sets is called a basis for  $\tau$  if every member of  $\tau$  is a union of elements from  $\mathcal{B}$ . Equivalently,  $U \in \tau$  if and only if, for each  $x \in U, \exists B \in \mathcal{B}$  such that  $x \in B$  and  $B \subset U$ .

3.2. Examples:

(i) Let  $X = \mathbb{R}$  with the usual topology and  $\mathcal{B}$  be the collection of open intervals.

- (ii) Similarly, if (X, d) is any metric space with  $\tau$  the metric topology. Then  $\mathcal{B}$  may denote the set of all balls (of various centers and radii).
- 3.3. Proposition: Let  $f : X \to Y$  be a function between two topological spaces, and suppose  $\mathcal{B}$  is a basis for  $\tau_Y$ . Then f is continuous if and only if  $f^{-1}(B) \in \tau_X$  for all  $B \in \mathcal{B}$

*Proof.* One direction is clear, so suppose  $f^{-1}(B) \in \tau_X$  for all  $B \in \mathcal{B}$ . WTS: f is continuous, so fix an open set  $U \in \tau_Y$  and we want to show  $f^{-1}(U) \in \tau_X$ . Fix  $x \in f^{-1}(U)$ , then  $f(x) \in U$ , so  $\exists B_x \in \mathcal{B}$  such that  $x \in B_x$ , and  $B_x \subset U$ . Hence,

$$V_x := f^{-1}(B_x) \in \tau_X$$
 and  $V_x \subset f^{-1}(U)$ 

This is true for any  $x \in f^{-1}(U)$  so

$$f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} V_x$$

Hence,  $f^{-1}(U) \in \tau_X$  as required.

- 3.4. Lemma: Let  $\mathcal{C}$  be a collection of subset of X. Then there is a unique topology  $\tau$  on X such that
  - (i)  $\mathcal{C} \subset \tau$
  - (ii) If  $\eta$  is any other topology on X such that  $\mathcal{C} \subset \eta$ , then  $\tau \subset \eta$ .

ie.  $\tau$  is the smallest topology containing C. This is called the topology generated by C.

*Proof.* Let  $\mathcal{F}$  be the set set of all topologies  $\eta$  on X such that  $\mathcal{C} \subset \eta$ . Then  $\mathcal{F} \neq \emptyset$  because  $\mathcal{P}(X) \in \mathcal{F}$ . Now set

$$\tau = \bigcap_{\eta \in \mathcal{F}} \eta$$

Then check that  $\tau$  is a topology that satisfies the required conditions.

- 3.5. Theorem: Let X be a set and  $\mathcal{B}$  be a collection of subsets of X such that
  - (a) For each  $x \in X, \exists B \in \mathcal{B}$  such that  $x \in B$
  - (b) If  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , then  $\exists B_3 \in \mathcal{B}$  such that  $x \in B_3$  and  $B_3 \subset B_1 \cap B_2$ .

Let  $\tau$  denote the topology generated by  $\mathcal{B}$ . Then  $\mathcal{B}$  is a basis for  $\tau$ .

*Proof.* Let  $\eta$  be the collection of all subsets of X that are unions of members of  $\mathcal{B}$ . Claim:  $\eta$  is a topology on X. The first three axioms hold trivially, and the last one follows from property (b) of  $\mathcal{B}$ .

Now clearly,  $\mathcal{B} \subset \eta$ , so that  $\eta \in \mathcal{F}$  of the previous proof. Hence,  $\tau \subset \eta$ . Furthermore, if  $\mu$  is any topology that contains  $\mathcal{B}$ , then  $\eta \subset \mu$  because  $\mu$  is closed under arbitrary unions. Hence,  $\eta \subset \tau$  as required.

(End of Day 6)

## 4. The Product Topology on $X \times Y$

4.1. Theorem: Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be two topological spaces. Then there is a unique topology on  $X \times Y$  whose basis are sets of the form

 $U \times V$ 

where  $U \in \tau_X$  and  $V \in \tau_Y$ . This is called the product topology on  $X \times Y$ , denoted by  $\tau_{X \times Y}$ 

*Proof.* Let  $\mathcal{B} = \{U \times V : U \in \tau_X, V \in \tau_Y\}$ . We check that  $\mathcal{B}$  satisfies the conditions of Theorem 3.5.

- (i) Clearly,  $X \times Y \in \mathcal{B}$
- (ii) If  $U_1, U_2 \in \tau_X$  and  $V_1, V_2 \in \tau_Y$ , then

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}$$

4.2. Theorem: Suppose  $(X, d_X)$  and  $(Y, d_Y)$  are metric spaces. Define  $d : (X \times Y)^2 \to \mathbb{R}$  by

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}\$$

Then

- (i) d is a metric on  $X \times Y$
- (ii) The metric topology induced by d coincides with the product topology on  $X\times Y$

*Proof.* Part (i) is trivial, so we prove (ii). Let  $\tau_d$  denote the metric topology and  $\tau_{X \times Y}$  denote the product topology.

• WTS:  $\tau_{X \times Y} \subset \tau_d$ : If  $U = B_X(a, \delta_1)$  and  $V = B_Y(b, \delta_2)$  are open balls in X and Y respectively, consider

$$W = U \times V$$

We claim that  $W \in \tau_d$ . To see this, fix  $(x, y) \in W$ , then  $x \in U, y \in V$ , so

$$d_X(x,a) < \delta_1$$
 and  $d_Y(y,b) < \delta_2$ 

Let  $r = \min\{\delta_1 - d_X(x, a), \delta_2 - d_Y(y, b)\} > 0$ . We claim that

 $B_d((x,y),r) \subset W$ 

So choose  $(u, v) \in B_d((x, y), r)$ , then d((u, v), (x, y)) < r, so that

$$d_X(u, x) < r$$
, and  $d_Y(v, y) < r$ 

Hence,

$$d_X(u,a) \le d_X(u,x) + d_X(x,a) < r + d_X(x,a) \le \delta_1 - d_X(x,a) + d_X(x,a) = \delta_1$$

Hence,  $u \in U$ . Similarly,  $v \in V$ , so that  $(u, v) \in W$ , proving the claim. Hence,

 $U \times V \in \tau_d$ 

for any open ball  $U \in \tau_{d_X}$  and  $V \in \tau_{d_Y}$ . But these open balls form a basis for  $\tau_{d_X}$  and  $\tau_{d_Y}$  respectively. Hence, by Lemma 4.2,

 $\tau_{X \times Y} \subset \tau_d$ 

• WTS:  $\tau_d \subset \tau_{X \times Y}$ : Let  $(a, b) \in X \times Y$  and r > 0. It suffices to show that

 $B_d((a,b),r) \subset \tau_{X \times Y}$ 

Note that  $(x, y) \in B_d((a, b), r)$  iff

$$d_X(x, a) < r$$
 and  $d_Y(y, b) < r$ 

Hence,

$$B_d((a,b),r) = B_X(a,r) \times B_Y(b,r) \in \tau_{X \times Y}$$

This is true for any open *d*-ball in  $X \times Y$ , so  $\tau_d \subset \tau_{X \times Y}$ .

4.3. Remark: Let  $X_1, X_2, X_3$  be three topological spaces, then we may define the product topology inductively as the product topology on  $(X_1 \times X_2) \times X_3$  where  $X_1 \times X_2$ has the product topology. Thus, basic open sets in  $X_1 \times X_2 \times X_3$  are of the form

$$U_1 \times U_2 \times U_3$$

where  $U_i$  are open in  $X_i$ . The same can be done for finitely many spaces  $X_1, X_2, \ldots, X_n$ .

4.4. Corollary: The metric topology on  $\mathbb{R}^n$  induced by the Euclidean metric is the same as the product topology.

*Proof.* Theorem 4.4+2.15.

4.5. Definition: Let X, Y be sets. The maps  $\pi_X : X \times Y \to X$  and  $\pi_Y : X \times Y \to Y$  given by

$$\pi_X(x,y) = x$$
 and  $\pi_Y(x,y) = y$ 

are called the projection maps.

- 4.6. Lemma:
  - (i) The maps  $\pi_X$  and  $\pi_Y$  are continuous if  $X \times Y$  is equipped with the product topology.

(ii) If  $\eta$  is a topology on  $X \times Y$  such that  $\pi_X$  and  $\pi_Y$  are both continuous, then  $\tau_{X \times Y} \subset \eta$ .

*Proof.* (i) If  $U \subset X$  is open, then

$$\pi_X^{-1}(U) = U \times Y \in \tau_{X \times Y}$$

and similarly for  $\pi_Y$ .

(ii) If  $\eta$  is a topology such that  $\pi_X$  and  $\pi_Y$  are continuous, then for any U, V open in X, Y respectively,

$$U \times V = \pi_X^{-1}(U) \cap \pi_Y^{-1}(V) \in \eta$$

Hence,  $\tau_{X \times Y} \subset \eta$ .

4.7. Theorem: Let  $f: Z \to X \times Y$  be a function. Then f is continuous if and only if  $\pi_X \circ f$  and  $\pi_Y \circ f$  are continuous.

*Proof.* If f is continuous then  $\pi_X \circ f$  and  $\pi_Y \circ f$  are continuous by the previous lemma and HW1. Conversely, suppose  $f_1 := \pi_X \circ f$  and  $f_2 := \pi_Y \circ f$  are continuous, and WTS: f is continuous. By Prop. 3.3, it suffices to show that  $f^{-1}(W)$  is open when  $W \subset X \times Y$  is a basic open set. So write  $W = U \times V$  where U and V are open in X and Y respectively. Then

$$f^{-1}(W) = \{z \in Z : f(z) \in U \times V\} = f_1^{-1}(U) \cap f_2^{-1}(V)$$

which is open by hypothesis.

## **5.** The Product Topology on $\prod X_{\alpha}$

Fix topological spaces  $(X_{\alpha}, \tau_{\alpha}), \alpha \in J$ , where J is a possibly infinite set.

5.1. Remark: The product topology on  $X \times Y$  has two definitions:

- (i) [Theorem 4.1]: The basis sets are of the form  $U \times V$  where  $U \in \tau_X, V \in \tau_Y$
- (ii) [Lemma 4.6]: It is the smallest topology that maps  $\pi_X$  and  $\pi_Y$  continuous.

### (End of Day 7)

5.2. Theorem: Let  $(X_{\alpha}, \tau_{\alpha})$  be a family of topological spaces, and let  $X = \prod X_{\alpha}$ . Let  $\pi_{\alpha} : X \to X_{\alpha}$  be the projection map. Let  $\mathcal{B}$  be the collection of finite intersections of the form

$$\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_i)$$

for some finite set  $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset J$  and open sets  $U_i \in \tau_{\alpha_i}$ . Then there is a unique topology  $\tau_p$  on X which has  $\mathcal{B}$  as a basis. This is called the product topology on X.

*Proof.* We once again check the conditions of Theorem 3.5:

- (i) If  $x \in X$  then  $x \in \prod X_{\alpha} = \pi_{\alpha_1}^{-1}(X_{\alpha_1})$ (ii) If  $B_1 := \bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(U_i)$  and  $B_2 = \bigcap_{j=1}^m \pi_{\beta_j}^{-1}(V_j)$ , then  $B_1 \cap B_2 \in \mathcal{B}$
- 5.3. Lemma: Let  $\{X_{\alpha}\}$  and X as above, and let  $\tau_p$  denote the product topology.
  - (i) Each  $\pi_{\alpha} : (X, \tau_p) \to (X_{\alpha}, \tau_{\alpha})$  is continuous.
  - (ii) If  $\eta$  is a topology on X such that each  $\pi_{\alpha} : (X, \eta) \to (X_{\alpha}, \tau_{\alpha})$  is continuous, then  $\tau_p \subset \eta$ .

*Proof.* (i) If  $U_{\alpha} \in \tau_{\alpha}$ , then  $\pi_{\alpha}^{-1}(U_{\alpha}) \in \tau_p$  by definition.

- (ii) If  $\eta$  is a topology as above, then for any  $\alpha \in J$ , and  $U_{\alpha} \in \tau_{\alpha}$ ,  $\pi_{\alpha}^{-1}(U_{\alpha}) \in \eta$ . By taking finite intersections, any basic open set in  $\tau_p$  is in  $\eta$ . Hence,  $\tau_p \subset \eta$ .
- 5.4. Theorem: Let  $f : Z \to X$  be a function. Then f is continuous iff  $\pi_{\alpha} \circ f$  is continuous for each  $\alpha \in J$

*Proof.* One direction is clear from HW1 and the previous Lemma. For the other, suppose  $\pi_{\alpha} \circ f$  is continuous for each  $\alpha \in J$  and WTS: f is continuous. Then by 3.3, it suffices to show that

$$f^{-1}(U) \in \tau_Z$$

for any basic open set  $U \subset X$ . Hence, we write  $U = \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_i)$ , whence

$$f^{-1}(U) = \bigcap_{i=1}^{n} (\pi_{\alpha_i} \circ f)^{-1}(U_i) \in \tau_Z$$

5.5. Theorem: Let  $(X_{\alpha}), \tau_{\alpha}$  be a family of topological spaces, and let  $X = \prod X_{\alpha}$ . Let  $\mathcal{B}$  be the collection of sets of the form

## $\prod U_{\alpha}$

where  $U_{\alpha} \in \tau_{\alpha}$  for each  $\alpha \in J$ . Then there is a unique topology  $\tau_B$  on X which has  $\mathcal{B}$  as a basis. This is called the box topology on X.

*Proof.* Identical to Theorem 5.2.

5.6. Remark/Example:

(i) If J is finite, then the product and box topologies on X coincide.

(ii) The basic open sets of  $\tau_B$  are of the form

## $\prod U_{\alpha}$

where  $U_{\alpha} \in \tau_{\alpha}$  are any open sets. However, the basic open sets in  $\tau_p$  are of the form

## $\prod U_{\alpha}$

where  $U_{\alpha} = X_{\alpha}$  for all but finitely many  $\alpha \in J$ 

- (iii) In general,  $\tau_p \subset \tau_B$ .
- (iv) If J is infinite, they may not coincide. Example: In  $\mathbb{R}^{\omega}$ ,

$$U := \prod_{n=1}^{\infty} (-1/n, 1/n)$$

is open in the box topology, but not in the product topology.

*Proof.* Consider  $0 \in U$ . If  $U \in \tau_p$ , then there must be a basic open set B such that  $0 \in B$  and  $B \subset U$ . But if B is a basic open set, then  $\exists n_1, n_2, \ldots, n_k \in \mathbb{N}$  and open sets  $U_i \subset \mathbb{R}$  such that

$$B = \bigcap_{i=1}^{k} \pi_i^{-1}(U_i) = U_{n_1} \times U_{n_2} \times \dots U_{n_k} \times \mathbb{R} \times \mathbb{R} \times \dots$$

Let  $n = \max\{n_i : 1 \le i \le k\} + 1$ , and  $y = (0, 0, 0, \dots, 1, 0, 0, \dots)$ , where 1 occurs in the  $n^{th}$  stage, then  $y \in B$ , but  $y \notin U$ . Hence, B is not a subset of U, so  $U \notin \tau_p$ .

## 6. Closed Sets

- 6.1. Definition: Let  $(X, \tau)$  be a topological space. A subset  $A \subset X$  is said to be closed if  $X \setminus A$  is open.
- 6.2. Examples:
  - (i) [a, b] is closed in  $\mathbb{R}$
  - (ii)  $A = \{(x, y) \in \mathbb{R}^2 : x \ge 0, \text{ and } y \ge 0\}$  is closed in  $\mathbb{R}^2$  because  $\mathbb{R}^2 \setminus A = \mathbb{R} \times (-\infty, 0) \cup (-\infty, 0) \times \mathbb{R}$
  - (iii) If  $\tau$  is the discrete topology, then every subset of X is closed.
  - (iv) If  $\tau$  is the co-finite (or finite complement) topology on  $\mathbb{R}$ , then the only closed sets are finite sets and  $\mathbb{R}$ .
- 6.3. Lemma: Let X be a topological space. Then
  - (i)  $\emptyset$  and X are closed in X

- (ii) If  $\{F_{\alpha}\}$  are closed in X, then so is  $\bigcap F_{\alpha}$
- (iii) If  $F_1, F_2$  are closed in X, then so is  $F_1 \cup F_2$
- 6.4. Theorem: Let  $Y \subset X$ . A set  $A \subset Y$  is closed in Y (wrt the subspace topology) if and only if  $\exists F \subset X$  closed in X such that  $A = F \cap Y$ *Proof.* HW
- 6.5. Corollary: Let  $Y \subset X$ . If  $A \subset Y$  is closed in Y, and Y is closed in X, then A is closed in X.

### (End of Day 8)

- 6.6. Definition: Let  $A \subset X$ 
  - (i) The interior of A, int(A) is the union of all open sets contained in A.
  - (ii) The closure of A,  $\overline{A}$ , is the intersection of all open sets containing A.

#### 6.7. Remark:

- (i)  $int(A) \subset A \subset \overline{A}$
- (ii) A is open iff int(A) = A and A is closed iff  $A = \overline{A}$
- (iii)  $\operatorname{int}(A)$  is the largest open set contained in A. ie. If  $U \subset A$  is open in X, then  $U \subset \operatorname{int}(A)$ .
- (iv) Similarly,  $\overline{A}$  is the smallest closed set containing A. If  $F \subset X$  is closed and  $A \subset F$ , then  $\overline{A} \subset F$ .
- (v) If  $A \subset Y \subset X$ , we write  $cl_X(A)$  and  $cl_Y(A)$  to denote the closures of A with respect to X and Y respectively.
- 6.8. Lemma: Let  $A \subset Y \subset X$ . Then  $cl_Y(A) = cl_X(A) \cap Y$

*Proof.* Note that

$$cl_Y(A) = \bigcap \{F \subset Y : F \text{ closed, and } A \subset F\}$$

By Theorem 6.4,

$$cl_Y(A) = \bigcap \{ G \cap Y : G \subset X \text{ closed in } X, \text{ and } A \subset G \}$$

which is clearly  $cl_X(A) \cap Y$ .

- 6.9. Theorem: Let  $A \subset X$  and  $x \in X$ .
  - (i)  $x \in \overline{A}$  iff, for every open set U containing  $x, U \cap A \neq \emptyset$ .
  - (ii) If the topology on X has a basis  $\mathcal{B}$ , then  $x \in \overline{A}$  iff, for every basic open set  $B \in \mathcal{B}, B \cap A \neq \emptyset$ .

Note: An open set U containing a point x is called a neighbourhood of x.

*Proof.* We only prove (i): If  $x \in \overline{A}$ , let U be an open set containing x. If  $x \in A$ , then  $U \cap A \neq \emptyset$  so there is nothing to prove. If  $x \notin A$ , suppose  $U \cap A = \emptyset$ . Then  $F := X \setminus U$  is closed, and  $A \subset F$ . By Remark 6.7,  $\overline{A} \subset F$ , so that  $\overline{A} \cap U = \emptyset$ , whence  $x \notin \overline{A}$ . This is a contradiction.

Conversely, suppose every open set U containing x has the property that  $U \cap A \neq \emptyset$ . WTS:  $x \in \overline{A}$ . By definition,

$$\overline{A} = \bigcap \{F : F \subset X \text{ closed, and } A \subset F\}$$

So choose  $F \subset X$  closed such that  $A \subset F$ . WTS:  $x \in F$ . Suppose  $x \notin F$ , then  $x \in U := X \setminus F$ , which is open. Hence,  $U \cap A \neq \emptyset$ . However,  $A \subset F$ , so this is impossible. Hence,  $x \in F$  as required.

- 6.10. Corollary: Let (X, d) be a metric space and  $A \subset X$ . Then  $x \in \overline{A}$  if and only if there is a sequence  $(x_n) \subset A$  such that  $x_n \to x$ .
  - *Proof.* (i) Suppose there is a sequence  $(x_n) \subset A$  such that  $x_n \to x$ , then, for any open set U containing  $x, \exists \epsilon > 0$  such that  $B(x, \epsilon) \subset U$ . Then  $\exists N \in \mathbb{N}$  such that  $x_n \in B(x, \epsilon)$  for all  $n \geq N$ . Hence,  $U \cap A \neq \emptyset$ , and so  $x \in \overline{A}$
  - (ii) Conversely, suppose  $x \in \overline{A}$ . Fix  $n \in \mathbb{N}$  and  $U_n := B(x, 1/n)$ . Then  $U_n \cap A \neq \emptyset$  so  $\exists x_n \in A$  such that  $d(x, x_n) < 1/n$ . It follows that  $x_n \to x$ .
- 6.11. Definition: Let  $(X, \tau)$  be a topological space and  $A \subset X$ . A point  $x \in X$  is said to be a limit point of A if, for every open set U containing  $x, U \cap A$  contains a point of A other than x. Equivalently,

$$x \in (A \setminus \{x\})$$

Write A' for the set of limit points of A.

6.12. Examples:

- (i) If  $A \subset \mathbb{R}$  is a finite set, then A has no limit points. Similarly,  $\mathbb{Z} \subset \mathbb{R}$  has no limit points.
- (ii) Let  $\tau$  be the co-finite topology on  $\mathbb{R}$ , and  $A = \mathbb{Z}$ , and let  $x \in \mathbb{R}$  be any point. If U is an open neighbourhood of x, then  $U \cap (\mathbb{Z} \setminus \{x\}) \neq \emptyset$  because U contains all but finitely many points of  $\mathbb{R}$ . Hence, every point of  $\mathbb{R}$  is a limit point of  $\mathbb{Z}$
- (iii) If A = [0, 1], then every point of A is a limit point of A.
- (iv) If  $A = \{1/n : n \in \mathbb{N}\}$ , then 0 is the only limit point of A.

*Proof.* If  $x \in A'$ , then

(a) If x < 0, then U := (x - |x|/2, x + |x|/2) is a neighbourhood of x, and  $U \cap A = \emptyset$ . Hence,  $x \notin A'$ .

- (b) If x > 1, then a similar argument shows that  $x \notin A'$ .
- (c) If  $1 \ge x > 0$ , and  $x \notin A$ , then  $\exists N \in \mathbb{N}$  such that

$$\frac{1}{N+1} < x < \frac{1}{N}$$

So if  $\delta = \min\{1/N - x, x - \frac{1}{N+1}\}$ , then  $U := (x - \delta/2, x + \delta/2)$  is an open neighbourhood of x such that  $U \cap A = \emptyset$ 

(d) If  $1 \ge x > 0$  and  $x \in A$ , then x = 1/N of some  $N \in \mathbb{N}$ . Once again,

$$\frac{1}{N+1} < x < \frac{1}{N-1}$$

so a similar argument shows that  $x \notin A'$ 

- (e) If x = 0, and U is an open set containing 0, then  $\exists \delta > 0$  such that  $(-\delta, +\delta) \subset U$ . Choose  $N \in \mathbb{N}$  such that  $1/N < \delta$ , so that  $1/N \in U$ , so that  $U \cap (A \setminus \{0\}) \neq \emptyset$ . Hence,  $0 \in A'$ .
- 6.13. Theorem:  $\overline{A} = A \cup A'$ 
  - *Proof.* (i)  $\overline{A} \subset A \cup A'$ : Let  $F := A \cup A'$  and  $U := X \setminus F$ . We claim that U is open. To see this, fix  $x \in X \setminus F$ . Then by definition,  $\exists$  a neighbourhood V of x such that  $V \cap (A \setminus \{x\}) = \emptyset$ . Furthermore,  $x \notin A$ , so that  $V \cap A = \emptyset$ . Hence,  $V \subset U$ , so that U is open. Hence, F is closed, and since  $A \subset F$ , it follows that  $\overline{A} \subset F$ .
  - (ii)  $A \cup A' \subset \overline{A}$ : If  $x \in A$ , then  $x \in \overline{A}$ . Also, if  $x \in A'$ , then  $x \in \overline{A}$  by definition. Hence,  $A \cup A' \subset \overline{A}$ .

(End of Day 9)

- 6.14. Corollary: A set A is closed iff it contains all its limit points.
- 6.15. Example: Let  $X = \mathbb{R}^{\omega}$  with the box topology, and

$$A := \{ (x_n) \in X : x_n > 0 \quad \forall n \in \mathbb{N} \}$$

and let 0 = (0, 0, ...). Then

(i)  $0 \in \overline{A}$ : If U is any basic open set containing 0, then

$$U = \prod U_r$$

where  $U_n \subset \mathbb{R}$  is open and contains 0. Hence,  $\exists x_n \in U_n$  such that  $x_n > 0$ , so that  $x := (x_n) \in A \cap U$ . Hence,  $A \cap U \neq \emptyset$ .

- (ii) Let  $x^m = (x_n^m)$  be a sequence in A. Then consider the diagonal  $a_n := x_n^n > 0$ , and the open set  $U_n = (-a_n, a_n) \subset \mathbb{R}$ . Define  $U := \prod U_n$ , so that  $0 \in U$ . However,  $x^m \notin U$  for all  $m \in \mathbb{N}$ . Hence, there is no sequence in A that converges to 0.
- (iii) Hence, the box topology on  $\mathbb{R}^{\omega}$  is not induced by a metric.
- 6.16. Definition: Let  $A \subset X$ 
  - (i) A is said to be dense in X if  $\overline{A} = X$ . Equivalently,  $U \cap A \neq \emptyset$  for any open set  $U \subset X$
  - (ii) X is said to be separable if it has a countable dense subset.

#### 6.17. Examples:

(i)  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , so  $\mathbb{R}$  is separable.

*Proof.* If  $x \in \mathbb{R}, \delta > 0$ , then  $(x - \delta, x + \delta) \cap \mathbb{Q} \neq \emptyset$ . By Theorem 6.9,  $\overline{\mathbb{Q}} = \mathbb{R}$ 

(ii) If X, Y are topological spaces and A, B are dense in X and Y respectively. Then  $A \times B$  is dense in  $X \times Y$ 

*Proof.* If  $U \subset X$  and  $V \subset Y$  are open, then  $U \cap A \neq \emptyset, V \cap B \neq \emptyset$ . Hence,  $(U \times V) \cap (A \times B) \neq \emptyset$  as required.

- (iii) Hence,  $\mathbb{R}^n$  is separable because  $\mathbb{Q}^n$  is dense in it.
- (iv)  $\mathbb{R}^{\omega}$  is separable with respect to the product topology because

$$A = \{ (x_n) \in \mathbb{R}^{\omega} : \exists N \in \mathbb{N} \text{ such that } x_n = 0 \forall n \ge N, x_n \in \mathbb{Q} \}$$

is dense in  $\mathbb{R}^{\omega}$ 

*Proof.* Let

$$A_N = \{ (x_n) : x_n \in \mathbb{Q}, x_n = 0 \quad \forall n \ge N \}$$

Then  $A_N \cong \mathbb{Q}^{N-1}$ , so  $A_N$  is countable. Hence,  $A = \bigcup A_N$  is also countable. Now if U is a basic open set in  $\mathbb{R}^{\omega}$ , then write  $U = \prod U_n$ , where  $U_n = \mathbb{R}$  for all  $n \ge N$ . Then  $U_i \cap \mathbb{Q} \ne \emptyset$  for all  $1 \le i \le N$ , so choose  $x_i \in U_i \cap \mathbb{Q}$ . Then

 $x = (x_1, x_2, \dots, x_N, 0, 0, \dots)$ 

is in  $U \cap A$ . Hence,  $U \cap A \neq \emptyset$ , so  $\overline{A} = \mathbb{R}^{\omega}$ 

(v)  $\mathbb{R}^{\omega}$  with the box topology is not separable.

*Proof.* Suppose  $A = \{y^n\}$  is a countable subset of  $\mathbb{R}^{\omega}$ , we show that A is not dense. For each  $n \in \mathbb{N}$ , write

$$y^n = (y_1^n, y_2^n, \dots, y_m^n, \dots)$$

Now,  $y_n^n \in \mathbb{R}$ , so choose an open set  $U_n \subset \mathbb{R}$  such that  $y_n^n \notin U_n$ . Then  $U := \prod U_n$  is open in  $\mathbb{R}^{\omega}$  and has the property that  $y^n \notin U$  for all  $n \in \mathbb{N}$ . Hence,  $A \cap U = \emptyset$  as required.  $\Box$ 

### 6.18. Theorem: Let $f: X \to Y$ be a function. Then TFAE:

- (i) f is continuous.
- (ii) For every  $A \subset X, f(\overline{A}) \subset \overline{f(A)}$
- (iii)  $f^{-1}(B)$  is closed in X whenever B is closed in Y.
- *Proof.* (i) (i)  $\Rightarrow$  (ii): Suppose f is continuous and  $y \in f(\overline{A})$ , then WTS:  $y \in \overline{f(A)}$ . Write y = f(x) for some  $x \in \overline{A}$ , and choose an open set U such that  $y \in U$ . Then  $f^{-1}(U)$  is an open neighbourhood of x. Hence,  $f^{-1}(U) \cap A \neq \emptyset$ , so choose  $z \in f^{-1}(U) \cap A$ . Then  $f(z) \in U \cap f(A)$ . Hence  $U \cap f(A) \neq \emptyset$  so that  $y \in \overline{f(A)}$ .
- (ii) (ii)  $\Rightarrow$  (iii): Suppose *B* is closed, WTS:  $A := f^{-1}(B)$  is closed. We have  $f(A) = f(f^{-1}(B)) \subset B$  so if  $x \in \overline{A}$ , then

$$f(x) \in f(\overline{A}) \subset \overline{f(A)} \subset \overline{B} = B$$

Hence,  $x \in f^{-1}(B) = A$ . Hence,  $\overline{A} \subset A$  whence  $A = \overline{A}$  is closed.

(iii) (iii)  $\Rightarrow$  (i): Take complements and apply the hypothesis.

(End of Day 10)

### 7. Continuous Functions

- 7.1. Definition: A function  $f: X \to Y$  is called a
  - (i) open map if f(U) is open whenever  $U \subset X$  is open.
  - (ii) homeomorphism if f is bijective, continuous, and  $f^{-1}: Y \to X$  is also continuous. Equivalently, f is bijective, continuous and an open map.

7.2. Examples:

- (i)  $f : \mathbb{R} \to \mathbb{R}$  given by f(x) = 2x + 3 is a homeomorphism because  $g(y) := \frac{1}{2}(y-3)$  is the inverse.
- (ii) Let  $f: (-1, 1) \to \mathbb{R}$  given by  $f(x) = x/(1-x^2)$ . Then f is a homeomorphism with inverse

$$g(y) := \frac{2y}{1 + (1 + 4y^2)^{1/2}}$$

(iii) Let  $Q = [-1,1]^2 \subset \mathbb{R}^2$  and  $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$  be the square and the disc in  $\mathbb{R}^2$ . Define  $f : D \to Q$  by f(0,0) = (0,0) and if  $(x,y) \neq (0,0)$ , then

$$f(x,y) = \frac{\sqrt{x^2 + y^2}}{\max\{|x|, |y|\}}(x,y)$$

and  $g: Q \to D$  by g(0,0) = (0,0) and if  $(x,y) \neq (0,0)$ , then

$$g(x,y) = \frac{\max\{|x|, |y|\}}{\sqrt{x^2 + y^2}}(x,y)$$

- (iv) Let  $f: [0,1) \to S^1$  be  $f(t) = (\cos(t), \sin(t))$ . Then f is bijective and continuous, but not a homeomorphism, because if U = [0, 1/4), then  $p := f(0) \in f(U)$  is not an interior point of f(U).
- 7.3. Theorem (Rules for constructing Continuous functions) Let X, Y, Z be topological spaces.
  - (i) (Constant function): If  $f: X \to Y$  maps X to a single point  $y_0 \in Y$ , then f is continuous.
  - (ii) (Inclusion): If  $Y \subset X$  has the subspace topology, then the inclusion map  $\iota: Y \to X$  is continuous.
  - (iii) (Composites): If  $f: X \to Y$  and  $g: Y \to Z$  are continuous, then  $g \circ f: X \to Z$  is continuous.
  - (iv) (Restricting the domain): If  $f : X \to Y$  is continuous and  $A \subset X$  has the subspace topology, then  $f|_A : A \to Y$  is continuous.
  - (v) (Restricting the range): Suppose  $f : X \to Y$  is continuous, and  $A \subset Y$  has the subspace topology. If  $f(X) \subset A$ , then the function  $g : X \to A$  given by f is continuous.
  - (vi) (Expanding the range): Suppose  $f : X \to Y$  is continuous, and  $Y \subset Z$  has the subspace topology, then  $f : X \to Z$  is continuous.
  - *Proof.* (i) If U is an open set, then  $f^{-1}(U) = X$  if  $y_0 \in U$  and  $f^{-1}(U) = \emptyset$  if  $y_0 \notin Y$ . In either case,  $f^{-1}(U)$  is open.
  - (ii) If  $U \subset X$  is open, then  $\iota^{-1}(U) = U \cap Y$ , which is open in Y by definition.
  - (iii) HW1.
  - (iv)  $f|_A = f \circ \iota$  where  $\iota : A \to X$  is the inclusion map. So apply (iii).
  - (v) If  $U \subset A$  is open, then  $U = V \cap A$  for some open set  $V \subset X$ . Then  $g^{-1}(U) = f^{-1}(V) \cap f^{-1}(A) = f^{-1}(V) \cap X = f^{-1}(V)$ , which is open in X.
  - (vi) If  $U \subset Z$  is open, then  $f^{-1}(U) = f^{-1}(U \cap Y)$ , which is open in X.

### 7.4. Theorem (Pasting Lemma):

(i) Let  $X = \bigcup_{\alpha \in J} U_{\alpha}$  where  $U_{\alpha}$  is open, and let  $f : X \to Y$  such that  $f|_{U_{\alpha}} : U_{\alpha} \to Y$  is continuous for each  $\alpha \in J$ . Then  $f : X \to Y$  is continuous.

(ii) Let  $X = A \cup B$  where A and B are closed. Let  $f : A \to Y$  and  $g : B \to Y$  be continuous functions such that f(x) = g(x) for all  $x \in A \cap B$ . Then  $h: X \to Y$  given by

$$h(x) = \begin{cases} f(x) & : x \in A \\ g(x) & : x \in B \end{cases}$$

is a well-defined continuous function from X to Y.

*Proof.* (i) If  $V \subset Y$  is open, then

$$f^{-1}(V) = \bigcup_{\alpha \in J} f^{-1}(V) \cap U_{\alpha} = \bigcup_{\alpha \in J} f|_{U_{\alpha}}^{-1}(V)$$

(ii) If  $C \subset Y$  is a closed set, then [Check!]

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

which is closed.

- 7.5. Example:
  - (i) Define  $h : \mathbb{R} \to \mathbb{R}$  by

$$h(x) = \begin{cases} 0 & : x \le 0\\ x & : x \ge 0 \end{cases}$$

defines a continuous function.

(ii) Let  $f, g: X \to \mathbb{R}$  be continuous functions. Then

$$h_1(x) := \min\{f(x), g(x)\} \text{ and } h_2(x) := \max\{f(x), g(x)\}$$

are continuous functions [HW]

(iii) (Part (ii) of the Pasting Lemma fails for infinitely many closed sets). Let  $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ , and  $A_0 = \{0\}, A_i = \{1/i\}$  for  $i \in \mathbb{N}$ . Define  $f_i : A_i \to \mathbb{R}$  by

$$f_i = \begin{cases} 0 & : i = 0\\ 1 & : i \neq 0 \end{cases}$$

Then each  $f_i$  is continuous, and  $A_i \cap A_j = \emptyset$  so they agree on the intersections. However, the function  $f: X \to \mathbb{R}$  obtained by pasting them is not continuous.

### (End of Day 11)

7.6. Example: (Stereographic Projection). Consider  $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ , and fix the north pole N = (0, 0, 1). We claim

$$S^2 \setminus \{N\} \cong \mathbb{R}^2$$

Consider the plane passing through the equatorial circle. Fix  $P = (x, y, z) \in S^2$ . Draw a line from N through P, and let it meet the plane at the point Q := (u, v, 0). Now taking ratios, we get

$$\frac{x}{y} = \frac{u}{v}$$
$$\frac{y}{1-z} = v$$
$$x^{2} + y^{2} + z^{2} = 1$$

Solving, we get

$$x = \frac{2u}{1 + u^2 + v^2}, u = \frac{x}{1 - z}$$
$$y = \frac{2v}{1 + u^2 + v^2}, v = \frac{y}{1 - z}$$
$$z = \frac{1 - u^2 - v^2}{1 + u^2 + v^2}$$

This gives a function

$$F: S^2 \setminus \{N\} \to \mathbb{R}^2 \text{ and } G: \mathbb{R}^2 \to S^2 \setminus \{N\}$$

Note that the map

$$(u,v) \mapsto 1 + u^2 + v^2$$

is continuous from  $\mathbb{R}^2 \to \mathbb{R} \setminus \{0\}$  and

 $t \mapsto 1/t$ 

is continuous from  $\mathbb{R} \setminus \{0\}$  to  $\mathbb{R}$ . Hence, by composition (See Example 2.7), both F and G are continuous, and inverses of each other. Hence, they are homeomorphisms.

7.7. Remark: The stereographic projection has the property that it preserves angles (such a map is called a conformal map). This is the same property that the mercator projection also has.

## 8. The Quotient Topology

- 8.1. Remark: Many spaces are constructed from other spaces by gluing, ie. by identifying parts of the space to obtain another space.
  - (i) A cylinder is obtained from a rectangle by identifying one pair of opposite edges.
  - (ii) The torus is obtained from a rectangle in  $\mathbb{R}^2$  by identifying both pairs of opposite edges.

- (iii) Consider X to be the union of two discs in  $\mathbb{R}^2$ . If we identify the boundary of one with the boundary of the other, we obtain the sphere  $S^2$ .
- 8.2. Definition: Let X be a set.
  - (i) An equivalence relation on X is a subset  $R \subset X \times X$  such that, for all  $x, y, z \in X$ ,
    - (a) (Reflexive):  $(x, x) \in R$
    - (b) (Symmetric):  $(x, y) \in R \Rightarrow (y, x) \in R$
    - (c) (Transitive):  $\{(x, y), (y, z)\} \subset R \Rightarrow (x, z) \in R$
    - We write  $x \sim y$  iff  $(x, y) \in R$ .
  - (ii) For  $x \in X$ , write

$$[x] := \{ y \in X : y \sim x \}$$

for the equivalence class of x. Note that  $[x] \cap [y] = \emptyset$  or [x] = [y]. Hence the equivalence classes partition X.

- (iii) Write  $X/ \sim = \{[x] : x \in X\}$  to be the set of equivalence classes of  $(X, \sim)$ , and let  $p : X \to X^*$  be the map  $x \mapsto [x]$ .
- 8.3. Examples:
  - (i) If  $X = \bigsqcup_{\alpha \in J} A_{\alpha}$  is a partition of X. Write  $x \sim y$  iff  $\exists \alpha \in J$  such that  $\{x, y\} \subset A_{\alpha}$ . Then this is an equivalence relation whose equivalence classes are precisely the  $A_{\alpha}$ .
  - (ii) Let  $A \subset X$ . Define  $x \sim y$  iff  $\{x, y\} \subset A$ . Then  $\sim$  is an equivalence relation whose equivalence classes are either A or singleton sets. In this case, we write

$$X/A := X/\sim$$

- (iii) If X = [0, 1], then define  $0 \sim 1$  and  $x \nsim y$  if  $\{x, y\} \neq \{0, 1\}$ . Then  $X / \sim$  can be thought of as gluing the end-points of X.
- (iv) If  $X = \mathbb{R}$ , write  $x \sim y$  iff  $x y \in \mathbb{Z}$ .

(v) If  $X = [0, 1]^2$ , write

$$(x,0) \sim (x,1), \text{ for } 0 \le x \le 1$$
  
 $(0,y) \sim (1,y), \text{ for } 0 \le y \le 1$ 

This gives equivalence classes

$$\begin{split} [(x,y)] &= \{(x,y)\} : 0 < x, y < 1 \\ [(x,0)] &= \{(x,0), (x,1)\} : 0 < x < 1 \\ [(0,y)] &= \{(0,y), (1,y)\} : 0 < y < 1 \\ [(0,0)] &= \{(0,0), (1,0), (0,1), (1,1)\} \end{split}$$

ie. Opposite edges of the square are identified, and the vertices collapse to a single point.

8.4. Lemma: Let X be a topological space, and Y any set. Suppose  $p: X \to Y$  is a function. Define

$$\tau_Y := \{ U \subset Y : p^{-1}(U) \in \tau_X \}$$

Then

- (i)  $\tau_Y$  is a topology on Y,
- (ii)  $p: X \to Y$  is a continuous function.
- (iii) If  $\eta$  is any topology on Y such that  $p: X \to (Y, \eta)$  is continuous, then  $\eta \subset \tau_Y$ . ie.  $\tau_Y$  is the largest topology that makes p continuous.

*Proof.* (i) To see that  $\tau_Y$  is a topology.

- (a)  $\emptyset = p^{-1}(\emptyset)$  and  $X = p^{-1}(Y)$ , so  $\emptyset, Y \in \tau_Y$
- (b) If  $\{U_{\alpha} : \alpha \in J\} \subset \tau_Y$ , then

$$p^{-1}(\bigcup U_{\alpha}) = \bigcup p^{-1}(U_{\alpha}) \in \tau_X$$

so  $\bigcup U_{\alpha} \in \tau_Y$ .

- (c) Similarly,  $\tau_Y$  is closed under finite intersection.
- (ii) Obvious.
- (iii) Suppose  $\eta$  is as above, then for any  $U \in \eta, p^{-1}(U) \in \tau_X$ , so  $U \in \tau_Y$  by definition. Hence,  $\eta \subset \tau_Y$ .
- 8.5. Definition: Let X be a set and ~ an equivalence relation of X. Let  $p: X \to X/ \sim$  be the map  $x \mapsto [x]$ . The quotient topology on  $X/ \sim$  is the topology induced by p as in the above lemma. i.e. A set  $U \subset X/ \sim$  is open iff

$$\bigcup_{[x]\in U} [x]$$

is open in X.

8.6. Examples:

(i) If X = [0, 1] with  $0 \sim 1$ . Then  $U = \{[x] : 0 \le x < 1/4\}$  is not an open set because

$$\bigcup_{[x]\in U} [x] = [0, 1/4) \cup \{1\}$$

whereas  $U = \{ [x] : 0 \le x < 1/4, \text{ or } 3/4 < x \le 1 \}$  is an open set.

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(ii) Similarly, if  $X = [0, 1]^2$  with the relation in Example 8.3, then (draw picture of open set bounded by an edge, and not having a counterpart on the opposite edge)

8.7. (Universal Property of Quotient Spaces): Let X be a set with an equivalence relation  $\sim$ , let  $X/\sim$  be given the quotient topology, and let  $p: X \to X/\sim$  be the natural map. Let Y be a topological space, and  $f: X \to Y$  be a function such that

$$x \sim x' \Rightarrow f(x) = f(x')$$

Then  $\exists$  a unique function  $\overline{f}: X/ \sim \to Y$  such that

$$f = \overline{f} \circ p$$

Furthermore, f is continuous iff  $\overline{f}$  is continuous.

*Proof.* (i) Given  $f: X \to Y$  as above, define  $\overline{f}: X^* \to Y$  by

$$\overline{f}([x]) := f(x)$$

This is well-defined and satisfies  $\overline{f} \circ p = f$ . Furthermore, if  $g: X/ \sim \to Y$  is any other function such that  $g \circ p = f$ . Then  $g \circ p = \overline{f} \circ p$ . But p is surjective, so  $g = \overline{f}$ , so  $\overline{f}$  is unique.

- (ii) Suppose  $\overline{f}$  is continuous, then  $\underline{f} = \overline{f} \circ p$  is continuous by 8.4. Conversely, suppose f is continuous. WTS:  $\overline{f}$  is continuous. So choose an open set  $U \subset Y$ , then WTS:  $\overline{f}^{-1}(U) \subset X/\sim$  is open. By definition, this is equivalent to asking if  $p^{-1}(\overline{f}^{-1}(U) = (\overline{f} \circ p)^{-1}(U)$  is open in X, which is true.

### 8.8. Example:

(i) Let X = [0, 1] with  $0 \sim 1$ , then  $X^* \cong S^1$ 

*Proof.* Define  $f: X \to S^1$  by  $f(x) = e^{2\pi i x}$ , then f is continuous, and f(0) = f(1). Hence, we get a continuous function  $\overline{f}: X/ \sim \to S^1$  as above. We want to construct an inverse  $g: S^1 \to X/ \sim$ . Write

$$A_1 = \{ z \in S^1 : \operatorname{Im}(z) \ge 0 \}, \text{ and } A_2 = \{ z \in S^1 : \operatorname{Im}(z) \le 0 \}$$

Then  $A_1$  and  $A_2$  are closed sets and  $A_1 \cap A_2 = \{\pm 1\}$ . We now use the pasting lemma. Given  $z \in A_1, \exists$  unique  $t \in [0, 1/2]$  such that  $z = e^{2\pi i t}$ . Define  $h_1 : A_1 \to [0, 1]$  by  $h_1(z) = t$ . Similarly, if  $z \in A_2, \exists$  unique  $t' \in [1/2, 1]$  such that  $z = e^{2\pi i t'}$ , so define  $h_2(z) = t'$ . Note that  $h_1$  and  $h_2$  are continuous, but do not agree on  $A_1 \cap A_2$  because

$$h_1(1) = 0$$
, but  $h_2(1) = 1$ 

Now define  $g_i : A_i \to X/ \sim$  by  $g_i = p \circ h_i$ . Then  $g_i$  are continuous (because the  $h_i$  are continuous), and they agree on  $A_1 \cap A_2$ . Hence by pasting lemma, they define a continuous function  $g : S^1 \to X/ \sim$ . Now note that

$$g \circ f([t]) = g(f(t)) = g(e^{2\pi i t}) = [t]$$

and similarly,

$$(\overline{f} \circ g)(z) = z \quad \forall z \in S^1$$

Hence,  $\overline{f}$  is a homeomorphism.

- (ii) If  $X = \mathbb{R}$  and  $x \sim y$  iff  $x y \in \mathbb{Z}$ , then define  $f : \mathbb{R} \to S^1$  by  $f(x) = e^{2\pi i x}$ . As above, we get a homeomorphism  $\mathbb{R}/\sim \cong S^1$ .
- (iii) Similarly, if  $X = [0, 1]^2$  with the equivalence relation in 8.3(v), then  $X/ \sim \cong S^1 \times S^1$ . This is the torus.

(End of Day 13)

(iv) Let  $D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ . Then  $S^1 \subset D^2$ . We claim

$$D^2/S^1 \cong S^2$$

*Proof.* (a) Write  $D^2 = int(D^2) \sqcup S^1$ . Now define  $f_1 : \mathbb{R}^2 \to int(D^2)$  by

$$f_1(x,y) = \frac{1}{\sqrt{x^2 + y^2} + 1}(x,y)$$

Then  $f_1$  is a homeomorphism. Let  $f_2 : \mathbb{R}^2 \to S^2 \setminus \{N\}$  be the inverse of the stereographic projection, so  $\widehat{f} = f_2 \circ f_1 : int(D^2) \to S^2 \setminus \{N\}$  is a homeomorphism.

(b) Define  $f: D^2 \to S^2$  by

$$f(x) = \begin{cases} \widehat{f}(x) & : x \in int(D^2) \\ N & : x \in S^1 \end{cases}$$

We claim that f is continuous. It suffices to check continuity on  $S^1$ , so fix  $x_0 \in S^1$  and an open set  $U \subset S^2$  containing  $N = f(x_0)$ . Then  $\exists \delta > 0$  such that  $B_{\mathbb{R}^3}(N, \delta) \cap S^2 \subset U$ . By definition of the stereographic projection,  $\exists R > 0$  such that

$$\sqrt{x^2 + y^2} > R \Rightarrow f_2(x, y) \in U$$

Hence,  $\exists 0 < r < 1$  such that

$$\sqrt{x^2 + y^2} > r \Rightarrow \widehat{f}(x, y) \in U$$

Hence,  $f^{-1}(U)$  contains the set

$$V = \{(x, y) \in D^2 : x^2 + y^2 > r^2\}$$

which is open in  $D^2$  and contains  $x_0$ 

(c) Thus, f is continuous. Clearly,  $x \sim y$  if and only if f(x) = f(y), so by 8.7, f induces a map

$$\overline{f}: D^2/S^1 \to S^2$$

This map is both continuous and bijective. We will show later this is enough to conclude that  $\overline{f}$  is a homeomorphism.

### 8.9. Definition:

(i) Consider

$$S^{n} = \{(x_{1}, x_{2}, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum x_{i}^{2} = 1\}$$

Define  $\overline{x} \sim \overline{y}$  iff  $\overline{y} = -\overline{x}$  (antipodal points are identified). Then we define

$$\mathbb{R}P^n := S^n / \sim$$

This is called the real projective space.

- (ii) Consider  $X = [0, 1]^2$ , and define  $\sim$  by  $(0, y) \sim (1, 1 y)$ . The quotient space  $X/\sim$  is called the Mobius strip.
- (iii) Let  $X = [0,1]^2$  and define  $\sim$  by  $(0,y) \sim (1,1-y)$  and  $(x,0) \sim (x,1)$ . The quotient space  $X/\sim$  is called the Klein bottle.

# **III.** Properties of Topological Spaces

### 1. The Hausdorff property

1.1. Definition: A topological space X is said to be Hausdorff  $(T_2)$  if, for each  $x, y \in X$ and distinct point, then  $\exists$  open sets U, V such that  $x \in U, y \in V$  and  $U \cap V = \emptyset$ .

1.2. Examples:

(i) Every metric space is Hausdorff.

*Proof.* If  $x, y \in X$  such that  $x \neq y$ , then  $\delta := d(x, y) > 0$ , so let  $U = B(x, \delta/2)$ and  $V = B(y, \delta/2)$ 

(ii) If X is Hausdorff, and  $Y \subset X$ , then Y is Hausdorff.

*Proof.* If  $x, y \in Y$  are distinct, then  $\exists U, V \subset X$  open such that  $x \in U, y \in V$  and  $U \cap V = \emptyset$ . So let  $U' = U \cap Y$  and  $V' = V \cap Y$ .

(iii) If X and Y are Hausdorff, then so is  $X \times Y$ .

*Proof.* If  $(x_1, y_1) \neq (x_2, y_2)$ , then assume WLOG that  $x_1 \neq x_2$ , so  $\exists U, V \subset X$  open such that  $U \cap V = \emptyset$  and  $x_1 \in U, x_2 \in V$ . Now consider  $U' = U \times Y, V' = V \times Y$ . Then  $U' \cap V' = \emptyset$  and  $(x_1, y_1) \in U', (x_2, y_2) \in V'$ .

- (iv) Similarly if each  $X_{\alpha}$  is Hausdorff, then so is  $\prod X_{\alpha}$  in either the product or the box topology.
- (v) If X has the indiscrete topology, then it is not Hausdorff.
- (vi) If  $\mathbb{R}$  has the co-finite topology, then it is not Hausdorff.

*Proof.* Any two open sets must intersect non-trivially.

- 1.3. Definition: A topological space X is said to be  $T_1$  is singleton sets are closed in X. Equivalently, if  $x \neq y$  are distinct points, then  $\exists$  an open set U such that  $x \in U$ and  $y \notin U$ .
- 1.4. Examples:
  - (i) If X is  $T_2$ , then it is  $T_1$

*Proof.* If  $x \in X$ , then WTS:  $X \setminus \{x\}$  is open. But if  $y \in X \setminus \{x\}$ , then by the Hausdorff property,  $\exists V$  open such that  $y \in V$  and  $V \subset X \setminus \{x\}$ . Hence,  $X \setminus \{x\}$  is open as required.

(ii)  $\mathbb{R}$  with the co-finite topology is  $T_1$  but not  $T_2$
*Proof.* If  $x \in \mathbb{R}$ , then by definition,  $\mathbb{R} \setminus \{x\}$  is an open set, so  $\{x\}$  is closed.  $\Box$ 

- (iii) If X has the indiscrete topology and  $|X| \ge 2$ , then X is not  $T_1$
- 1.5. Theorem: Let X be Hausdorff, and  $(x_n) \subset X$ . Then  $(x_n)$  can converge to atmost one point in X.

*Proof.* If  $x_n \to x$ , and  $x \neq y$ , then choose neighbourhoods U, V such that  $x \in U, y \in V$  and  $U \cap V = \emptyset$ . Then  $\exists N \in \mathbb{N}$  such that  $x_n \in U$  for all  $n \geq N$ . Hence, at most finitely many  $x_j$  may lie in V. Hence,  $(x_n)$  does not converge to y.  $\Box$ 

1.6. Example: Recall that if  $\mathbb{R}$  has the co-finite topology, and  $x_n = n$ , then for any open set  $U \subset \mathbb{R}, \exists N \in \mathbb{N}$  such that  $x_n \in U$  for all  $n \geq \mathbb{N}$ . Hence,  $x_n \to a$  for all  $a \in \mathbb{R}$ .

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- 1.7. Remark/Example:
  - (i) Let X be a topological space and  $X^*$  be a quotient space of X. Then a set  $A \subset X^*$  is closed iff

 $\bigcup_{[x]\in A} [x]$ 

is closed in X. Hence,  $X^*$  is  $T_1$  if and only each [x] is closed in X.

(ii) For example, all the spaces constructed in the previous section are  $T_1$ . However, if  $A = \mathbb{Q} \subset \mathbb{R}$ , then  $\mathbb{R}/\mathbb{Q}$  (the topological space) is not  $T_1$  because  $\mathbb{Q}$ is not closed in  $\mathbb{R}$ . Hence, it is not true that if X is Hausdorff, then  $X^*$  is Hausdorff.

# 2. Connectedness

- 2.1. Definition: Let X be a topological space.
  - (i) A separation of X is a pair  $\{U, V\}$  of non-empty open sets such that  $X = U \cup V$ and  $U \cap V = \emptyset$ .
  - (ii) A space X is said to be connected if it does not have a separation.
  - (iii) A set  $A \subset X$  is called cl-open if it is both closed an open.
- 2.2. Lemma: X is connected iff the only sets in X that are both open and closed are  $\emptyset$  and X (ie. X has no non-trivial cl-open sets)

*Proof.* If X has a non-trivial cl-open set U, then  $V := X \setminus U$  is cl-open, and  $\{U, V\}$  is a separation of X. Conversely, if X is not connected, then it has a separation  $\{U, V\}$  of disjoint non-empty sets. Then U is a non-trivial cl-open set.  $\Box$ 

#### 2.3. Example:

- (i) If X has the indiscrete topology, then X is connected.
- (ii) If X has the discrete topology and  $|X| \ge 2$ , then X is disconnected.

(iii)  $\mathbb{R}$  is connected.

- (iv)  $\mathbb{Q} \subset \mathbb{R}$  is not connected.
- 2.4. Lemma: If  $A \subset X$  is connected, and  $A \subset B \subset \overline{A}$ , then B is connected. In particular,  $\overline{A}$  is connected.

*Proof.* If  $A \subset B \subset \overline{A}$  has a separation  $\{U, V\}$ , then  $U' := U \cap A, V' := V \cap A$  are disjoint open subsets of A. Furthermore,  $U' \neq \emptyset$  because  $U \subset \overline{A}$  is open (by II.6.9). Similarly,  $V' \neq \emptyset$ , so  $\{U', V'\}$  is a separation of A. This is a contradiction.  $\Box$ 

2.5. Theorem: Any interval in  $\mathbb{R}$  is connected. In particular,  $\mathbb{R}$  is connected.

*Proof.* By the previous lemma, it suffices to consider closed intervals Y = [a, b]. Suppose  $\{U, V\}$  is a separation of Y, then  $U = U' \cap Y, V = V' \cap Y$  for some open sets  $U', V' \subset \mathbb{R}$ . Assume WLOG that  $a \in U$ . Since U is open in  $Y, \exists \delta > 0$  such that  $[a, a + \delta) \subset U$ . Define

$$c := \sup A$$
, where  $A := \{x \in [a, b] : [a, x] \subset U\}$ 

Note that c > a by the above argument. Claim:  $c \in U$ .

*Proof.* For each  $\epsilon > 0, c - \epsilon$  is not an upper bound for the set A, so  $\exists x \in A$  such that

$$c - \epsilon < x < c$$

Now  $[a, x] \subset U$ , so Hence,  $(c - \epsilon, c + \epsilon) \cap U \neq \emptyset$ . Hence,  $c \in cl_{\mathbb{R}}(U)$  by II.6.9. But Y is closed in  $\mathbb{R}$ , so  $c \in cl_Y(U)$  (II.6.8). But U is closed in Y, so  $c \in U$ .  $\Box$ 

Claim: c = b.

*Proof.* Suppose c < b, then since  $c \in U$  and U is open in  $Y, \exists \delta > 0$  such that  $[c, c + \delta) \subset U \cap Y$ . Hence,  $[a, c + \delta/2] \subset U$ , which contradicts the fact that  $c = \sup A$ . Hence, c = b.

Thus,  $[a, b] \subset U$ , so that V is empty.

2.6. Theorem: The only connected subsets of  $\mathbb{R}$  are intervals.

*Proof.* Suppose  $Y \subset \mathbb{R}$  is connected is not an interval. Then  $\exists a < c < b$  such that  $\{a, b\} \subset Y$  and  $c \notin Y$ . Hence,  $U := (-\infty, c) \cap Y$  and  $V := (c, \infty) \cap Y$  form a separation of Y.

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2.7. Theorem: Let X be a topological space and  $\{A_{\alpha} : \alpha \in J\}$  be a collection of connected sets such that

 $\bigcap A_{\alpha} \neq \emptyset$ 

Then  $A := \bigcup A_{\alpha}$  is connected.

*Proof.* Let  $\{U, V\}$  be a separation of A, then for any  $\beta \in J$ ,  $\{U \cap A_{\beta}, V \cap A_{\beta}\}$  are two disjoint cl-open sets in  $A_{\beta}$ . By 2.2, either  $U \cap A_{\beta} = A_{\beta}$  or  $V \cap A_{\beta} = A_{\beta}$ . ie. either  $A_{\beta} \subset U$  or  $A_{\beta} \subset V$ . Let

$$J_1 := \{ \alpha \in J : A_\alpha \subset U \} \text{ and } J_2 = \{ \alpha \in J : A_\alpha \subset V \}$$

Since  $\{U, V\}$  is a separation of A, it follows that  $J_1, J_2$  are both non-empty. However, if  $x \in \cap A_{\alpha}$ , then  $x \in U \cap V$ . This contradicts the fact that  $U \cap V = \emptyset$ .  $\Box$ 

2.8. Theorem: Let X, Y be connected, then  $X \times Y$  is connected.

*Proof.* Fix  $a \in X, b \in Y$ , then  $Y_a := \{a\} \times Y \cong Y$  is connected, and  $X_b := X \times \{b\}$  is connected. Furthermore,  $X_a \cap Y_b = \{(a, b)\} \neq \emptyset$ . Hence,  $X_b \cup Y_a$  is connected by the previous lemma. Now consider  $A_b := X_b \cup Y_a, b \in Y$ . Then  $A_b$  is connected, and

$$X \times Y = \bigcap A_b = Y_a \neq \emptyset$$

So by the previous theorem,  $X \times Y$  is connected.

2.9. Example:

(i) Let  $X = \mathbb{R}^{\omega}$  with the product topology, then X is connected.

Proof. Write

$$X_n = \{ (x_1, x_2, \dots, x_n, 0, 0, \dots) : x_i \in \mathbb{R} \} \subset X$$

Then  $X_n \cong \mathbb{R}^n$ , so  $X_n$  is connected by the previous theorems and induction. Furthermore,  $\bigcap X_n = \{\overline{0}\} \neq \emptyset$ . Hence,

$$A := \bigcup_{n=1}^{\infty} X_n$$

is connected. We claim:  $X = \overline{A}$ . Fix  $x = (x_n) \in X$  and an open set U containing x. Then we may assume that

$$U := \prod_{n=1}^{\infty} U_n$$

where  $U_n = \mathbb{R}$  for all  $n \ge N$ . Then for

$$y := (x_1, x_2, \ldots, x_N, 0, 0, \ldots)$$

we have  $y \in A$  and  $y \in U$ , so  $U \cap A \neq \emptyset$ . Hence,  $\overline{A} = X$ , so X is connected by 2.4

(ii) Let  $X = \mathbb{R}^{\omega}$  with the box topology, then X is disconnected.

*Proof.* Let

$$A := \{ (x_n) \in \mathbb{R}^{\omega} : \exists M \in \mathbb{N} \text{ such that } |x_n| \le M \quad \forall n \in \mathbb{N} \}$$

be the set of all bounded sequences. Then  $A \neq \emptyset$  and  $A \neq X$ . We claim that A is cl-open, which would prove that  $\mathbb{R}^{\omega}$  is disconnected.

• To see that A is open, fix  $x = (x_n) \in A$ , and consider

$$V := \prod_{n=1}^{\infty} (x_n - 1, x_n + 1)$$

Then V is open, and if  $y = (y_n) \in V$ , then

$$|y_n| < |x_n| + 1$$

so  $(y_n) \in A$ .

• To see that A is closed, fix  $x = (x_n) \notin A$ , and

$$V := \prod_{n=1}^{\infty} (x_n - 1, x_n + 1)$$

If  $y = (y_n) \in V$  is bounded, then  $|x_n| \leq |y_n| + 1$  would imply that  $x \in A$ . This is a contradiction, so  $V \subset X \setminus A$ . Hence,  $X \setminus A$  is open, so A is closed.

2.10. Theorem: Let  $f: X \to Y$  be a continuous function. If X is connected, then so is f(X) (i.e. the continuous image of a connected set is connected).

*Proof.* If f(X) has a separation  $\{U, V\}$ , then  $\{f^{-1}(U), f^{-1}(V)\}$  would be open sets, and

$$X = f^{-1}(f(X)) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

and

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$$

so  $\{f^{-1}(U), f^{-1}(V)\}$  would be a separation of X. Since X is connected, this cannot happen.

- 2.11. Corollary: If X is connected, and ~ and equivalence relation on X, then  $X/\sim$  is connected.
- 2.12. (Intermediate Value Theorem): Let  $f : [a, b] \to \mathbb{R}$  be a continuous function and  $d \in \mathbb{R}$  such that f(a) < d < f(b). Then  $\exists c \in [a, b]$  such that f(c) = d.

*Proof.* By the previous theorems, f([a, b]) is a connected subset of  $\mathbb{R}$ , and is hence an interval. In particular,  $f(a), f(b) \in f([a, b])$ , so  $d \in f([a, b])$ . This implies the result.

2.13. Theorem:  $\mathbb{R}^n \cong \mathbb{R}$  iff n = 1

(It is true that  $\mathbb{R}^n \cong \mathbb{R}^m$  implies that n = m, but that is much harder to prove.)

*Proof.* Assume n > 1 and  $f : \mathbb{R}^n \to \mathbb{R}$  is a homeomorphism. We will show that  $\mathbb{R}^n \setminus \{0\}$  is connected, so  $f(\mathbb{R}^n \setminus \{0\}) = f(\mathbb{R}^n) \setminus \{f(0)\}$  must be connected. But

$$f(\mathbb{R}^n \setminus \{0\}) = f(\mathbb{R}^n) \setminus \{f(0)\} = \mathbb{R} \setminus \{c\} = (-\infty, c) \bigsqcup (c, \infty)$$

which is disconnected. This is a contradiction

(End of Day 16)

### 3. Path Connectedness

3.1. Definition: Let X be a topological space.

- (i) A path between two points  $x, y \in X$  is a continuous function  $f : [0, 1] \to X$  such that f(0) = x, f(1) = y.
- (ii) A space X is said to be path connected if any two points in X are connected by a path.
- 3.2. Remark: Every interval [a, b] is homeomorphic to [0, 1] (via the map  $t \mapsto at + (1 t)b$ ), so we may as well write  $f : [a, b] \to X$  is the above definition.
- 3.3. Theorem: A path connected space is connected.

Proof. If  $\{U, V\}$  is a separation for X, then choose  $x \in U, y \in V$ . By hypothesis, there is path  $f : [0, 1] \to X$  such that f(0) = x, f(1) = y. Consider  $U' := f^{-1}(U)$  and  $V' := f^{-1}(V)$ . Then these are non-empty open sets and  $[0, 1] = f^{-1}(X) = f^{-1}(U) \cup f^{-1}(V)$ , so [0, 1] must be disconnected. This contradicts 2.5.

3.4. Theorem: If  $f : X \to Y$  is continuous, and X is path connected, then f(X) is path connected.

*Proof.* Given  $u, v \in f(X)$ , write u = f(x), v = f(y) for some  $x, y \in X$ . Let  $g: [0,1] \to X$  be a path from x to y, then  $f \circ g$  is path from u to v.  $\Box$ 

- 3.5. Corollary: If X is path connected, then any quotient space on X is path connected.
- 3.6. Definition: A set  $X \subset \mathbb{R}^n$  is said to be convex if, for any  $x, y \in X$  and  $0 \le t \le 1$ , the point  $z := tx + (1 t)y \in X$ .
- 3.7. Lemma: Any convex subset of  $\mathbb{R}^n$  is path connected. In particular,  $\mathbb{R}^n$ , and every (closed or open) ball in  $\mathbb{R}^n$  is path connected.

*Proof.* Consider the straight line path  $f: [0,1] \to X$  by f(t) := tx + (1-t)y and check that this is continuous.

3.8. Let X be a topological space and  $\{A_{\alpha} : \alpha \in J\}$  be a collection of path connected sets such that, for any two  $\alpha, \beta \in J, \exists \gamma \in J$  such that

$$A_{\alpha} \cap A_{\gamma} \neq \emptyset$$
 and  $A_{\beta} \cap A_{\gamma} \neq \emptyset$ 

Then  $A := \bigcup A_{\alpha}$  is path connected.

Proof. Fix  $x, y \in A$ , then  $\exists \alpha, \beta \in J$  such that  $x \in A_{\alpha}, y \in A_{\beta}$ . Let  $\gamma \in J$  as in the hypothesis, and  $z_1 \in A_{\alpha} \cap A_{\gamma}, z_2 \in A_{\beta} \cap A_{\gamma}$ . Since  $A_{\alpha}$  is path connected,  $\exists f_1 : [0,1] \to A_{\alpha}$  continuous such that  $f_1(0) = x, f_1(1) = z_1$ . Similarly,  $\exists f_2 : [1,2] \to A_{\gamma}$ such that  $f_2(1) = z_1, f_2(2) = z_2$ , and  $\exists f_3 : [2,3] \to A_{\beta}$  such that  $f_3(2) = z_2$  and  $f_3(3) = y$ . Define  $h : [0,3] \to A$  by

$$h(x) = \begin{cases} f_1(x) & : 0 \le x \le 1\\ f_2(x) & : 1 \le x \le 2\\ f_3(x) & : 2 \le x \le 3 \end{cases}$$

Then h is continuous by pasting lemma and II.7.3, and h(0) = x, h(3) = y. So by 3.2, A is path connected.

#### 3.9. Examples:

(i) If n > 1, then  $\mathbb{R}^n \setminus \{0\}$  is path connected.

*Proof.* For each  $1 \leq i \leq n$ , let

 $A_i := \{\overline{x} \in \mathbb{R}^n : x_i > 0\}, \text{ and } B_i := \{\overline{x} \in \mathbb{R}^n : x_i < 0\}$ 

Then  $A_i$  and  $B_i$  are convex (check!) and satisfy the hypotheses of 3.8. Hence,

$$\mathbb{R}^n \setminus \{0\} = \bigcup A_i \cup B_i$$

is path connected.

(ii)  $S^n \subset \mathbb{R}^{n+1}$  is path connected.

*Proof.* The map  $g : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$  given by  $x \mapsto x/d(x,0)$  is a continuous surjective map. So apply 3.4.

- (iii) The following quotient spaces are all path connected: The Torus, The Mobius strip, the Klein bottle, the real projective space.
- 3.10. Theorem: If each  $X_{\alpha}$  is path connected, then  $\prod X_{\alpha}$  is path connected with the product topology.

Proof. Given  $x = (x_{\alpha}), y = (y_{\alpha}) \in X := \prod X_{\alpha}$ , for each  $\beta \in J$ , there is a path  $f_{\beta} : [0,1] \to X_{\beta}$  such that  $f(0) = x_{\beta}$  and  $f(1) = y_{\beta}$ . Define  $f : [0,1] \to X$  by  $f(t) = (f_{\alpha}(t))$ , then f is continuous because each component of f is continuous. And clearly f(0) = x, f(1) = y, so X is path connected.  $\Box$ 

3.11. Remark: Note that the above result is not true with the box topology:  $\mathbb{R}^{\omega}$  is not connected with the box topology, so cannot be path connected. (See Example 2.9)

(End of Day 17)

3.12. Example (The Topologists' Sine Curve): Define

$$S := \{ (x, \sin(1/x)) : 0 < x \le 1 \} \subset \mathbb{R}^2$$

and let  $X = \overline{S}$ . Then note that

$$X = S \cup \{0\} \times [-1, 1]$$

Then X is connected, but not path connected.

*Proof.* The map  $f: (0,1] \to S$  given by  $x \mapsto (x, \sin(1/x))$  is continuous, and (0,1] is connected. Hence, S is connected (Note: In fact, S is path connected). By Lemma 2.4, X is connected. We claim there is no path from (0,0) to any point of S. Suppose  $f: [0,1] \to X$  is such a path, consider

$$A = \{t \in [0,1] : f(t) \in \{0\} \times [-1,1]\}$$

and let  $a := \sup(A)$ . By hypothesis, a < 1. Consider  $f|_{[a,1]}$ :  $[a,1] \to X$  and write f(t) = (x(t), y(t)). Then x(0) = 0 and x(t) > 0 for all t > a, so that  $y(t) = \sin(1/x(t))$  for all t > a. We claim:  $\exists (t_n) \subset [a,1]$  such that  $t_n \to a$  and  $y(t_n) = (-1)^n$ .

For  $n \in \mathbb{N}$  fixed, choose 0 < u < x(a+1/n) such that  $\sin(1/u) = (-1)^n$ . By the intermediate value theorem,  $\exists a < t_n < a+1/n$  such that  $f(t_n) = (t_n, (-1)^n)$ . This proves the claim.

Hence,  $t_n \to 0$  and  $f(t_n) = (t_n, (-1)^n)$  does not converge. Hence, f is not continuous.

- 3.13. Remark:
  - (i) The above example also shows that even if A is path connected, then  $\overline{A}$  may not be path connected (compare with 2.4)
  - (ii) There are two other examples similar to the topologists' sine curve:
    - (a) The deleted infinite broom: For  $n \in \mathbb{N}$ , let  $L_n$  denote the line segment in  $\mathbb{R}^2$  connecting (0,0) to (1,1/n). Let

$$S := \bigcup_{n=1}^{\infty} L_n, \text{ and } X := S \setminus \{(0,1)\}$$

Then S is called the infinite broom, and X the deleted infinite broom. Once again, X is connected, but not path connected. (b) The deleted comb space: Define

$$D := ([0,1] \times \{0\}) \cup \bigcup_{n=1}^{\infty} (\{1/n\} \times [0,1]) \cup [0,1]$$

and  $X := D \setminus \{(0, 1)\}$ . Then D is called the comb space, and X the deleted comb space. Once again, X is connected, but not path connected.

#### (End of Day 18)

# 4. Local Connectedness

- 4.1. Definition: Let X be a topological space. Write  $x \sim y$  if there is a connected subspace  $A \subset X$  such that  $\{x, y\} \subset A$ .
- 4.2. Lemma: The above relation is an equivalence relation, and the equivalence classes are the maximal connected subsets of X (ie. if C is an equivalence class, and B is a connected set such that  $C \subset B$ , then C = B). These equivalence classes are called the connected components of X.

*Proof.* That this is an equivalence class is easy to see. For any  $x \in X$ ,

$$[x] = \{y \in X : x \sim y\}$$
  
=  $\{y \in X : \exists A_y \text{ connected, such that } \{x, y\} \subset A_y\}$   
=  $\bigcup_{y \in [x]} A_y$ 

Each  $A_y$  is connected, and  $\bigcap A_y \supset \{x\} \neq \emptyset$ , so by 2.7, [x] is connected. Furthermore, if B is a connected set such that  $[x] \subset B$ , and  $y \in B$ , then  $\{x, y\} \subset B$ , so by definition,  $y \in [x]$ . Hence, [x] is maximal as well.

- 4.3. Definition: Let X be a topological space. Write  $x \sim_h y$  if there is a path  $f : [0,1] \to X$  such that f(0) = x, f(1) = y.
- 4.4. Lemma: The above relation is an equivalence relation, and the equivalence classes are the maximal path connected subsets of X. These are called the path components of X.

*Proof.* To show that  $\sim_h$  is an equivalence relation:

- (i)  $x \sim x$ : Consider the constant path
- (ii)  $x \sim y \Rightarrow y \sim x$ : If  $f : [0,1] \rightarrow X$  is such that f(0) = x, f(1) = y, take g(s) := f(1-s), then g is continuous, g(0) = y, g(1) = y.
- (iii) If  $x \sim y, y \sim z$ : To show that  $x \sim z$ , simply use the pasting lemma as in 3.8 to join the two paths.

That the equivalence classes are path connected, and maximal is exactly as in 4.2.

- 4.5. Examples:
  - (i) If X is connected, it has only one component.
  - (ii) If  $X = \mathbb{Q}$ , then the connected components are singletons.

*Proof.* If  $A \subset X$  has at least two points, then  $\exists a, b \in A$  and  $x \in \mathbb{R} \setminus \mathbb{Q}$  such that a < x < b. Hence,  $U := (-\infty, x) \cap A$  and  $V := (x, \infty) \cap A$  forms a separation of A, so A is disconnected. Hence, the only connected sets are singletons.

- 4.6. Definition.
  - (i) A topological space X is said to be locally connected if, for each  $x \in X$  and each open set  $U \ni x, \exists$  an open neighbourhood  $V \subset U$  of x that is connected.
  - (ii) We define locally path connected similarly.

#### (End of Day 19)

- 4.7. Examples:
  - (i) Locally path connected implies locally connected.
  - (ii)  $A = (0, 1) \sqcup (2, 3)$  is locally (path) connected, but not connected.
  - (iii) If  $A = \{0\} \cup \{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$ , then A is not locally connected because, for any  $1 > \delta > 0, B(0, \delta) \cap A$  is a finite set, and hence disconnected.
  - (iv) However, connected does not imply local connectedness: Consider the topologists' sine curve X from 3.12, and  $x = (0, 1) \in X$ . Fix  $\delta < 1$  and consider  $U = B(x, \delta) \cap X$ . Then U is a disjoint union of infinitely many line segments  $U = \sqcup L_n$ . Each such  $L_n$  is a cl-open set in U, so U is disconnected.
  - (v) Similarly, path connectedness does not imply local path connectedness: Define

$$X = \bigcup_{n=1}^{\infty} \left\{ \left(\frac{1}{n}, y\right) \right\} : y \in \mathbb{R} \right\} \cup \left\{ (0, y) : y \in \mathbb{R} \right\} \cup \left\{ (x, 0) : x \in \mathbb{R} \right\}$$

Then X is clearly path connected, but if  $x = (0,1) \in X$ , and  $\delta < 1$ , then  $U = B(x, \delta) \cap X$  is once again a disjoint union of line segments. Hence, U is not path connected either.

- 4.8. Lemma:
  - (i) If X is locally connected, then components are open sets. Hence each component is cl-open.
  - (ii) If X is locally path connected, then each path component is open in X. Hence, each path component is cl-open.

*Proof.* We prove (i), because (ii) is identical: If C is a component of x and  $x \in C$ , then  $\exists$  a connected neighbourhood U of x. It follows that  $U \subset C$ , so C is open. Now if each component is open, and X is a disjoint union of components, then each component must also be closed.

- 4.9. Theorem: Let X be a topological space.
  - (i) Every path component is contained in a connected component of X.
  - (ii) If X is locally path connected, then the components and path components coincide.

*Proof.* (i) is obvious, so we prove (ii): Let P be a path component, and  $x \in P$ , then  $P \subset C_x$ , the connected component of x. Also, P is a cl-open set in X, so P is cl-open in  $C_x$ . Since  $C_x$  is connected, it follows that  $P = C_x$ .

- 4.10. Corollary: If X is connected and locally path connected, then it is path connected.
- 4.11. Examples:
  - (i) If  $X \subset \mathbb{R}^n$  is open, then it is locally path connected.

*Proof.* Let  $x \in X$ , then  $\exists$  a *n*-cell  $V := \prod_{i=1}^{n} (a_i, b_i) \subset X$  such that  $x \in V$ . But each  $(a_i, b_i) \subset \mathbb{R}$  is path connected by 3.7, so V is path connected by 3.10.

(ii) More generally, if X is locally connected, and  $Y \subset X$  is open, then Y is locally connected.

(End of Day 20)

## 5. Compactness

- 5.1. Remark: Consider some nice properties of the interval [0, 1]:
  - (i) If  $f:[0,1] \to \mathbb{R}$  is continuous, then f is bounded.
  - (ii) If  $f : [0,1] \to \mathbb{R}$  is continuous, then it is uniformly continuous. ie. For all  $\epsilon > 0, \exists \delta > 0$  such that  $|x y| < \delta$  implies  $|f(x) f(y)| < \epsilon$ .
  - (iii) Every sequence in [0, 1] has a convergent subsequence.

Note that these properties are also shared by other sets, for instance, finite sets. Compactness is a generalization of finiteness in the context of topological spaces.

- (iv) Example: If  $f: (0,1) \to \mathbb{R}$  is given by f(x) = 1/x, then f is not uniformly continuous, and is not bounded. ie. [0,1] should be compact, but (0,1) should not.
- 5.2. Definition: Let X be a topological space.

- (i) A collection  $\mathcal{U}$  of subsets of X is called an open cover for X if every member of  $\mathcal{U}$  is open, and, for each  $x \in X, \exists U \in \mathcal{U}$  such that  $x \in U$ .
- (ii) Let  $\mathcal{U}$  and  $\mathcal{V}$  be open covers of X. We say  $\mathcal{V}$  is a subcover of  $\mathcal{U}$  if  $\mathcal{V} \subset \mathcal{U}$ .

#### 5.3. Examples:

- (i)  $\{X\}$  is an open cover for X. Similarly, the topology  $\tau$  (or any basis of  $\tau$ ) is an open cover for X.
- (ii) If  $\mathcal{U}$  is an open cover for X, and  $\mathcal{W} \subset \tau$  is any collection of open sets, then  $\mathcal{U} \cup \mathcal{W}$  is an open cover, and  $\mathcal{U}$  is a subcover of  $\mathcal{U} \cup \mathcal{W}$ .
- (iii) If X is a metric space. For each  $x \in X$ , choose  $\delta_x > 0$ . Then  $\mathcal{U} := \{B(x, \delta_x) : x \in X\}$  is an open cover for X.
- (iv) If  $\mathcal{U}$  is an open cover for X, and  $\mathcal{V}$  is an open cover for Y, then  $\mathcal{W} := \{U \times V : U \in \mathcal{U}, V \in \mathcal{V}\}$  is an open cover for  $X \times Y$ .
- (v) If  $\mathcal{U}$  is an open cover for X, and  $X^*$  is any quotient space of X, then  $\mathcal{V} := \{\pi(U) : U \in \mathcal{U}\}$  is an open cover for  $X^*$  (where  $\pi : X \to X^*$  denotes the quotient map).
- 5.4. Definition: A topological space X is said to be compact if, whenever  $\mathcal{U}$  is an open cover for X,  $\exists$  finitely many elements  $\mathcal{V} := \{U_1, U_2, \ldots, U_n\} \subset \mathcal{U}$  such that  $\mathcal{V}$  is an open cover for X. ie. Every open cover of X has a finite subcover.
- 5.5. Examples:
  - (i) Any finite set is compact.

*Proof.* If  $\mathcal{U}$  is an open cover for X, then  $\mathcal{U} \subset \mathcal{P}(X)$ , which is itself finite.  $\Box$ 

(ii) (0,1) is not compact.

*Proof.* Let  $U_n := (1/n, 1)$ , then  $\{U_n\}$  is an open cover without a finite subcover.

5.6. Theorem:  $[0,1] \subset \mathbb{R}$  is compact.

*Proof.* Let  $\mathcal{U}$  be an open cover for [0, 1]. Since  $0 \in [0, 1]$ ,  $\exists U \in \mathcal{U}$  such that  $0 \in U$ . Hence,  $\exists \delta > 0$  such that  $[0, \delta) \subset U$ . Now define

 $A := \{x \in [0,1] : [0,x] \text{ is contained in finitely many elements of } \mathcal{U}\}\$ 

Then, by the above argument,  $\delta/2 \in A$ . So define

$$c := \sup(A)$$

We claim that c = 1. If c < 1, then  $c \in [0, 1]$ , so  $\exists V \in \mathcal{U}$  such that  $c \in V$ . Hence,  $\exists \delta > 0$  such that  $(c - \delta, c + \delta) \subset V$ . Since  $c = \sup(A), c - \delta$  is not an upper bound for A. Hence,  $\exists x \in A$  such that

$$c - \delta < x \le c$$

Now, [a, x] is covered by finitely many members of  $\mathcal{U}$ , say  $\{U_1, U_2, \ldots, U_k\}$ . Also,  $[x, c+\delta/2] \subset (c-\delta, c+\delta) \subset V$ . Hence,  $[a, c+\delta/2]$  is covered by  $\{U_1, U_2, \ldots, U_k, V\}$ . In particular,

$$c + \delta/2 \in A$$

contradicting the fact that  $c = \sup(A)$ . Thus, c = 1, and the proof is complete.  $\Box$ 

5.7. Theorem: A closed subspace of a compact space is compact.

*Proof.* Let  $Y \subset X$  be a closed and X compact. Let  $\mathcal{U}$  be an open cover for Y. Then for each  $V \in \mathcal{U}, \exists V' \subset X$  open such that  $V = V' \cap Y$ . Consider

$$\mathcal{U}' := \{ V' : V \in \mathcal{U} \} \bigcup \{ X \setminus Y \}$$

This is an open cover for X, so has a finite subcover  $\mathcal{V} \subset \mathcal{U}'$ . Consider

$$\{W \cap Y : W \in \mathcal{V}\}$$

then this is a cover of Y that is finite, and a subcover of  $\mathcal{U}$  [Check!]

5.8. (The tube lemma): Let X, Y be topological spaces with Y compact. Let  $x_0 \in X$ , and suppose  $N \subset X \times Y$  is open such that

$$x_0 \times Y \subset N$$

Then  $\exists W \subset X$  open such that  $x_0 \in W$  and

$$W \times Y \subset N$$

Note: A set of the form  $W \times Y$  is called a tube about  $x_0 \times Y$ 

*Proof.* For each  $(x_0, y) \in x_0 \times Y$ , choose a basic open set  $U_y \times V_y$  such that  $(x_0, y) \in U_y \times V_y$  and

$$U_y \times V_y \subset N$$

The collection  $\{U_y \times V_y : y \in Y\}$  forms an open cover for  $x_0 \times Y \cong Y$ . Hence, it has a finite subcover

$$\{U_1 \times V_1, U_2 \times V_2, \dots, U_n \times V_n\}$$

Consider  $W := U_1 \cap U_2 \cap \ldots \cap U_n$ , then if  $x \in W$  and  $y \in Y$ , then  $\exists 1 \leq i \leq n$  such that  $(x_0, y) \in U_i \times V_i \subset N$ . Hence,  $(x, y) \in U_i \times V_i$ , so

$$(x,y) \in N$$

So  $W \times Y \subset N$ 

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5.9. Theorem: The finite product of compact spaces is compact.

*Proof.* By induction, we prove it for two spaces, so let X, Y be compact, and let  $\mathcal{U} = \{U_{\alpha}\}$  be an open cover for  $X \times Y$ . Fix  $x_0 \in X$ , then  $\mathcal{U}$  is an open cover for  $x_0 \times Y$ . Since  $x_0 \times Y \cong Y$  is compact, it has a finite subcover  $\{U_1, U_2, \ldots, U_n\}$ . Let

$$N := U_1 \cup U_2 \cup \ldots \cup U_n$$

then N is an open set containing  $x_0 \times Y$ . Let  $W \subset X$  be an open set such that

 $W\times Y\subset N$ 

as in the previous lemma. Then  $W \times Y$  is covered by finitely many sets of  $\mathcal{U}$ , namely  $\{U_1, U_2, \ldots, U_n\}$ .

Hence, for each  $x \in X$ , there is an open neighbourhood  $W_x$  of x such that  $W_x \times Y$  is covered by finitely many elements of  $\mathcal{U}$ . Now the collection  $\{W_x : x \in X\}$  forms an open cover for X, so has a finite subcover  $\{W_1, W_2, \ldots, W_n\}$ . Now each  $W_i \times Y$  is covered by finitely many elements of  $\mathcal{U}$ , so

$$\bigcup_{i=1}^{n} W_i \times Y$$

is covered by finitely many elements of  $\mathcal{U}$ . But

$$X \times Y \subset \bigcup_{i=1}^{n} W_i \times Y$$

so this completes the proof.

5.10. Definition: A collection C of subsets of X is said to have the finite intersection property if, for each finite subcollection  $\{C_1, C_2, \ldots, C_n\} \subset C$ , the intersection

$$C_1 \cap C_2 \cap \ldots \cap C_n$$

is non-empty.

5.11. Theorem: Let X be a topological space, then X is compact iff, for every collection  $\mathcal{C}$  of closed sets with the finite intersection property,

$$\bigcap_{C \in \mathcal{C}} C \neq \emptyset$$

*Proof.* Define  $\mathcal{U}$  by

$$\mathcal{U} := \{X \setminus C : C \in \mathcal{C}\}$$

Then

(i)  $\mathcal{U}$  is a collection of open sets.

(ii)  $\mathcal{U}$  is an open cover for X if and only if

$$\bigcap_{C \in \mathcal{C}} C = \emptyset$$

(iii) A finite subcollection  $\{U_1, U_2, \ldots, U_n\}$  of  $\mathcal{U}$  covers X if and only if, the corresponding subcollection  $C_i := X \setminus U_i$  has the property that

$$C_1 \cap C_2 \cap \ldots \cap C_n = \emptyset$$

Now suppose X is compact: If  $\mathcal{C}$  has the finite intersection property and

$$\bigcap_{C \in \mathcal{C}} C = \emptyset$$

then  $\mathcal{U}$  is a cover for X. By compactness, it must have a finite subcover. By (iii), this would violate the finite intersection property.

The converse is similar.

5.12. Corollary: Let X be a compact topological space. Let  $\{C_i\}$  be a sequence of non-empty closed subsets of X such that

$$C_1 \supset C_2 \supset \ldots \supset C_i \supset C_{i+1} \supset \ldots$$

(Such a sequence is called a nested sequence of closed sets.) Then

$$\bigcap_{n\in\mathbb{N}}C_n\neq\emptyset$$

## **6.** Compact Subsets of $\mathbb{R}^n$

6.1. Example: Fix real numbers  $a_i < b_i$  for  $1 \le i \le n$ , then

$$X := \prod_{i=1}^{n} [a_i, b_i]$$

is compact in  $\mathbb{R}^n$ . Such a set is called a *n*-cell.

*Proof.* Any set of the form  $[a, b] \subset \mathbb{R}$  is homeomorphic to [0, 1] via the map

$$t \mapsto tb + (1-t)a$$

Hence, [a, b] is compact. Hence, X is compact by Theorem 5.9.

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6.2. Definition: Let X be a metric space and  $Y \subset X$ . Y is said to be bounded if  $\exists M > 0$  such that

$$d(x,y) \le M \quad \forall x,y \in Y$$

By the triangle inequality, this is equivalent to:  $\exists x_0 \in X$  and M' > 0 such that

$$d(x_0, y) \le M' \quad \forall y \in Y$$

6.3. Lemma: Let X be a metric space and  $Y \subset X$  be a compact set, then Y is bounded.

*Proof.* Fix  $x_0 \in Y$ . Then consider

$$\mathcal{U} := \{ B(x_0, r) \cap Y : r > 0 \}$$

If  $y \in Y$ , then  $\exists r > 0$  such that  $d(x_0, y) < r$ , so  $\mathcal{U}$  is an open cover for Y. Hence it has a finite subcover  $\{B(x_0, r_1) \cap Y, \ldots, B(x_0, r_n) \cap Y\}$ . Let

$$M := \max\{r_i : 1 \le i \le n\} > 0$$

Then for any  $y \in Y, \exists 1 \leq i \leq n$  such that  $y \in B(x_0, r_i) \cap Y$ , so  $d(x_0, y) < r_i \leq M$ . Hence, Y is bounded.

Recall: Let X be a set. Two metrics  $d_1$  and  $d_2$  on X are said to be equivalent if  $\exists K, M > 0$  such that

$$Kd_1(x,y) \le d_2(x,y) \le Md_1(x,y) \quad \forall x, y \in X$$

Note: If a set  $Y \subset X$  is bounded with respect to  $d_1$ , then it is bounded with respect to  $d_2$  and vice versa.

6.4. Lemma: Let X be a Hausdorff space and  $Y \subset X$  compact, then Y is closed.

*Proof.* If  $x \notin Y$ , then for each  $y \in Y$ ,  $\exists$  open sets  $U_y$  and  $V_y$  such that  $x \in U_y, y \in V_y$ and  $U_y \cap V_y = \emptyset$ . Now  $\{V_y : y \in Y\}$  is an open cover of Y, which must have a finite subcover  $\{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}$ . Set

$$U := \bigcap_{i=1}^{n} U_{y_i}$$

Then U is open,  $x \in U$ , and  $U \cap V_{y_i} = \emptyset$  for all i. Hence,  $U \cap Y = \emptyset$ , so  $U \subset Y^c$ , whence  $Y^c$  is open.

6.5. (Heine-Borel Theorem): Let  $X \subset \mathbb{R}^n$ , then X is compact if and only if X is both closed and bounded (wrt the Euclidean metric).

*Proof.* If X is compact, X is closed and bounded by the previous two lemmas. If X is closed and bounded, and is non-empty, fix  $x_0 \in X$ , then

$$X - x_0 := \{a - x_0 : a \in X\}$$

is homeomorphic to X and contains 0. To show that X is compact, it suffices to show that  $X - x_0$  is compact, so we may assume WLOG that  $0 \in X$ . Since X is bounded with respect to the Euclidean metric, it is bounded with respect to the sup-metric because they are equivalent (Example II.2.14). Hence,  $\exists M > 0$  such that

$$\max\{|y_i|: 1 \le i \le n\} = d_{\infty}(0, y) \le M \quad \forall y \in X$$

Hence, if  $y \in X$ , then  $|y_i| \leq M$  for all  $1 \leq i \leq n$ . i.e. X is contained in the set

$$Z := \prod_{i=1}^{n} [-M, M]$$

Now Z is compact because it is an n-cell. Since  $X \subset Z$  and X is closed in  $\mathbb{R}^n$ , X is closed in Z (Why?). Hence X is compact by 5.8.

6.6. Example: Let  $X = \mathbb{Z}$  with the discrete metric

$$d(x,y) = \begin{cases} 1 & : x \neq y \\ 0 & : x = y \end{cases}$$

Then X is closed and bounded, but not compact. Hence, the above theorem does not hold for all metric spaces.

(End of Day 22)

# 7. Continuous Functions on Compact Sets

7.1. Theorem: Let  $f: X \to Y$  be a continuous function, and X compact. Then f(X) is compact.

*Proof.* If  $\mathcal{U}$  is an open cover for f(X), then

$$\mathcal{V} := \{ f^{-1}(U) : U \in \mathcal{U} \}$$

is an open cover for X [Check!]. Let  $\{f^{-1}(U_1), f^{-1}(U_2), \ldots, f^{-1}(U_n)\}$  be a finite subcover of  $\mathcal{V}$ , then  $\{U_1, U_2, \ldots, U_n\}$  is a finite subcover of f(X) [Check!].  $\Box$ 

- 7.2. Theorem: If X is compact and  $X^*$  is a quotient space of X, then  $X^*$  is compact. *Proof.* The quotient map  $\pi : X \to X^*$  is surjective and continuous, so the previous theorem applies.
- 7.3. Definition: Let  $f: X \to \mathbb{R}$  be a function.
  - (i) We say that f is bounded below if  $\exists m \in \mathbb{R}$  such that  $f(x) \ge m$  for all  $x \in X$ .
  - (ii) Similarly, we define f to be bounded above.
  - (iii) If f is bounded below, we say that f attains its infimum at a point  $x_0 \in X$  if

$$f(x_0) \le f(x) \quad \forall x \in X$$

(iv) We say that f attains its supremum at  $x_1$  if

$$f(x) \le f(x_1) \quad \forall x \in X$$

The points  $x_0$  and  $x_1$  (if they exist, and they need not be unique) are called extreme points of f.

#### 7.4. Example:

- (i) Let  $f: (0,1) \to \mathbb{R}$  be given by f(x) = 1/x, then f is not bounded above.
- (ii) Let  $f : \mathbb{R} \to \mathbb{R}$  by  $f(x) = e^{-x}$ , then f is bounded below, but it does not attain its infimum 0.
- 7.5. (Extreme Value Theorem): Let X be compact and  $f: X \to \mathbb{R}$  continuous, then  $\exists x_0, x_1 \in X$  such that

$$f(x_0) \le f(x) \le f(x_1) \quad \forall x \in X$$

*Proof.* Since f(X) is compact, by the Heine-Borel theorem, it is closed and bounded. In particular,

$$m := \inf\{f(x) : x \in X\}$$

exists and is finite. m is a limit point of f(X) and f(X) is closed, so  $m \in f(X)$ . Hence,  $\exists x_0 \in X$  such that  $f(x_0) = m$ . The proof for the upper bound is analogous.

7.6. Theorem: Let  $f : X \to Y$  be a continuous, bijective function. If X is compact, and Y is Hausdorff, then f is a homeomorphism.

*Proof.* We want to show that f is an open map. It suffices to show that f is a closed map. If  $F \subset X$  is closed, then F is compact. Hence, f(F) is compact in Y, so f(F) is closed in Y.

(i) This completes the proof from Example II.8.8,

$$D^2/S^1 \cong S^2$$

(ii) In the Mid-Sem Exam (Q. 5), we had

$$A := \{(x, y) : 1 \le \sqrt{x^2 + y^2} \le 2\}$$

and we had constructed a continuous bijective function  $f: S^1 \times [1,2] \to A$ . Note that  $S^1 \times [1,2]$  is compact and A is Hausdorff, so f is a homeomorphism.

7.8. Definition: Let (X, d) be a metric space and  $A \subset X$ . Given  $x \in X$ , define the distance of x from A as

$$d(x, A) := \inf\{d(x, y) : y \in A\}$$

7.9. Lemma: The function  $p: X \to \mathbb{R}$  given by p(x) := d(x, A) is a continuous function. Furthermore, p(x) = 0 if and only if  $x \in \overline{A}$ 

*Proof.* (i) If  $x_1, x_2 \in X, y \in A$ 

$$d(x_1, A) \le d(x_1, y) \le d(x_1, x_2) + d(x_2, y)$$

This is true for all  $y \in A$ , so

$$d(x_1, A) \le d(x_1, x_2) + d(x_2, A)$$

 $\mathbf{SO}$ 

$$d(x_1, A) - d(x_2, A) \le d(x_1, x_2)$$

By symmetry,  $d(x_2, A) - d(x_1, A) \le d(x_1, x_2)$  so

$$|d(x_1, A) - d(x_2, A)| \le d(x_1, x_2)$$

From this continuity follows [Why?]

(ii) Suppose  $x \in \overline{A}$ , then  $\exists y_n \in A$  such that  $d(x, y_n) \to 0$ . Hence, d(x, A) = 0. Conversely, if d(x, A) = 0, then for each  $n \in \mathbb{N}$ , 1/n is not a lower bound for the set

$$\{d(x,y): y \in A\}$$
So  $\exists y_n \in A$  such that  $d(x,y_n) < 1/n$ . Clearly,  $y_n \to x$ , so  $x \in \overline{A}$ 

7.10. Definition: Let (X, d) be a metric space and  $A \subset X$ . The diameter of A is defined as

$$\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\}$$

7.11. (Lebesgue Number Lemma): Let  $\mathcal{U}$  be an open cover of a metric space (X, d). If X is compact,  $\exists \delta > 0$  such that if  $A \subset X$  such that  $\operatorname{diam}(A) < \delta$ , then  $\exists U \in \mathcal{U}$  such that  $A \subset U$ .

Any number  $\delta$  as above is called a Lebesgue number for the cover  $\mathcal{U}$ . Note if  $\delta$  is a Lebesgue number for  $\mathcal{U}$  and  $\delta' < \delta$ , then  $\delta'$  is also a Lebesgue number for  $\mathcal{U}$ 

*Proof.* Let  $\{U_1, U_2, \ldots, U_n\}$  be a finite subcover of  $\mathcal{U}$  and define  $A_i := X \setminus U_i$ . Define  $f : X \to \mathbb{R}$  by

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, A_i)$$

Then f is continuous by the previous lemma, so it must attain its minimum at some point  $x \in X$ . Now,  $\exists U_i$  such that  $x \in U_i$ , so  $x \notin A_i$  so by the previous lemma,  $d(x, A_i) > 0$ , whence f(x) > 0, so if  $\delta := f(x)$ , then

$$f(y) \ge \delta \quad \forall y \in X$$

Now if A is a set of diameter less than  $\delta$ , then fix  $x_0 \in A$ , then

$$A \subset B(x_0, \delta)$$

Now, assume that  $d(x_0, A_m)$  is the maximum of  $\{d(x_0, A_1), d(x_0, A_2), \ldots, d(x_0, A_n)\}$ . Then

$$\delta \le f(x_0) \le d(x_0, C_m)$$

Hence, for each  $y \in C_m$ ,  $d(x_0, y) \ge \delta$ , whence

$$B(x_0,\delta) \subset X \setminus C_m = U_m \Rightarrow A \subset U_m$$

#### (End of Day 23)

7.12. Definition: Let  $f : X \to Y$  be a continuous function between two metric spaces. We say that f is uniformly continuous if, for each  $\epsilon > 0, \exists \delta > 0$  such that

$$d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon$$

- 7.13. Example: Let  $f : (0,1) \to \mathbb{R}$  given by f(x) = 1/x, then f is not uniformly continuous.
- 7.14. Theorem: Let  $f: X \to Y$  be a continuous function between metric spaces. If X is compact, then f is uniformly continuous.

*Proof.* Consider  $\epsilon > 0$  and set

$$\mathcal{V} := \{ B(y, \epsilon/2) : y \in Y \}$$

Then  $\mathcal{V}$  is an open cover for Y, so

$$\mathcal{U} := \{ f^{-1}(B(y, \epsilon/2)) : y \in Y \}$$

is an open cover for X. Let  $\delta > 0$  be a Lebesgue number for  $\mathcal{U}$ . Then if  $x_1, x_2 \in X$  such that  $d_X(x_1, x_2) < \delta$ , then  $A := \{x_1, x_2\}$  has diameter  $< \delta$ , so  $\exists y \in Y$  such that

$$A \subset f^{-1}(B(y, \epsilon/2))$$

Hence,  $\{f(x_1), f(x_2)\} \subset B(y, \epsilon/2)$  so by the triangle inequality,

$$d_Y(f(x_1), f(x_2)) < \epsilon$$

# 8. Compactness in Metric Spaces

8.1. Definition: Let X be a topological space.

- (i) X is said to be sequentially compact if, for any sequence  $(x_n) \subset X$ , there is a subsequence  $(x_{n_k})$  of  $(x_n)$  that converges to a point in X.
- (ii) Recall: If  $A \subset X$ . A point  $x \in X$  is called a limit point of A if, for each open set U containing  $x, U \cap (A \setminus \{x\}) \neq \emptyset$
- (iii) X is said to be limit point compact if every infinite subset of X has a limit point in X.
- 8.2. Lemma: If X is compact, then it is limit point compact.

*Proof.* Let  $A \subset X$  be an infinite set, and suppose A has no limit point. Then, for each  $x \in X$ , there is an open set  $U_x$  containing x such that  $U_x \cap (A \setminus \{x\}) = \emptyset$ . Then,  $\mathcal{U} := \{U_x : x \in X\}$  is an open cover for X which has a finite subcover  $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$ . Then each  $U_{x_1}$  contains atmost one point of A (possibly  $x_i$ ). Hence A is finite.

8.3. Example: Let  $Y = \{1, 2\}$  with the indiscrete topology  $\tau_Y = \{\emptyset, Y\}$ , and let

$$X := \mathbb{N} \times Y$$

with the product topology, where  $\mathbb{N}$  is given the usual discrete topology. Then X is limit point compact but not compact.

*Proof.* If  $A \subset X$  is any non-empty set, and assume that  $(n, 1) \in A$ . If U is an open set containing (n, 2), then U contains a basic open neighbourhood  $W = \{n\} \times Y$ , so

$$(n,1) \in W \cap (A \setminus \{(n,2)\})$$

whence  $U \cap (A \setminus \{(n, 1)\}) \neq \emptyset$ .

However, the open cover  $\{\{n\} \times Y : n \in \mathbb{N}\}$  does not have a finite subcover, so X is not compact.

8.4. Lemma: Let X be Hausdorff,  $A \subset X$  and  $x \in X$  a limit point of A. Then for any open neighbourhood U of  $x, U \cap (A \setminus \{x\})$  is infinite.

*Proof.* Suppose  $U \cap (A \setminus \{x\})$  is finite, then write  $U \cap A = \{a_1, a_2, \ldots, a_n\}$ . For each i, there are open sets  $V_i, W_i$  such that  $x \in V_i$  and  $a_i \in W_i$  such that  $V_i \cap W_i = \emptyset$ . If

$$V := \bigcap_{i=1}^{n} V_i$$

Then V is an open set containing x and  $V \cap (A \setminus \{x\}) = \emptyset$ , so x cannot be a limit point of A.

8.5. Definition: A metric space X is said to be totally bounded if, for each  $\epsilon > 0$ , there are finitely many points  $\{x_1, x_2, \ldots, x_n\} \subset X$  such that

$$\{B(x_i,\epsilon): 1 \le i \le n\}$$

covers X. Such a collection of open set is called an  $\epsilon$ -net of X.

8.6. Lemma: If X is sequentially compact, then it is totally bounded.

*Proof.* Suppose X is not totally bounded, then  $\exists \epsilon > 0$  for which there is no finite epsilon net. In particular, if  $x_1 \in X$ , then  $X \neq B(x_1, \epsilon)$ , so  $\exists x_2 \in X$  such that

$$d(x_1, x_2) \ge \epsilon$$

Now,  $\{B(x_1, \epsilon), B(x_2, \epsilon)\}$  is not an open cover for X, so  $\exists x_3 \in X$  such that

 $d(x_3, x_1)\epsilon$  and  $d(x_3, x_2) \ge \epsilon$ 

Thus proceeding, we obtain a sequence  $(x_n) \subset X$  such that if m > n, then

 $d(x_m, x_n) \ge \epsilon$ 

Such a sequence cannot have a convergent subsequence [Why?] contradicting the fact that X is sequentially compact.  $\Box$ 

8.7. (Lebesgue Number Lemma - II): If X is a sequentially compact metric space and  $\mathcal{U}$  is an open cover for X, then  $\exists \delta > 0$  such that, for any  $y \in X, \exists U \in \mathcal{U}$  such that  $B(y, \epsilon) \subset U$ .

Proof. Suppose  $\mathcal{U}$  does not have a Lebesgue number, then  $\delta = 1/n$  does not work. So  $\exists x_n \in X$  such that  $B(x_n, 1/n)$  is not contained in any single member of  $\mathcal{U}$ . Then  $(x_n)$  has a convergent subsequence  $x_{n_k} \to x$ . Now  $x \in X$ , so  $\exists U \in \mathcal{U}$  such that  $x \in U$ . Choose  $\delta > 0$  such that  $B(x, \delta) \subset U$ , then  $\exists n_k \in \mathbb{N}$  such that

$$d(x_{n_k}, x) < \delta/2$$
 and  $1/n_k < \delta/2$ 

Then by the triangle inequality

$$B(x_{n_k}, 1/n_k) \subset B(x, \delta) \subset U$$

This contradicts the assumption on the  $x_n$ .

(End of Day 24)

8.8. Theorem: If X is a metric space, then TFAE:

(i) X is compact

- (ii) X is limit point compact.
- (iii) X is sequentially compact.

*Proof.* (i)  $\Rightarrow$  (ii): Lemma 8.2.

(ii)  $\Rightarrow$  (iii): If  $(x_n) \subset X$  is a sequence, then let  $A := \{x_n\}$ . If A is finite, then there is a subsequence  $(n_k) \subset \mathbb{N}$  such that  $x_{n_k}$  is constant, and hence convergent. Suppose A is infinite, then it has a limit point x. In particular,

$$B(x,1) \cap (A \setminus \{x\}) \neq \emptyset$$

so choose  $n_1 \in \mathbb{N}$  such that  $x_{n_1} \in B(x, 1)$ . Now,

$$B(x, 1/2) \cap (A \setminus \{x\}) \neq \emptyset$$

By the previous lemma,  $B(x, 1/2) \cap (A \setminus \{x\})$  is infinite. In particular,

$$B(x, 1/2) \cap (A \setminus \{x, x_1, x_2, \dots, x_{n_1}\}) \neq \emptyset$$

So  $\exists n_2 > n_1$  such that

$$x_{n_2} \in B(x, 1/2) \cap (A \setminus \{x\})$$

Thus proceeding, for each  $k \in \mathbb{N}$ , we choose  $n_k > n_{k-1}$  such that

$$x_{n_k} \in B(x, 1/k) \cap (A \setminus \{x\})$$

Now  $d(x, x_{n_k}) < 1/k$ , so  $x_{n_k} \to x$ .

- (iii)  $\Rightarrow$  (i): If X is sequentially compact, choose an open cover  $\mathcal{U}$  of X. By the Lebesgue Number Lemma II,  $\exists \delta > 0$  such that any ball of radius  $\delta$  is contained in a single member of  $\mathcal{U}$ . However, X is totally bounded by Lemma 8.6, so finitely many balls  $\{B(x_1, \delta), B(x_2, \delta), \ldots, B(x_n, \delta)\}$  cover X. Hence, finitely many members of  $\mathcal{U}$  cover X.
- 8.9. (Bolzano-Weierstrass): Every bounded sequence in  $\mathbb{R}^n$  has a convergent subsequence.

*Proof.* If  $(x_m) \subset \mathbb{R}^n$  is bounded, then  $\exists M \geq 0$  such that

$$(x_m) \subset \prod_{i=1}^n [-M, M] =: Z$$

Z is compact, so it is sequentially compact.

8.10. Example: Let

$$X := \{ (x_n) \in \mathbb{R}^{\omega} : (x_n) \text{ is bounded} \}$$

Define a metric on X by

$$d(\overline{x},\overline{y}) := \sup\{|x_n - y_n| : n \in \mathbb{N}\}\$$

This is a well-defined metric on X. Now consider  $e^n$  to be the standard basis vector in X. Then  $d(e^n, 0) = 1$ , so  $\{e^n\}$  is a bounded sequence in X. However,  $e^n$  does not have a convergent subsequence because  $d(e^n, e^m) = 1$  if  $n \neq m$ .

# 9. Local Compactness

- 9.1. Definition: A topological space X is said to be locally compact if, for each  $x \in X$ , there is an open neighbourhood V of x such that  $\overline{V}$  is compact.
- 9.2. Examples:
  - (i) Every compact space is locally compact.
  - (ii)  $\mathbb{R}$  is locally compact because every closed interval  $[a, b] = \overline{(a, b)}$  is compact.
  - (iii)  $\mathbb{Q}$  is not locally compact because if  $V \subset \mathbb{Q}$  is open, then  $\exists a < b$  in  $\mathbb{R}$  such that  $(a,b) \cap \mathbb{Q} \subset V$ . If  $s \in \mathbb{R} \setminus \mathbb{Q}$  is an irrational such that a < s < b, then there is a sequence  $(x_n) \subset V$  that converges to s in  $\mathbb{R}$ , so  $(x_n)$  cannot have a convergent subsequence. Hence,  $\overline{V}$  cannot be compact.
  - (iv)  $\mathbb{R}^{\omega}$  with the product topology is not locally compact, because if V is a nonempty open set, then V contains an open set of the form

$$(a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \ldots$$

If  $\overline{V}$  were compact, then

$$[a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] \times \mathbb{R} \times \mathbb{R} \times \ldots$$

would be compact, but it is not [Check! Use the fact that  $\mathbb{R}$  is not compact].

- 9.3. Theorem: Let X be a topological space, then  $\exists$  a compact space Y such that
  - (i)  $X \subset Y$
  - (ii)  $Y \setminus X$  is a singleton.

*Proof.* Define  $Y := X \sqcup \{\infty\}$  as a new set, and define  $\tau_Y$  as the collection of sets U satisfying one of the two following properties:

- (i)  $U \subset X$  is open in X
- (ii)  $\infty \in U$  and  $Y \setminus U$  is compact in X

We show that  $\tau_Y$  is a topology on Y, and that Y is compact.

- (i)  $\emptyset \in \tau_Y$  because  $\emptyset \in \tau_X$
- (ii)  $Y \in \tau_Y$  because  $Y \setminus Y = \emptyset$  is compact in X
- (iii) If  $\{U_{\alpha}\}$  is a collection of members of  $\tau_Y$ , we set  $U := \bigcup U_{\alpha}$  consider two cases:
  - (a) If  $\infty \notin U$ , then  $U \in \tau_X$  so  $U \in \tau_Y$
  - (b) If  $\infty \in U$ , then choose  $I \subset J$  such that  $\infty \in U_{\beta}$  iff  $\beta \in I$ , so  $U_{\beta} = Y \setminus C_{\beta}$  for all  $\beta \in J$ , where  $C_{\beta} \subset X$  is compact, then

$$\bigcup_{\alpha \in J} U_{\alpha} = \left(\bigcup_{\beta \in I} (Y \setminus C_{\beta})\right) \cup \left(\bigcup_{\gamma \in I^c} U_{\gamma}\right)$$

Now  $\bigcap_{\beta \in I} C_{\beta}$  is compact, so

$$\bigcup_{\beta \in I} Y \setminus C_{\beta}$$

is in  $\tau_Y$ , so  $U \in \tau_Y$ .

- (iv) If  $U_1, U_2 \in \tau_Y$ , we WTS:  $U_1 \cap U_2 \in \tau_Y$ . Consider cases again:
  - (a) If  $\infty \notin U_1 \cup U_2$ , then  $U_1 \cap U_2 \in \tau_X \subset \tau_Y$
  - (b) If  $\infty \in U_1, \infty \notin U_2$ , then  $U_1 = Y \setminus C$  for  $C \subset X$  compact, so

$$U_1 \cap U_2 = (Y \setminus C) \cap U_2 = (X \setminus C) \cap U_2 \in \tau_X \subset \tau_Y$$

- (c) Similarly if  $\infty \in U_2 \setminus U_1$
- (d) If  $\infty \in U_1 \cap U_2$ , then  $U_i = (Y \setminus C_i)$  as above, so

$$U_1 \cap U_2 = Y \setminus (C_1 \cup C_2)$$

but  $C_1 \cup C_2$  is compact in X.

We now show that Y is compact: Suppose  $\mathcal{U}$  is an open cover for Y, then  $\exists U \in \mathcal{U}$  such that  $\infty \in U$ , so  $U = Y \setminus C$  for some compact  $C \subset X$ . There are finitely many elements  $\{U_1, U_2, \ldots, U_n\}$  of  $\mathcal{U}$  that cover C, so

$$\{U_1, U_2, \ldots, U_n\} \cup \{U\}$$

covers Y.

(End of Day 25)

9.4. Lemma: If X is a locally compact and Hausdorff, then the space constructed above is Hausdorff.

*Proof.* If  $x, y \in Y$  with  $x \neq y$ . If  $x, y \in X$ , then we use the fact that X is Hausdorff to produce open sets as required. So assume  $y = \infty$ , then choose a neighbourhood V of x such that  $\overline{V}$  is compact. Then  $U := X \setminus \overline{V}$  is an open neighbourhood of y and  $U \cap V = \emptyset$ . So Y is Hausdorff.  $\Box$ 

- 9.5. Theorem: If X is locally compact and Hausdorff, and suppose  $Y_1$  and  $Y_2$  are two spaces such that
  - (i) Both  $Y_1$  and  $Y_2$  are compact.
  - (ii)  $X \subset Y_1$  and  $X \subset Y_2$
  - (iii)  $Y_1 \setminus X$  is a singleton and  $Y_2 \setminus X$  is a singleton.

Then there is a homeomorphism  $p: Y_1 \to Y_2$  such that  $p|_X = \mathrm{id}_X$ 

*Proof.* Suppose  $Y_1 \setminus X = \{y_1\}$  and  $Y_2 \setminus X = \{y_2\}$ , then define  $p: Y_1 \to Y_2$  by

$$p(z) = \begin{cases} z & : z \in X \\ y_2 & : z = y_1 \end{cases}$$

Then p is clearly a well-defined bijection. Also, if  $U \subset Y_2$  is an open set such that  $U \subset X$ , then  $p^{-1}(U) = U \subset Y_1$  is open. If  $U \subset Y_2$  is open and  $\infty \in Y_2$ , then  $F := Y_2 \setminus U = X \setminus U$  is closed in  $Y_2$ . But  $Y_2$  is compact, so F is compact in  $Y_2$ . Since  $F \subset X$ , F is compact in X. But  $X \subset Y_1$ , so F is compact in  $Y_1$ . But  $Y_1$  is Hausdorff, so F is closed in  $Y_1$ . Hence,  $Y_1 \setminus F = p^{-1}(U)$  is open in  $Y_1$ . Hence, p is continuous. But  $p: Y_1 \to Y_2$  is a continuous bijection from a compact space to a Hausdorff space, so it is a homeomorphism.

- 9.6. Definition: Given a locally compact Hausdorff space, we have shown that  $\exists$  a compact space Y such that  $X \subset Y$  and  $Y \setminus X$  is a singleton. Furthermore, Y is unique in the sense of 9.5. This space Y is called the one-point compactification of X, and is denoted by  $X^+$ .
- 9.7. Example: If  $X = \mathbb{R}^n$ , then  $X^+ \cong S^n$

Proof. The stereographic projection gives a continuous injective map  $p: X \to S^n$ , and is a homeomorphism onto its range  $p(X) = S^n \setminus \{N\}$ . Identifying X with p(X), we see that  $S^n$  satisfies the conditions of Theorem 9.3. By Theorem 9.5,  $S^n \cong X^+$ .

Note: For  $n = 2, S^2$ , thought of as  $(\mathbb{R}^2)^+$  is referred to as the Riemann sphere.

# **IV. Separation Axioms**

# 1. Regular Spaces

Assume that all spaces are  $T_1$ : Singleton sets are closed.

- 1.1. Definition: A topological space X is said to be regular if, for any closed set  $A \subset X$  and any  $x \notin A$ , there are open sets  $U, V \subset X$  such that  $A \subset U, x \in V$  and  $U \cap V = \emptyset$ .
- 1.2. Example:
  - (i) Every regular space is Hausdorff.
  - (ii) Let  $K = \{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$  and define a topology on  $\mathbb{R}$  as follows: Define

$$\mathcal{B}_1 := \{ \text{ open intervals in } \mathbb{R} \}$$
$$\mathcal{B}_2 := \{ (a, b) \setminus K : a < b \text{ in } \mathbb{R} \}$$

Then  $\mathcal{B} := \mathcal{B}_1 \cup \mathcal{B}_2$  forms a basis for a topology on  $\mathbb{R}$  (HW 4), which we denote by  $\tau_K$ . Then  $\mathbb{R}_K := (\mathbb{R}, \tau_K)$  is Hausdorff but not regular.

*Proof.*  $\mathbb{R}_K$  is Hausdorff because distinct points can be separated by open intervals. To see that  $\mathbb{R}_K$  is not regular, note that K is closed in  $\mathbb{R}_K$  and  $0 \notin K$ . However, if U is an open set containing 0, then U must contain a basic open set around 0. It cannot contain sets of the form (-r, r) because they intersect K. So suppose  $(-r, r) \setminus K \subset U$ . Let  $n \in \mathbb{N}$  such that 1/n < r. Let V be an open set containing K and choose a basic open set (a, b) around 1/n contained in V. Then

$$1/n \in (a, b)$$
 and  $1/n < r \Rightarrow ((a, b) \setminus K) \cap (-r, r) \neq \emptyset$ 

Hence,  $U \cap V \neq \emptyset$ , so K and 0 cannot be separated.

(End of Day 26)

1.3. Theorem: Every compact Hausdorff space is regular.

*Proof.* If X is compact and  $A \subset X$  closed,  $x \notin A$ , then A is compact. For each  $y \in A$ , there are open sets  $U_y, V_y$  such that  $x \in U_y, y \in V_y$  and  $U_y \cap V_y = \emptyset$ . Now  $\{V_y \cap A : y \in A\}$  forms an open cover for A. Choose a finite subcover  $\{V_{y_i} \cap A : 1 \leq i \leq n\}$  and consider

$$U := \bigcap_{i=1}^{n} U_{y_i}$$
 and  $V := \bigcup_{i=1}^{n} V_{y_i}$ 

Then U and V are open,  $A \subset V, x \in U$  and  $U \cap V = \emptyset$ .

1.4. Theorem: X is regular iff, for each  $x \in X$  and an open neighbourhood U of x, there is an open neighbourhood V of x such that  $\overline{V} \subset U$ .

*Proof.* Suppose X is regular, and  $x \in X, U$  an open neighbourhood of X. Then,  $X \setminus U$  is closed and does not contain x, so there are open sets V, W such that  $x \in V, X \setminus U \subset W$  and  $V \cap W = \emptyset$ . We claim that  $\overline{V} \subset U$ . If  $y \notin U$ , then  $y \in W$  and  $W \cap V = \emptyset$ , so  $y \notin \overline{V}$ . Hence,  $\overline{V} \subset U$ .

Conversely, suppose the given condition holds and  $x \in X, A \subset X$  closed and  $x \notin A$ . Then  $U := X \setminus A$  is an open set containing x, so there is an open set V such that  $\overline{V} \subset U$ . Then  $W := X \setminus \overline{V}$  is open, contains A and  $V \cap W = \emptyset$ .

1.5. Corollary: Every subspace of a regular space is regular.

*Proof.* If  $Y \subset X$ , where X is regular, suppose U is an open neighbourhood of x in Y, then  $U = U' \cap Y$  for some open set  $U' \subset X$ . Choose  $V' \subset X$  open such that  $\overline{V'} \subset U'$ . Now take  $V := V' \cap Y$ , which is open in Y, contains x and by II.6.8,

$$cl_Y(V) = cl_X(V) \cap Y \subset cl_X(V') \cap Y \subset U' \cap Y = U$$

1.6. Corollary: Every locally compact Hausdorff space is regular.

*Proof.* Let X be locally compact and Hausdorff, and  $X \subset X^+$  its one point compactification.  $X^+$  is regular, so X must also be regular.

1.7. Corollary: Any product of regular spaces is regular.

*Proof.* Suppose  $X_{\alpha}$  is regular for all  $\alpha \in J$ , and  $X := \prod_{\alpha \in J} X_{\alpha}$ . Let  $x := (x_{\alpha} \in X$  and  $U \subset X$  an open neighbourhood of x. Then we may assume that U is a basic open set of the form

$$U_{\alpha_1} \times U_{\alpha_2} \times \ldots \times U_{\alpha_n} \times \prod_{\beta} X_{\beta}$$

Now  $x_{\alpha_i} \in U_{\alpha_i}$ , so there are open sets  $V_{\alpha_i}$  such that  $\overline{V_{\alpha_i}} \subset U_{\alpha_i}$ . Then

$$V := V_{\alpha_1} \times V_{\alpha_2} \times \ldots \times V_{\alpha_n} \times \prod_{\beta} X_{\beta}$$

is an open neighbourhood of x such that  $\overline{V} \subset U$  [Why?]

# 2. Normal Spaces

- 2.1. Definition: A topological space X is said to be normal if, whenever A and B are disjoint closed sets, there are open sets U, V such that  $A \subset U, B \subset V$  and  $U \cap V = \emptyset$
- 2.2. Lemma: X is normal iff, given a closed set  $A \subset X$  and an open set U containing A, there is an open set V containing A such that  $\overline{V} \subset U$

Proof. HW.

2.3. Theorem: Every metric space is normal.

*Proof.* If  $A, B \subset X$  are disjoint closed sets. For each  $a \in A$ ,  $a \notin B$ , so  $\exists \epsilon_a > 0$  such that  $B(a, \epsilon_a) \subset X \setminus B$ . Define

$$U := \bigcup_{a \in A} B(a, \epsilon_a/2)$$

Then U is open and it contains A. Similarly, define

$$V := \bigcup_{b \in B} B(b, \epsilon_b/2)$$

where  $\epsilon_b$  is chosen as above. Then, if  $z \in U \cap V$ , then  $\exists a \in A, b \in B$  such that

$$z \in B(a, \epsilon_a/2) \cap B(b, \epsilon_b/2)$$

Assume WLOG that  $\epsilon_a \leq \epsilon_b$ , then by triangle inequality,

$$d(a,b) \le d(a,z) + d(z,b) < \frac{\epsilon_a}{2} + \frac{\epsilon_b}{2} \le \epsilon_a$$

Hence,  $B(a, \epsilon_a) \cap B \neq \emptyset$  contradicting the choice of  $\epsilon_a$ .

2.4. Theorem: Every compact Hausdorff space is normal.

*Proof.* Let X be a compact Hausdorff space and  $A, B \subset X$  disjoint closed sets. By 1.3, X is regular, so for each  $a \in A$ , there are open sets  $U_a$  and  $V_a$  such that

$$a \in U_a, B \subset V_a \text{ and } U_a \cap V_a = \emptyset$$

So  $\{U_a : a \in A\}$  is an open cover for A. But A is compact, so there is a finite subcover  $\{U_{a_1}, U_{a_2}, \ldots, U_{a_k}\}$ . Define

$$U := \bigcup_{i=1}^{k} U_{a_i} \text{ and } V := \bigcap_{i=1}^{n} V_{a_i}$$

Then U, V are open,  $A \subset U, B \subset V$  and  $U \cap V = \emptyset$  [Check!].

(End of Day 27)

2.5. Theorem: A closed subspace of a normal space is normal.

*Proof.* If  $Y \subset X$  is closed and X is normal. We use Lemma 2.2. Suppose  $A \subset Y$  is closed and  $U \subset Y$  an open set such that  $A \subset U$ . Then write  $U = U' \cap Y$  for some open set  $U' \subset X$ . Since A is closed in Y and Y is closed in X, A is closed in X. Hence, there is an open set  $V' \subset X$  such that  $A \subset V'$  and  $\overline{V'} \subset U'$ . Now set

$$V := V' \cap Y$$

Then  $A \subset V$  and by II.6.8,

$$cl_Y(V) = cl_X(V) \cap Y \subset cl_X(V') \cap Y \subset U' \cap Y = U$$

#### 2.6. Examples:

- (i) Every normal space is regular. Hence, every normal space is Hausdorff.
- (ii) Let  $X = \mathbb{R}$  with the topology whose basis are sets of the form

[a,b)

where  $-\infty < a < b \leq \infty$  (See Quiz 1). This topology is denoted by  $\tau_{\ell}$  and it contains the usual topology. It follows that  $\mathbb{R}_{\ell} := (\mathbb{R}, \tau_{\ell})$  is normal.

- (iii)  $X := \mathbb{R}_{\ell} \times \mathbb{R}_{\ell}$  is thus a product of regular spaces, so it is regular. However, it is not normal [without proof]. Hence,
  - (a) The product of normal spaces is not necessarily normal.
  - (b) This is an example of a space that is regular but not normal.
- 2.7. (Urysohn's Lemma for metric spaces): Let (X, d) be a metric space and  $A, B \subset X$  disjoint closed sets. Then  $\exists f : X \to [0, 1]$  continuous such that

$$f(x) = 0 \quad \forall x \in A \text{ and } f(y) = 1 \quad \forall y \in B$$

*Proof.* Recall that  $x \mapsto d(x, A)$  is continuous and d(x, A) = 0 iff  $x \in \overline{A}$ . Define  $f: X \to [0, 1]$  by

$$f(x) = \frac{d(x,A)}{d(x,A) + d(x,B)}$$

Note that the denominator is non-zero because  $A \cap B = \emptyset$ . Now check that f satisfies the required properties.

- 2.8. Lemma: Let X be a normal space and  $A, B \subset X$  disjoint closed sets. Let  $P := \mathbb{Q} \cap [0, 1]$ , then there is a sequence of open sets  $\{U_p : p \in P\}$  such that
  - (i)  $A \subset U_0$  and  $U_1 = X \setminus B$
  - (ii) For all  $p, q \in P, p < q \Rightarrow \overline{U_p} \subset U_q$

*Proof.* Define  $U_1 := X \setminus B$ . Since  $A \subset U_1$ , define  $U_0$  by Lemma 2.2 such that

$$A \subset U_0$$
 and  $\overline{U_0} \subset U_1$ 

Now arrange P in a sequence  $\{0, 1, p_1, p_2, \ldots\}$ . We wish to define  $U_{p_1}$ : Note that  $0 < p_1 < 1$  and  $\overline{U_0} \subset U_1$ , so by Lemma 2.2, there is an open set  $U_{p_1}$  such that

$$\overline{U_0} \subset U_{p_1}$$
 and  $\overline{U_{p_1}} \subset U_1$ 

Now we proceed by induction. Having define  $\{U_0, U_1, U_{p_1}, \ldots, U_{p_n}\}$ , we wish to define  $U_{p_{n+1}}$ . Since  $0 < p_{n+1} < 1$ , choose an immediate predecession  $p_i$  and an immediate successor  $p_j$  among  $\{0, 1, p_1, p_2, \ldots, p_n\}$ . Note that  $\overline{U_{p_i}} \subset U_{p_j}$ . So by Lemma 2.2, there is an open set  $U_{p_{n+1}}$  such that

$$\overline{U_{p_i}} \subset U_{p_{n+1}}$$
 and  $\overline{U_{p_{n+1}}} \subset U_{p_j}$ 

By induction, we define  $U_p$  for all  $p \in P$  satisfying (i) and (ii).

2.9. Lemma: Let X be a normal space and  $A, B \subset X$  disjoint closed sets. Let  $\{U_p : p \in \mathbb{Q} \cap [0,1]\}$  be a sequence of open sets as in the previous lemma. Define  $U_p = \emptyset$  if p < 0 and  $U_q = X$  if q > 1. Now define  $f : X \to \mathbb{R}$  by

$$f(x) := \inf \mathbb{Q}(x)$$

where  $\mathbb{Q}(x) := \{ p \in \mathbb{Q} \cap [0, 1] : x \in U_p \}.$ 

- (i)  $f(x) \in [0, 1]$  for all  $x \in X$ .
- (ii) For any  $r \in \mathbb{Q}, x \in \overline{U_r} \Rightarrow f(x) \leq r$ , and
- (iii)  $x \notin U_r \Rightarrow f(x) \ge r$

*Proof.* Note that f is well-defined because, for any  $x \in X$ ,  $x \in U_p$  for all p > 1, so  $(1, \infty) \cap \mathbb{Q} \subset \mathbb{Q}(x)$ . Hence,  $f(x) \leq 1$ . Similarly,  $x \notin U_p$  for all p < 0. Hence,  $f(x) \geq 0$ .

If  $x \in \overline{U_r}$ , then for any  $p > r, x \in U_p$ . Hence,

$$(r,\infty) \cap \mathbb{Q} \subset \mathbb{Q}(x)$$

Since the infimum of a subset is greater than the infimum of a super set,  $f(x) \leq r$ . Similarly, if  $x \notin U_r$ , then  $x \notin U_s$  for all s < r. Hence,

$$\mathbb{Q}(x) \subset (r,\infty) \cap \mathbb{Q}$$

As before, this implies  $f(x) \ge r$ 

2.10. (Urysohn's Lemma): Let X be a normal space and  $A, B \subset X$  disjoint closed sets. Then  $\exists f : X \to [0, 1]$  continuous such that

$$f(x) = 0 \quad \forall x \in A \text{ and } f(y) = 1 \quad \forall y \in B$$

*Proof.* Let  $\{U_p : p \in \mathbb{Q}\}$  and  $f : X \to \mathbb{R}$  defined as above. For any  $x \in X$ , and  $r < 0, x \notin U_r$ , so  $f(x) \ge 0$ . Similarly,  $f(x) \le 1$ . Furthermore, if  $x \in A$ , then  $x \in U_0$ , so f(x) = 0. Similarly, f(y) = 1 for all  $y \in B$ . It suffices to show that f is continuous.

Fix  $x_0 \in X$  and U an open set containing  $f(x_0)$ . WTS:  $\exists$  an open set  $V \subset X$  containing  $x_0$  such that  $f(V) \subset U$ . Choose  $c, d \in \mathbb{R}$  such that  $(c, d) \subset U$ . Now there exists  $p, r \in \mathbb{Q}$  such that  $[p, r] \subset (c, d) \subset U$ , and let

$$V := U_r \setminus \overline{U_p}$$

Note that V is open, and if  $z \in V$ , then  $z \in U_r$  and  $z \notin \overline{U_p}$ . So by the previous lemma,

$$p \le f(x) \le r$$

Hence,  $f(V) \subset U$  as required.

#### (End of Day 28)

2.11. Corollary: Let X be a normal space and  $A, B \subset X$  disjoint closed sets. Given  $a, b \in \mathbb{R}$  with  $a < b, \exists f : X \to [a, b]$  continuous such that

$$f|_A = a$$
 and  $f|_B = b$ 

*Proof.* Simply compose the function  $g: X \to [0, 1]$  produced by Urysohn's lemma with the map  $[0, 1] \to [a, b]$  given by

$$t \mapsto (1-t)a + tb$$

# 3. Tietze's extension Theorem

3.1. Definition: Let (X, d) be a metric space.

- (i) A sequence  $(x_n) \subset X$  is said to be Cauchy if, for each  $\epsilon > 0, \exists N \in \mathbb{N}$  such that  $d(x_n, x_m) < \epsilon$  for all  $n, m \ge N$ .
- (ii) X is said to be complete if every Cauchy sequence in X converges to a point in X.
- 3.2. Examples:
  - (i) Every convergent sequence is Cauchy.
  - (ii) Let  $X = \mathbb{Q}^c$ , and  $x_n := \sqrt{2}/n$ , then  $(x_n)$  is Cauchy, but does not converge to a point in X. Hence X is not complete.
  - (iii) X = (0, 1) is not complete because (1/n) is Cauchy but not convergent.

3.3. Lemma: Let (X, d) be a metric space and  $(x_n) \subset X$  Cauchy. Then  $(x_n)$  is bounded. ie.  $\exists x_0 \in X$  and  $M \ge 0$  such that  $d(x_n, x_0) \le M$  for all  $n \in \mathbb{N}$ 

*Proof.* Fix  $\epsilon = 1$ , then  $\exists N \in \mathbb{N}$  such that

$$d(x_n, x_m) < 1 \quad \forall n, m \ge 1$$

For  $x_0 \in X$  fixed, let

$$M := \max\{d(x_0, x_i) : 1 \le i \le N\} + 1$$

Then for any  $n \in \mathbb{N}$ , if  $n \leq N$ , then  $d(x_n, x_0) \leq M$ . And if  $n \geq N$ , then

$$d(x_n, x_0) \le d(x_n, x_N) + d(x_N, x_0) \le M$$

3.4. Lemma: Let (X, d) be a metric space and  $(x_n)$  a Cauchy sequence. If  $(x_n)$  has a convergent subsequence, then  $(x_n)$  converges.

*Proof.* Suppose  $x_{n_k} \to x$  is a convergent subsequence. For any  $\epsilon > 0$ , choose  $N \in \mathbb{N}$  such that

$$d(x_n, x_m) < \epsilon/2 \quad \forall n, m \ge N$$

Now choose  $K \in \mathbb{N}$  such that

$$d(x_{n_i}, x) < \epsilon/2 \quad \forall i \ge K$$

Hence,  $N_0 := \max\{N, n_K\}$  has the property that

$$d(x_n, x) < \epsilon \quad \forall n \ge N_0$$

3.5. Lemma: Every sequence in  $\mathbb{R}$  has a monotone subsequence.

*Proof.* Let  $(x_n) \subset \mathbb{R}$  and suppose  $(x_n)$  has no monotone increasing subsequence. We show that  $(x_n)$  has a monotone decreasing subsequence. We claim:  $\exists n_1 \in \mathbb{N}$  such that  $x_n < x_{n_1}$  for all  $n > n_1$ .

*Proof.* Suppose not, then set  $n_1 = 1$ . Then  $\exists n_2 > n_1$  and  $x_{n_2}$  such that  $x_{n_2} > x_{n_1}$ . Similarly,  $\exists n_3 > n_2$  such that  $x_{n_3} > x_{n_2}$  and so on. Thus, we produce a subsequence  $(x_{n_k})$  that is strictly increasing. This contradicts the assumption that  $(x_n)$  has no increasing subsequence.

Now choose  $n_1 \in \mathbb{N}$  such that  $x_n < x_{n_1}$  for all  $n > n_1$ . Now consider the subsequence  $\{x_{n_1}, x_{n_1+1}, x_{n_1+2}, \ldots\}$ . By the same argument as above,  $\exists n_2 > n_1$  such that  $x_n < x_{n_2}$  for all  $n > n_2$ . In particular,

$$x_{n_2} < x_{n_1}$$

and

$$x_n < x_{n_2} \quad \forall n > n_2$$

Thus proceeding (by induction) there is a subsequence  $(x_{n_k})$  that is strictly decreasing.

3.6. Theorem:  $\mathbb{R}$  is complete.

*Proof.* Let  $(x_n) \subset \mathbb{R}$  be Cauchy, then by the previous lemmas,  $(x_n)$  is bounded and has a monotone subsequence. But every monotone bounded subsequence in  $\mathbb{R}$  is convergent (to its supremum or infimum). Some the previous lemma applies.  $\Box$ 

- 3.7. Definition: Let X be a topological space and (Y, d) a metric space.
  - (i) A function  $f: X \to Y$  is said to be bounded if f(X) is a bounded subset of Y (i.e  $\exists y_0 \in X$  and  $M \ge 0$  such that  $d(f(x), y_0) \le M$  for all  $x \in X$ .
  - (ii) Let  $C_b(X,Y)$  denote the set of all continuous, bounded functions  $f: X \to Y$
- 3.8. Theorem: Define  $d_{\infty}: C_b(X,Y) \times C_b(X,Y) \to \mathbb{R}$  by

$$d_{\infty}(f,g) := \sup\{d(f(x),g(x)) : x \in X\}$$

Then this defines a metric on  $C_b(X, Y)$ .

Proof. HW

(End of Day 29)

3.9. Theorem: If (Y, d) is a complete metric space, and  $(C_b(X, Y), d_\infty)$  is complete. *Proof.* Let  $(f_n) \subset C_b(X, Y)$  be a Cauchy sequence. For any  $x \in X$ ,

$$d(f_n(x), f_m(x)) \le d_{\infty}(f_n, f_m)$$

Hence,  $(f_n(x))$  is Cauchy in Y. Hence,  $\exists z_x \in Y$  such that  $f_n(x) \to z_x$ . Define  $f: X \to Y$  by  $f(x) = z_x$ . We claim that f is continuous and bounded.

(i) Since  $(f_n)$  is Cauchy, it is bounded. Hence,  $\exists M \ge 0$  such that

$$\sup_{x \in X} d(f_n(x), 0) \le M \quad \forall n \in \mathbb{N}$$

For any  $x \in X$  fixed,  $f_n(x) \to f(x)$ . Hence,  $d(f(x), 0) \leq M$  [Why?]. Hence, f is bounded.

(ii) To see that  $f_n \to f$  wrt  $d_{\infty}$ : Fix  $\epsilon > 0$ , then  $\exists N \in \mathbb{N}$  such that

$$d_{\infty}(f_n, f_m) < \epsilon/2 \quad \forall n, m \ge M$$

Hence for  $x \in X$  fixed,

$$d(f_n(x), f_m(x)) < \epsilon/2 \quad \forall n, m \ge N$$

Let  $m \to \infty$ , then

$$d(f_n(x), f(x)) \le \epsilon/2 \quad \forall n \ge N$$

Hence,  $d_{\infty}(f_n, f) < \epsilon \quad \forall n \ge N$ . Hence,  $f_n \to f$  in  $d_{\infty}$ 

(iii) To see that f is continuous: Let  $x_0 \in X$  and  $\epsilon > 0$ , then  $\exists N \in \mathbb{N}$  such that

$$d_{\infty}(f_n, f) < \epsilon/3 \quad \forall n \ge N$$

Since  $f_N$  is continuous,  $\exists U \subset X$  open such that  $x_0 \in U$  and

$$d(f_N(y), f_N(x_0)) < \epsilon/3 \quad \forall y \in U$$

Hence, for all  $y \in U$ ,

$$d(f(y), f(x_0)) < \epsilon$$

- 3.10. Corollary: Let X be any topological space. The set  $C_b(X) := C_b(X, \mathbb{R})$  is a complete metric space with respect to the metric

$$d_{\infty}(f,g) := \sup_{x \in X} |f(x) - g(x)|$$

3.11. (Tietze's Extension Theorem): Let X be a normal topological space and  $Y \subset X$  closed. Let  $f: Y \to \mathbb{R}$  be a continuous function, then  $\exists h: X \to \mathbb{R}$  continuous such that

$$h(y) = f(y) \quad \forall y \in Y$$

(h is called a continuous extension of f)

*Proof.* Assume first that f is bounded and

$$c := \sup\{|f(y)| : y \in Y\}$$

Define

$$E_0 := \{x \in X : f(x) \le -c/3\} = f^{-1}(-\infty, -c/3]$$
  
$$F_0 := \{x \in X : f(x) \ge c/3\} = f^{-1}[c/3, \infty)$$

Then  $E_0$  and  $F_0$  are disjoint closed sets. By Corollary 2.11,  $\exists g_0 : X \to \mathbb{R}$  such that

$$-c/3 \le g_0(x) \le c/3 \quad \forall x \in X$$

and

$$g_0|_{E_0} = -c/3$$
 and  $g_0|_{F_0} = c/3$ 

Hence,

$$|g_0(x)| \le c/3 \quad \forall x \in X$$
  
$$|f(y) - g_0(y)| \le 2c/3 \quad \forall y \in Y$$

Let  $f_1 := f - g_0$ . Then by the above argument,  $\exists g_1 : X \to \mathbb{R}$  continuous such that

$$|g_1(x)| \le 2c/9 \quad \forall x \in X$$
  
$$|f(y) - g_0(y) - g_1(y)| \le 4c/9 \quad \forall y \in Y$$

Thus proceeding, we obtain a sequence  $(g_n)$  of continuous functions such that

$$|g_n(x)| \le 2^n c/3^{n+1} \quad \forall x \in X$$
  
$$|f(y) - h_n(y)| \le 2^{n+1} c/3^{n+1} \quad \forall y \in Y$$

where  $h_n := g_0 + g_1 + \ldots + g_n$ . Now note that if m > n,

$$|h_n(x) - h_m(x)| = \left| \sum_{i=m+1}^n g_i(x) \right|$$
  
$$\leq \sum_{i=m+1}^n |g_i(x)|$$
  
$$\leq \sum_{i=m+1}^n \frac{2^i c}{3^{i+1}} \leq \frac{2^{m+1} c}{3^{m+1}}$$

Hence,

$$d_{\infty}(h_n, h_m) \le \frac{2^{m+1}c}{3^{m+1}}$$

Since the RHS goes to zero,  $(h_n)$  form a Cauchy sequence in  $C_b(X, \mathbb{R})$ . By the previous lemma,  $\exists h \in C_b(X, \mathbb{R})$  such that  $h_n \to h$ . Now if  $y \in Y$ , then

$$|f(y) - h_n(y)| \le \frac{2^{n+1}c}{3^{n+1}}$$

Letting  $n \to \infty$ , we see that h = f on Y.

Now suppose f is not bounded. Let  $g : \mathbb{R} \to (-1, 1)$  be a homeomorphism (is there one?). Now define  $\tilde{f} := g \circ f$ . Now  $\tilde{f}$  is bounded, so  $\exists \tilde{h} : X \to \mathbb{R}$  continuous such that  $\tilde{h}|_Y = \tilde{f}$ . Now define  $h := g^{-1} \circ \tilde{h}$ , and check that h satisfies the required conditions.

(End of Day 30)

# 4. Urysohn Metrization Theorem

- 4.1. Definition: A topological space  $(X, \tau)$  is said to be metrizable if there exists a metric d on X such that  $\tau = \tau_d$ .
- 4.2. Theorem:  $\mathbb{R}^{\omega}$  with the product topology is metrizable.

*Proof.* Let  $\overline{d} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$  be the metric given by

$$d(a, b) = \min\{|a - b|, 1\}$$

Define  $D: \mathbb{R}^{\omega} \times \mathbb{R}^{\omega} \to \mathbb{R}$  by

$$D(x,y) := \sup\left\{\frac{\overline{d}(x_i,y_i)}{i}\right\}$$

Then [Check!] that D is a metric on  $\mathbb{R}^{\omega}$ . We claim that the product topology  $\tau_p$  on  $\mathbb{R}^{\omega}$  coincides with  $\tau_D$ 

(i)  $\tau_p \subset \tau_D$ : Let U be a basic open set in  $\tau_p$  of the form

$$U := U_1 \times U_2 \times \ldots \times U_n \times \mathbb{R} \times \mathbb{R} \times \ldots$$

Let  $x = (x_i) \in U$ , so for  $1 \le i \le n, x_i \in U_i$ , so  $\exists \epsilon_i > 0$  such that

 $(x_i - \epsilon_i, x_i + \epsilon_i) \subset U_i$ 

Assume  $\epsilon_i < 1$  for all *i*, and let  $\epsilon := \min\{\epsilon_i/i : 1 \le i \le n\}$ , then we claim that

 $B_D(x,\epsilon) \subset U$ 

To see this, suppose  $y = (y_i) \in B_D(x, \epsilon)$ , then for  $1 \le i \le n$ ,

$$\frac{\overline{d}(x_i, y_i)}{i} \le D(x, y) < \epsilon$$

Hence,  $\overline{d}(x_i, y_i) \leq \epsilon_i < 1$ , so  $|x_i - y_i| < \epsilon_i$ . Hence,  $y_i \in U_i$  for all  $1 \leq i \leq n$ . Hence,  $y \in U$ , so

 $B_D(x,\epsilon) \subset U$ 

Thus, U is a union of sets of the form  $B_D(x, \epsilon)$ , and so  $U \in \tau_D$ . Since U is a generic basic open set, it follows that  $\tau_p \subset \tau_D$ .

(ii)  $\tau_D \subset \tau_p$ : Let  $U \in \tau_D$  be open, and  $x \in U$ . Then  $\exists \epsilon > 0$  such that  $B_D(x, \epsilon) \subset U$ . Choose  $N \in \mathbb{N}$  such that  $1/N < \epsilon$ , and consider

$$V := (x_1 - \epsilon, x_1 + \epsilon) \times \ldots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \ldots$$

We claim that  $V \subset B_D(x, \epsilon)$ . To see this, suppose  $y = (y_i) \in V$ , then for  $i \geq N$ ,

$$\frac{\overline{d}(x_i, y_i)}{i} \le \frac{1}{N}$$

because  $\overline{d}(x_i, y_i) \leq 1$ . Furthermore, if  $1 \leq i \leq N$ , then

$$\frac{\overline{d}(x_i, y_i)}{i} \le \frac{d(x_i, y_i)}{i} \le \frac{1}{Ni} < \epsilon$$

Hence,  $D(x, y) < \epsilon$ . This is true for any  $y \in V$ , so  $V \subset B_D(x, \epsilon) \subset U$ . Hence, U is a union of open sets in  $\tau_p$ , and so  $U \in \tau_p$ . Thus,  $\tau_D \subset \tau_p$  as well.
- 4.3. Definition: A topological space is called second countable if it has a countable basis.
- 4.4. Example:
  - (i)  $\mathbb{R}^n$  is second countable.
  - (ii) If  $\mathbb{R}$  is given the discrete metric, then it is not second countable.
  - (iii) Every second countable space is separable.

*Proof.* Let  $\{B_n : n \in \mathbb{N}\}$  be a countable basis for X. For each  $n \in \mathbb{N}$ , choose  $x_n \in B_n$  and let  $D := \{x_n : n \in \mathbb{N}\}$ . Then D is dense in X, because if U is any non-empty open set, then  $\exists n \in \mathbb{N}$  such that  $B_n \subset U$ , so  $x_n \in U$  which implies  $D \cap U \neq \emptyset$ .

(iv) Any separable metric space is second countable.

*Proof.* Let (X, d) be a separable metric space and  $A := \{x_n\}$  be a countable dense subset of X. Let  $B_{m,n} := B(x_m, 1/n)$ , then we claim that  $\mathcal{B} := \{B_{m,n}\}$  forms a basis for  $\tau_d$ .

- (a) If  $x \in X$ , then  $\exists x_m \in A$  such that  $d(x_m, x) < 1$ . Hence,  $x \in B_{m,1}$ . So  $\mathcal{B}$  covers X.
- (b) Furthermore, if  $x \in B_{m_1,n_1} \cap B_{m_2,n_2}$  then let  $\alpha := \min\{1/2n_1, 1/2n_2\}$ . Choose  $m_3 \in \mathbb{N}$  such that  $d(x, x_{m_3}) < \alpha$  and let  $n_3 \in \mathbb{N}$  such that  $1/n_3 < \alpha$ , then [Check!]

$$B_{m_3,n_3} \subset B_{m_1,n_1} \cap B_{m_2,n_2}$$

and  $x \in B_{m_3,n_3}$ 

- (c) Thus,  $\mathcal{B}$  forms a basis for some topology  $\tau$  on X. Since  $\mathcal{B} \subset \tau_d$ , it follows that  $\tau \subset \tau_d$ .
- (d) However, if  $U \in \tau_d$  and  $x \in U$ , then  $\exists \epsilon > 0$  such that  $B_d(x, \epsilon) \subset U$ . Now choose  $m \in \mathbb{N}$  such that  $d(x, x_m) < \epsilon/2$ , and let  $n \in \mathbb{N}$  such that  $1/n < \epsilon/2$ , then  $x \in B_{m,n}$  and  $B_{m,n} \subset B_d(x, \epsilon) \subset U$ . Hence, every  $U \in \tau_d$  is obtained as a union of elements of  $\mathcal{B}$ .

Hence,  $\mathcal{B}$  is a basis for  $\tau_d$ .

*Proof.* Let X be a regular space with a countable basis  $\mathcal{B}$ , and let  $A, B \subset X$  be two closed disjoint sets. WTS:  $\exists$  open sets U and V such that  $A \subset U, B \subset V$  and  $U \cap V = \emptyset$ .

(i) For each  $x \in A$ ,  $x \notin B$ , so there is an open sets U, V such that  $x \in U, B \subset V$ and  $U \cap V = \emptyset$ . Since X is regular, there is an open set W such that  $x \in W$ and  $\overline{W} \subset U$ . Choose a basic open set  $B_x \in \mathcal{B}$  such that  $x \in B_x$  and  $B_x \subset W$ . Thus,

$$B_x \cap B = \emptyset$$

Thus, we obtain an open cover  $\{B_x : x \in A\}$  for A which is countable, so we denote it by  $\{U_n : n \in \mathbb{N}\}$ . Note that

$$\overline{U_n} \cap B = \emptyset \quad \forall n \in \mathbb{N}$$

Similarly, we obtain an open cover  $\{V_n : n \in \mathbb{N}\}$  of B which is countable such that

$$\overline{V_n} \cap A = \emptyset \quad \forall n \in \mathbb{N}$$

(ii) If  $U := \bigcup U_n$  and  $V := \bigcup V_n$ , then  $A \subset U, B \subset V$ , but U and V need not be disjoint. So define

$$U'_n := U_n \setminus \left[\bigcup_{i=1}^n \overline{V_n}\right] \text{ and } V'_n := V_n \setminus \left[\bigcup_{i=1}^n \overline{U_n}\right]$$

Then each  $U'_n$  and  $V'_n$  is open.

(iii) If  $x \in A$ , then  $\exists n \in \mathbb{N}$  such that  $x \in U_n$ . But  $\overline{V_i} \cap A = \emptyset$  for all *i*. Hence,  $x \in U'_n$ . Thus,  $\{U'_n : n \in \mathbb{N}\}$  forms an open cover for A. Define

$$U' := \bigcup_{n=1}^{\infty} U'_n$$

Then  $A \subset U'$ . Similarly, if

$$V' := \bigcup_{n=1}^{\infty} V'_n$$

Then  $B \subset V'$ .

(iv) We claim that  $U' \cap V' = \emptyset$ . Suppose  $x \in U' \cap V'$ , then  $\exists n, m \in \mathbb{N}$  such that  $x \in U'_n$  and  $x \in V'_m$ . Assume n > m, then  $x \notin V_m$  by definition of  $U'_n$ . This is a contradiction, so  $U' \cap V' = \emptyset$ .

## (End of Day 31)

4.6. Lemma: Let X be a regular space with a countable basis. Then there is a sequence of functions  $f_n : X \to [0, 1]$  such that, for any  $x_0 \in X$  and open set U containing  $x_0, \exists n \in \mathbb{N}$  such that  $f_n(x_0) = 1$  and  $f_n = 0$  on  $X \setminus U$ .

*Proof.* Note that X is normal so Urysohn's lemma applies. Let  $\{B_n : n \in \mathbb{N}\}$  be a countable basis for X. Define

$$D := \{ (n, m) \in \mathbb{N} \times \mathbb{N} : \overline{B_n} \subset B_m \}$$

For each  $(n,m) \in D$ , Urysohn's lemma implies that there is a function  $g_{n,m} : X \to [0,1]$  such that

$$g_{n,m}|_{\overline{B_n}} = 1 \text{ and } g_{n,m}|_{X \setminus B_m} = 0$$

This collection  $\{g_{n,m}\} = \{f_n\}$  is countable, and it satisfies the required condition: If  $x_0 \in X$  and U is an open set such that  $x_0 \in U$ , then  $\exists$  a basic open set  $B_m$ such that  $x_0 \in B_m$  and  $B_m \subset U$ . Furthermore, by regularity,  $\exists$  a basic open set  $B_n$  such that  $x_0 \in B_n$  and  $\overline{B_n} \subset B_m$ . Now

$$g_{n,m}(x_0) = 1$$
 and  $g_{n,m}|_{X \setminus U} = 0$ 

- 4.7. (Urysohn's Metrization Theorem): Every regular space with a countable basis is metrizable.
  - *Proof.* (i) We construct a continuous function  $F : X \to \mathbb{R}^{\omega}$  as follows: Let  $\{f_n\}$  be a sequence as in the previous lemma, and define

$$F(x) := (f_n(x))$$

Then F is continuous because each coordinate function  $f_n$  is continuous.

- (ii) F is injective: If  $x \neq y$ , then there is an open set U such that  $x \in U$  and  $y \notin U$ . Choose  $n \in \mathbb{N}$  such that  $f_n(x) = 1$  and  $f_n|_{X \setminus U} = 0$ . In particular,  $f_n(y) = 0$ . Hence,  $F(x) \neq F(y)$ .
- (iii) Let Z := F(X). We claim that  $F : X \to Z$  is a homeomorphism. F is clearly surjective, so it suffices to show that F is an open map. Let  $U \subset X$  be an open set. WTS:  $F(U) \subset Z$  is open. Fix  $z \in F(U)$ , then  $\exists x \in U$  such that

$$F(x) = z$$

Choose  $n \in \mathbb{N}$  such that  $f_n(x) = 1$  and  $f_n|_{X \setminus U} = 0$ . Define

$$V := \pi_n^{-1}((0,\infty)) \subset \mathbb{R}^{\omega}$$

and set

$$W := V \cap Z$$

Then W is open in Z since V is open in  $\mathbb{R}^{\omega}$ . Furthermore,  $f_n(x) > 0$ , so  $z \in W$ . We claim:  $W \subset F(U)$ . To see this, fix  $y \in W$ , then  $\exists x' \in X$  such that F(x') = y. Now,  $\pi_n(y) > 0$ , but

$$\pi_n(y) = \pi_n(F(x')) = f_n(x')$$

Since  $f_n = 0$  on  $X \setminus U$ , it follows that  $x' \in U$ . Hence,  $x' \in F(U)$ . Thus,  $W \subset F(U)$ . Hence, every  $z \in F(U)$  is an interior point of F(U), so F(U) is open.

(iv) Thus,  $F: X \to Z$  is a homeomorphism. Since  $Z \subset \mathbb{R}^{\omega}$  and  $\mathbb{R}^{\omega}$  is metrizable, it follows that Z is metrizable, and so X is too.

4.8. Corollary: Every compact Hausdorff space with a countable basis is metrizable.

4.9. Example:

- (i) Every metric space is certainly regular, but need not have a countable basis (See 4.4(ii)).
- (ii) Let  $K = \{1/n : n \in \mathbb{N}\}$ . Define

 $\mathcal{B}_1 := \{ \text{ open intervals in } \mathbb{R} \text{ with rational end-points} \}$  $\mathcal{B}_2 := \{ (a, b) \setminus K : a < b \text{ in } \mathbb{Q} \}$ 

Then  $\mathcal{B} := \mathcal{B}_1 \cup \mathcal{B}_2$  forms a basis for a topology on  $\mathbb{R}$ , which we denote by  $\tau_K$ . Then  $\mathbb{R}_K := (\mathbb{R}, \tau_K)$  is Hausdorff, has a countable basis, but is not metrizable because it is not regular. Thus, regularity is necessary for Urysohn's metrization theorem to hold.

## 5. Imbedding of Manifolds

- 5.1. An *m*-manifold is a topological space X with a countable basis such that for each  $x \in X$ , there is a neighbourhood  $U_x$  of x such that  $U_x$  is homeomorphic with an open subset of  $\mathbb{R}^m$ .
- 5.2. Examples:
  - (i)  $\mathbb{R}^m$  is an *m*-manifold. So is any open subset of  $\mathbb{R}^m$ .
  - (ii) [0,1] is not a 1-manifold, because any neighbourhood of 0 is of the form  $[0,\delta)$ , which is not homeomorphic to an open subset of  $\mathbb{R}$ .
  - (iii)  $S^1$  is a 1-manifold. In general,  $S^m$  is an *m*-manifold (without proof)
  - (iv) A 1-manifold is called a curve, and a 2-manifold is called a surface.
  - (v) The torus  $S^1 \times S^1$  is a surface. In general, if X and Y are manifolds, then so is  $X \times Y$ .

(End of Day 32)

- 5.3. Theorem: Let X be an m-manifold. Then X is
  - (i) Locally path connected.
  - (ii) Locally compact.
  - (iii) Regular
  - (iv) Metrizable.
  - *Proof.* (i) Let  $x \in X$  and U an open neighbourhood of x. WTS:  $\exists V \subset U$  open such that  $x \in V$  and V is path connected. To see this, choose a neighbourhood  $U_x$  of x and a homeomorphism

$$g: U_x \to U'_x \subset \mathbb{R}^m$$

where  $U'_x$  is open in  $\mathbb{R}^m$ . Then  $U_x \cap U$  is open and

$$g|_{U_x \cap U} : U_x \cap U \to g(U'_x \cap U) \subset \mathbb{R}^m$$

is a homeomorphism. Since  $g(U'_x \cap U)$  is an open subset of  $\mathbb{R}^m$  containing g(x), and  $\mathbb{R}^m$  is locally path connected, there is an open set  $V' \subset g(U'_x \cap U)$  that is path connected and containing g(x). Then  $V := g^{-1}(V')$  is open, path connected, contains x and  $V \subset U$ .

- (ii) Local compactness is identical to part (i).
- (iii) Let  $x \in X$  and an open set U containing x. WTS:  $\exists V$  open such that  $x \in V$  and  $\overline{V} \subset U$ . Choose  $U_x$  open and a homeomorphism

$$g: U_x \to U'_x \subset \mathbb{R}^m$$

as before. Since  $U \cap U_x$  is open in  $U_x$ ,

$$g(U \cap U_x) \subset U'_x$$

is open and contains g(x). Since  $U'_x \subset \mathbb{R}^m$  and  $\mathbb{R}^m$  is regular,  $U'_x$  is regular by 1.5. Hence, there is an open set V' such that  $g(x) \in V'$  and

$$\overline{V'} \subset g(U \cap U_x)$$

Then  $V := g^{-1}(V')$  is open, contains x and since g is a local homeomorphism

$$\overline{V} = \overline{g^{-1}(V')} = g^{-1}(\overline{V'}) \subset g^{-1}(g(U \cap U_x)) \subset U \cap U_x \subset U$$

Hence, X is regular.

(iv) X has a countable basis, so Urysohn's metrization theorem applies.

- 5.4. Definition: Let X be a topological space.
  - (i) Let  $f: X \to \mathbb{R}$  be a function. The support of f is the set

$$\operatorname{supp}(f) := \{ x \in X : f(x) \neq 0 \}$$

- (ii) Let  $\mathcal{U} := \{U_1, U_2, \dots, U_n\}$  be an open cover for X. A partition of unity dominated by  $\mathcal{U}$  is a family of continuous functions  $f_i : X \to \mathbb{R}$  such that
  - (a)  $\operatorname{supp}(f_i) \subset U_i$  for all  $1 \leq i \leq n$
  - (b) For each  $x \in X$ ,  $f_1(x) + f_2(x) + \ldots + f_n(x) = 1$
- 5.5. Lemma: Let X be a normal space and  $\mathcal{U} := \{U_1, U_2, \dots, U_n\}$  be an open cover for X. Then there is an open cover  $\mathcal{V} := \{V_1, V_2, \dots, V_n\}$  such that

$$\overline{V_i} \subset U_i$$

for all  $1 \leq i \leq n$ 

*Proof.* We induct on n: If n = 1, then  $U_1 = X$  so take  $V_1 = U_1$ . If  $n \ge 2$ , note that

$$A := X \setminus \left[\bigcup_{i=2}^{n} U_i\right]$$

is closed and  $A \subset U_1$ . Since X is normal, there is an open set  $V_1$  such that

$$A \subset V_1$$
 and  $\overline{V_1} \subset U_2$ 

The collection  $\{V_1, U_2, \ldots, U_n\}$  now covers X. Proceeding by induction, suppose that we have produced a cover

$$\{V_1, V_2, \ldots, V_{k-1}, U_k, U_{k+1}, \ldots, U_n\}$$

such that  $\overline{V_i} \subset U_i$  for all  $1 \leq i \leq k-1$ . Let

$$A := X \setminus \left[ \left( \bigcup_{i=1}^{k-1} V_i \right) \cup \left( \bigcup_{j=k+1}^n U_j \right) \right]$$

Then A is closed and contained in  $U_k$ . Choose  $V_k$  open such that  $A \subset V_k$  and  $\overline{V_k} \subset U_k$ . Now  $\{V_1, V_2, \ldots, V_k, U_{k+1}, \ldots, U_n\}$  forms an open cover. Proceeding thus, we exhaust all  $U_i$ 's.

5.6. Theorem: Let X be a normal space and  $\mathcal{U}$  be a finite open cover for X. Then there is a partition of unity dominated by  $\mathcal{U}$ .

Proof. Let  $\mathcal{U} := \{U_1, U_2, \ldots, U_n\}$  be an open cover for X. Choose a cover  $\mathcal{V} := \{V_1, V_2, \ldots, V_n\}$  such that  $\overline{V_i} \subset U_i$  and an open cover  $\mathcal{W} := \{W_1, W_2, \ldots, W_n\}$  such that  $\overline{W_i} \subset V_i$  for all  $1 \leq i \leq n$ . By Urysohn's lemma, there exist function  $\psi_i : X \to [0, 1]$  such that

$$\psi_i|_{\overline{W_i}} = 1 \text{ and } \psi_i|_{X \setminus V_i} = 0$$

Then

$$\operatorname{supp}(\psi_i) \subset \overline{V_i} \subset U_i$$

For any  $x \in X, \exists 1 \leq i \leq n$  such that  $x \in W_i$ , so  $\psi_i(x) = 1$ . Hence, define  $f_i : X \to \mathbb{R}$  by

$$f_i(x) := \frac{\psi_i(x)}{\psi_1(x) + \psi_2(x) + \ldots + \psi_n(x)}$$

The denominator is never zero, so  $f_i$  is continuous, and is a partition of unity dominated by  $\mathcal{U}$ .

5.7. (Imbedding Theorem): Let X be a compact m-manifold, then  $\exists N \in \mathbb{N}$  and an injective map  $E : X \to \mathbb{D}^N$ 

$$F: X \to \mathbb{R}^N$$

such that  $F: X \to F(X)$  is a homeomorphism. (ie. F is an imbedding of X into  $\mathbb{R}^n$ )

*Proof.* For each  $x \in X, \exists$  an open set  $U_x$  that is homeomorphic to an open subset of  $\mathbb{R}^m$ . Choose a finite subcover  $\{U_1, U_2, \ldots, U_n\}$  and homeomorphisms

$$g_i: U_i \to V_i$$

where  $V_i \subset \mathbb{R}^m$  is open. Let  $\{f_1, f_2, \ldots, f_n\}$  be a partition of unity dominated by  $\mathcal{U}$ . Let  $A_i := \operatorname{supp}(f_i) \subset U_i$  and define  $h_i : X \to \mathbb{R}^m$  by

$$h_i(x) := \begin{cases} f_i(x)g_i(x) & : x \in U_i \\ 0 & : x \in X \setminus A_i \end{cases}$$

If  $x \in (X \setminus A_i) \cap U_i$ , then  $f_i(x) = 0$ , so both definitions agree. So by pasting lemma,  $h_i$  is continuous. Define

$$F: X \to \underbrace{\mathbb{R} \times \mathbb{R} \times \dots \mathbb{R}}_{n \text{ times}} \times \underbrace{\mathbb{R}^m \times \mathbb{R}^m \times \dots \times \mathbb{R}^m}_{n \text{ times}}$$

by

$$x \mapsto (f_1(x), f_2(x), \dots, f_n(x), h_1(x), h_2(x), \dots, h_n(x))$$

Then F is continuous. Suppose we show that F is injective, then since X is compact,

$$F: X \to F(X)$$

will be a homeomorphism. So suppose  $x, y \in X$  such that F(x) = F(y), then choose  $1 \leq i \leq n$  such that  $f_i(x) > 0$ . Then  $x \in U_i$  and  $f_i(x) = f_i(y) > 0$  and  $h_i(x) = h_i(y)$  implies that

$$g_i(x) = g_i(y)$$

But  $g_i: U_i \to V_i$  is a homeomorphism, so x = y as required.

(End of Day 33)

Review

(End of Day 34)

## **V. Instructor Notes**

- 0.1. This time, the semester was shortened by one week to accommodate an earlier convocation.
- 0.2. Therefore, I was unable to cover Tychonoff's theorem, which is unfortunate. I did however discuss the theorem, and how the tube lemma proof does not work. The only other topic on the syllabus I did not cover was Lindeloff spaces, which is not a major loss. One extra week and I would have done all of it.
- 0.3. The student response seemed alright, but no questions were forthcoming which made it harder to judge.

## Bibliography

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