# MTH 101 : Calculus of One Variable Semester 1, 2012-2013 

Dr. Prahlad Vaidyanathan

## Contents

I. Limits and Continuity ..... 2

1. Review ..... 2
2. Limits ..... 3
3. Continuity ..... 3
4. Intermediate Value Theorem ..... 4
5. Extreme Values of Continuous Functions ..... 4
6. Examples and Problems ..... 4
II. Differentiation ..... 5
7. Motivation, and Definitions ..... 5
8. Rules of Differentiation ..... 5
9. Mean-Value Theorem for Derivatives ..... 6
10. Curve Sketching ..... 6
11. Miscellaneous Topics ..... 7
III. Sequences and Series ..... 8
12. Sequences ..... 8
13. Infinite Series ..... 9
14. Comparison Tests ..... 10
15. Root and Ratio tests for series with non-negative terms ..... 10
16. Series with negative terms ..... 10
IV. Integration ..... 11
17. Definition of the Integral ..... 11
18. Integrable functions ..... 12
19. Properties of the Integral ..... 12
20. Fundamental Theorem of Calculus ..... 13
21. Logarithmic and Exponential functions ..... 14
22. Integration by Substitution ..... 15
23. Integration by Parts ..... 15
24. Integration by Partial Fractions ..... 16
25. Applications of Integration ..... 16
V. Instructor Notes ..... 17

## I. Limits and Continuity

## 1. Review

1.1. Definition, and examples of Real numbers
1.2. Definition of $\mathbb{R}^{2}$
1.3. Definition of a real function
1.4. Examples (with graphs)
(i) $f(x)=x^{2}$
(ii) $f(x)= \begin{cases}2 & :-1 \leq x \leq 1 \\ x & : x<-1 \text { or } x>1\end{cases}$
(iii) $f(x)= \begin{cases}x & : x \neq 2 \\ 1 & : x=2\end{cases}$

## 2. Limits

2.1. Motivation
2.2. Definition of absolute value function
2.3. $\epsilon-\delta$ definition of limit
2.4. Examples from 1.4. above (drawing horizontal $\epsilon$-strip and vertical $\delta$-strip)
(End of Day 1)
2.5. Definition of left and right limits

### 2.6. Examples

(i) Limit exists iff left/right limit exist and are equal.
(ii) $f(x)= \begin{cases}2 & :-1 \leq x \leq 1 \\ x & : x<-1 \text { or } x>1\end{cases}$

## 3. Continuity

3.1. Definition of continuous function
3.2. Examples from 1.4.
3.3. Theorem on algebra of limits
3.4. Squeezing principle

More examples (continuing from above)
(i) Constant function
(ii) Identity function
(iii) $f(x)=x^{n}$ and then polynomials

## 4. Intermediate Value Theorem

### 4.1. Motivation

4.2. Completeness of $\mathbb{R}$
4.3. Examples of suprema of sets
(i) $S=[0,1]$
(ii) $S=(0,1)$
(iii) $S=\{0,1,2\}$
(iv) $S=\mathbb{N}$
4.4. Lemma : If $f$ continuous and $f(p) \neq 0$, then there is a neighbourhood of $p$ on which $f$ takes the same sign as $f(p)$.
4.5. Bolzano's theorem and proof
(End of Day 4)
4.6. Intermediate Value Theorem
4.7. Applications of IVT
(i) $a>0$ and $n \in \mathbb{N} \Rightarrow \exists b>0$ such that $b^{n}=a$
(ii) Isolating roots of a real polynomial (dictionary search algorithm)

## 5. Extreme Values of Continuous Functions

5.1. Definition of bounded function on $[a, b]$
(End of Day 5)
5.2. Theorem : Continuous function on $[a, b]$ is bounded
5.3. Definition of infimum, $\sup (f)$ and $\inf (f)$
5.4. Theorem (without proof) : Continuous function on $[a, b]$ attains sup and inf.

## 6. Examples and Problems

6.1. If $f$ continuous at $a$, then $\lim _{x \rightarrow a} f(x)=f(a)$
6.2. Rational function is continuous at $a$ if denominator is non-zero.
(End of Day 6)
6.3. Limit of rational function if denominator is zero
(i) Cancellation : $\lim _{x \rightarrow 1} \frac{\left(x^{2}-1\right)}{(x-1)}$
(ii) Rationalize : $\lim _{x \rightarrow 0} \frac{\sqrt{x^{2}+9}-3}{x^{2}}$
(iii) Squeezing principle : $\lim _{x \rightarrow 0} x^{2} \sin \left(\frac{1}{x}\right)$

## II. Differentiation

## 1. Motivation, and Definitions

1.1. Velocity of a projectile with distance given by $f(t)=144 t-16 t^{2}$
1.2. Slope of tangent line
1.3. Definition of derivative as limit of difference quotient
1.4. Examples
(i) $f(t)=144 t-16 t^{2}$
(ii) $f(x)=x^{2}$
(iii) $f(x)=|x|$
(iv) $f(x)=c$, where $c$ is a constant
(v) $f(x)=m x+b$, a line
1.5. Theorem : $f$ differentiable at $p \Rightarrow f$ continuous at $p$

## 2. Rules of Differentiation

2.1. Derivatives of $f+g, f-g, f g, f / g$
2.2. Examples
(i) $f(x)=x^{2}$
(ii) $f(x)=x^{n}$ (Proof by induction)
(End of Day 8)
(iii) $g=c f$ for $c$ a constant
(iv) $f$ a polynomial
(v) $f(x)=1 / x^{m}$ (comparing with part (ii))
(vi) $f(x)=x^{r}$ for $r$ a real number (without proof)
(vii) $f(x)=\sin (x)$ and $g(x)=\cos (x)$ (without proof)
2.3. Definition of composition of two functions
2.4. Examples
(i) $f(x)=\sin \left(x^{2}\right)$
(ii) $f(x)=\sqrt{1+x^{2}}$
2.5. Theorem : Chain rule
2.6. Examples
(i) $f(x)=\sin \left(x^{2}\right)$
(ii) $f(x)=[\sin (x)]^{2}$
(iii) $f(x)=\sqrt{r^{2}+x^{2}}$
(iv) $f(x)=-\sqrt{r^{2}+x^{2}}$. Explanation of implicit differentiation using $y^{2}=r^{2}+x^{2}$.
(v) Find $y^{\prime}$ if $6 x^{2}+14 y^{2}=3 y^{3}+4 x y$
(vi) Find $y^{\prime}$ if $\sin (x+y)=y^{2} \cos (x)$
(End of Day 9)

## 3. Mean-Value Theorem for Derivatives

3.1. Definitions of relative and absolute extrema
3.2. Theorem : If $c$ is relative extremum, then $f^{\prime}(c)=0$
3.3. Examples
(i) $f(x)=x^{3}$, then $f^{\prime}(0)=0$, but 0 is not an extremum.
(ii) $f(x)=|x|$, then 0 is an extremum but $f^{\prime}(0)$ does not exist.
3.4. Rolle's theorem
(End of Day 10)
3.5. Motivation for, and proof of Mean Value Theorem
3.6. Examples
(i) $|\sin (x)-\sin (y)| \leq|x-y| \forall x, y \in \mathbb{R}$
(ii) $f(x)=x^{3}+x-1$ has exactly one real root.
3.7. A real polynomial of degree $n$ has atmost $n$ real roots.
(End of Day 11)

## 4. Curve Sketching

4.1. Theorem : If $f^{\prime}(x)>0, f$ is increasing. If $f^{\prime}(x)<0, f$ is decreasing, and if $f^{\prime}=0, f$ is constant.
4.2. Definition of critical point
4.3. Theorem : First Derivative Test
4.4. Example : Sketch the graph of $f(x)=2 x^{3}-15 x^{2}+36 x+4$
(i) Find $f^{\prime}$
(ii) Find critical points
(iii) Classify critical points using first derivative test
(iv) Sketch the graph
(End of Day 12)
4.5. Definitions
(i) Second derivative
(ii) Concave up and Concave down, Points of inflection
4.6. Theorem : Second derivative test
4.7. Definition :
(i) Vertical asymptote
(ii) Horizontal asymptote
4.8. Examples :
(i) $f(x)=2 x^{3}+3 x^{2}-1$ has no vertical/horizontal asymptotes.
(ii) $f(x)=2 x^{2} /\left(x^{2}-1\right)$
(iii) $f(x)=2 x^{3} /\left(x^{2}-1\right)$
(iv) $f(x)=2 x /\left(x^{2}-1\right)$ with explanation of horizontal asymptotes for rational functions.
(v) Sketch the graph of $f(x)=\left(2 x^{2}-8\right) /\left(x^{2}-1\right)$
i. Find domain of $f$
ii. Find $X$ and $Y$ intercepts
iii. Find vertical and horizontal asymptotes
iv. Find $f^{\prime}$, critical points and classify them using the table
v. Find $f^{\prime \prime}$, points of inflection, and concavity using the table
vi. Sketch the graph
(End of Day 13)

## 5. Miscellaneous Topics

5.1. Theorem : L'Hospital's Rule for indeterminate form $0 / 0$ if functions are $C^{1}$
5.2. Example : $\lim _{x \rightarrow 0} \sin (x) / x$
5.3. Notation : $d y / d x$ and description of chain rule in this notation.
5.4. Examples :
(i) Related rates problem where gas is pumped into a spherical balloon at a rate of 50 cubic centimetres, and you are asked to find the rate of change of the radius when the radius is 5 cm .
(ii) The square has the largest area amongst rectangles of a fixed perimeter.
(End of Day 14)
Practice Problems for Quiz 1
(End of Day 15)

## III. Sequences and Series

## 1. Sequences

### 1.1. Definition of sequence

1.2. Examples
(i) $a_{n}=n$
(ii) $a_{n}=1$ for all $n$
(iii) $a_{n}=1 / n$
(iv) $a_{n}=(-1)^{n}$
(v) $a_{1}=1$ and $a_{n+1}=1+\frac{1}{1+a_{n}}$ for $n \geq 1$
(vi) $a_{1}=a_{2}=1$ and $a_{n+2}=a_{n+1}+a_{n}$ for $n \geq 1$
1.3. Definition of convergent, divergent sequence
1.4. Examples
(i) $a_{n}=n$, then $\left\{a_{n}\right\}$ diverges
(ii) $a_{n}=1$ for all $n$, then $\lim _{n \rightarrow \infty} a_{n}=1$
(iii) $a_{n}=1 / n$ for all $n$, then $\lim _{n \rightarrow \infty} a_{n}=0$
(iv) $a_{n}=(-1)^{n}$, then $\left\{a_{n}\right\}$ diverges (without proof)
(End of Day 16)
1.5. Theorem : Algebra of limits (without proof)
1.6. Squeezing principle (without proof)
1.7. Corollary : If $\lim \left|a_{n}\right|=0$, then $\lim a_{n}=0$
1.8. Examples
(i) $a_{n}=(-1)^{n} / n$
(ii) $a_{n}=n!/ n^{n}$, then $0<a_{n} \leq 1 / n$
1.9. Definition of increasing, decreasing and monotonic sequence
1.10. Example : $a_{n}=n /\left(n^{2}+1\right)$, then $a_{n}$ is decreasing (using $A(x)=x /\left(x^{2}+1\right)$ )
1.11. Definition of bounded above, below and bounded sequence
1.12. Theorem : If $\left\{a_{n}\right\}$ increasing and bounded above, then $\lim a_{n}=\sup \left\{a_{n}\right\}$. If $\left\{a_{n}\right\}$ decreasing and bounded below, then $\lim a_{n}=\inf \left\{a_{n}\right\}$
1.13. Theorem : If $f$ continuous and $\lim a_{n}=L$, then $\left\{f\left(a_{n}\right)\right\}$ converges and $\lim f\left(a_{n}\right)=f(L)$.
1.14. Examples
(i) (Practice Problem \#7 for Quiz 1) Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$
f(x+y)=f(x)+f(y)
$$

for all $x, y \in \mathbb{R}$. If $f$ continuous, then $f(x)=x f(1)$ for all $x \in \mathbb{R}$.
(ii) Suppose $a_{1}=a$ and $a_{n+1}=f\left(a_{n}\right)$ where $f$ continuous. Then if $\lim a_{n}=L$, then $f(L)=L$.
1.15. Aside : (Logical symbols)
(i) $\forall$
(ii) $\exists$
(iii) $\in$
(End of Day 18)

## 2. Infinite Series

### 2.1. Examples

(i) $a_{n}=n$, then $\sum a_{n}$ diverges because $s_{n}=n(n+1) / 2$
(ii) $a_{n}=(1 / 2)^{n}$ then $\sum a_{n}=1$ because $s_{n}=1-(1 / 2)^{n}$
2.2. Definition of partial sum, and sum of the series

### 2.3. Examples

(i) $a_{n}=n$ for all $n$, then $\sum a_{n}$ diverges
(ii) $a_{n}=(1 / 2)^{n}$, then $\sum a_{n}=1$
(iii) (Geometric Series) $a_{n}=a r^{n-1}$, then $\sum a_{n}=a /(1-r)$ if $|r|<1$ and $\sum a_{n}$ diverges if $|r| \geq 1$
(iv) $a_{n}=1 /\left(n^{2}+n\right)$, then $\sum a_{n}=1$ because $a_{n}=1 / n-1 /(n+1)$
(v) (Telescoping Series) If $\lim b_{n}=L$ and $a_{n}=b_{n}-b_{n+1}$, then $\sum a_{n}=b_{1}-L$
(vi) If $|x|<1$, then $1 /(1-x)=1+x+x^{2}+x^{3} \ldots=\sum x^{n}$. Short explanation of power series and the fact that

$$
\frac{1}{(1-x)^{2}}=1+2 x+3 x^{2}+\ldots
$$

(End of Day 19)
(vii) (Harmonic Series) $a_{n}=1 / n$, then $\sum a_{n}$ diverges because there is a subsequence $s_{2^{k}}$ of $s_{n}$ such that $s_{2^{k}} \geq 1+k / 2$ (Proof by induction)
2.4. Pictorial description of a series
2.5. Test for divergence
2.6. Example
(i) $a_{n}=1 / n$, then $\lim a_{n}=0$, but $\sum a_{n}$ diverges.
2.7. If $\sum a_{n}=A$ and $\sum b_{n}=B$, then $\sum\left(a_{n} \pm b_{n}\right)=A \pm B$. Warning $\sum\left(a_{n} b_{n}\right) \neq A B$ and $\sum a_{n} / b_{n} \neq A / B$
(i) $A=\sum a_{n}$ and $B=\sum b_{n}$, then $A=\lim s_{n}$ where $s_{n}=a_{1}+a_{2}+\ldots+a_{n}$ and $B=\lim t_{n}$, where $t_{n}=\sum_{i=1}^{n} b_{i}$, then $A B=\lim \left(s_{n} t_{n}\right) \neq \sum\left(a_{n} b_{n}\right)$
(ii) If $a_{n}=b_{n}=(1 / 2)^{n}$, then $\sum a_{n}=\sum b_{n}=1$, but $\sum\left(a_{n} b_{n}\right)=1 / 3$
(End of Day 20)
2.8. If $\left\{a_{n}\right\}$ convergent, then $\left\{a_{n}\right\}$ bounded.

## 3. Comparison Tests

### 3.1. Comparison Test

3.2. Limit comparison test
(End of Day 21)
3.3. Theorem : $p$-series converges if $p>1$ (without proof) and diverges if $p \leq 1$
3.4. Examples : $a_{n}$ is given by rational functions in $n$, then compare with a $p$-series.

## 4. Root and Ratio tests for series with non-negative terms

4.1. Note : If $0 \leq a_{n} \leq x^{n}$ for some fixed $x<1$, then $\sum a_{n}$ converges by comparison test. This motivates these two tests
4.2. Root test
4.3. Examples
(i) $a_{n}=2^{n} / n^{n}$
(ii) If $a_{n}=\left(b_{n}\right)^{n}$, then apply Root test
(End of Day 22)
4.4. Ratio test
4.5. Examples
(i) $a_{n}=2^{n} / n$ !
(ii) If $a_{n}$ has $n$ ! or (constant) ${ }^{n}$, then apply Ratio test

## 5. Series with negative terms

5.1. Alternating Series Test
5.2. Example : $a_{n}=(-1)^{n} / n$
5.3. Definition of Absolute Convergence
5.4. Theorem : Absolute convergence implies convergence.
5.5. Example : $a_{n}=(-1)^{n} / n$ is convergent, but not absolutely convergent

## IV. Integration

## 1. Definition of the Integral

1.1. Idea : Area under the curve of a continuous function $f:[a, b] \rightarrow \mathbb{R}$ with $f \geq 0$
(i) Divide the interval $[a, b]$ into $n$ pieces $a=x_{0}<x_{1}<x_{2}<\ldots<x_{n}=b$
(ii) Consider the Riemann sum $A_{n}(f)=\sum_{i=0}^{n-1} f\left(x_{i}\right) \Delta_{x_{i}}$
(iii) Take a limit to get $A(f)=\lim _{n \rightarrow \infty} A_{n}(f)$
1.2. Example : Area of a right-triangle
1.3. Definition
(i) Partition of an interval $[a, b]$
(ii) Step function
(iii) Integral of a step function. Note explaining the $\int$ and $d x$ notation
1.4. Example
(i) If $s(x)=M$ for all $x \in[a, b]$, then $\int_{a}^{b} s(x) d x=M(b-a)$
(End of Day 24)
(ii) $A_{n}(f)=\int_{a}^{b} s(x) d x$ for some step function $s:[a, b] \rightarrow \mathbb{R}$
1.5. Remark: If $s, t$ step functions, then $s+t$ step functions
1.6. Additivity of the integral of step functions
1.7. Homogeneity
1.8. Comparison theorem (Proof is homework)
1.9. Additivity of interval of integration
1.10. Translation of interval of integration
1.11. Contraction/Expansion of interval of integration
(End of Day 25)
1.12. Definition : Suppose $f:[a, b] \rightarrow \mathbb{R}$ is bounded
(i) Lower integral of $f$
(ii) Upper integral of $f$
(iii) $f$ is integrable if $\bar{I}(f)=\underline{I}(f)$
1.13. Examples
(i) If $f \equiv M$, then $f$ is integrable and $\int_{a}^{b} f(x) d x=M(b-a)$
(ii) If $f$ is a step function, then $\int_{a}^{b} f(x) d x$ has the same meaning as before.
(iii) Consider

$$
f(x)= \begin{cases}0 & : 0 \leq x \leq 1, x \neq 1 / 2 \\ 1 & : x=1 / 2\end{cases}
$$

Then $f$ is integrable and $\int_{a}^{b} f(x) d x=0$.
1.14. Remark :
(i) Definition of Lower Riemann sum $L(f, \mathcal{P})$ of $f$ w.r.t a a partition $\mathcal{P}$. Note: $\underline{I}(f)=$ $\sup L(f, \mathcal{P})$
(ii) Definition of Upper Riemann sum $U(f, \mathcal{P})$. Note: $\bar{I}(f)=\inf U(f, \mathcal{P})$
(End of Day 26)

## 2. Integrable functions

2.1. Theorem : Suppose that $\forall \epsilon>0, \exists$ a partition $\mathcal{P}$ such that $U(f, \mathcal{P})-L(f, \mathcal{P})<\epsilon$, then $f$ is integrable.
2.2. Theorem : Monotonic functions are integrable
2.3. Theorem : If $f$ increasing, and $I$ is such that $L\left(f, \mathcal{P}_{n}\right)<I<U\left(f, \mathcal{P}_{n}\right)$ where $\mathcal{P}_{n}$ is the even partition of $[a, b]$ into $n$ equal pieces, then $\int_{a}^{b} f(x) d x=I$
2.4. Example:
(i) If $f(x)=x^{k}$, then $\int_{0}^{1} f(x) d x=1 /(k+1)$
(ii) If $f(x)=x^{1 / k}$, then $\int_{0}^{1} f(x) d x=k /(k+1)$
(End of Day 27)
2.5. Note: Visual proof of Theorem 2.2
2.6. Theorem : Continuous functions are integrable (Proof later)

## 3. Properties of the Integral

### 3.1. Additivity

3.2. Homogeneity
3.3. Example: Polynomials over $[0,1]$
(End of Day 28)
3.4. Comparison Theorem
3.5. Additivity of the interval of integration (Proof is exercise)
3.6. Translation invariance (Proof is exercise)
3.7. Invariance under contraction/expansion (Proof is exercise)
3.8. Example: $\int_{a}^{b} x^{n} d x$

Proof of Theorem 2.5 :
(i) Definition : Span of a function w.r.t. a partition $\operatorname{Span}(f, \mathcal{P})$
(ii) If $f$ continuous, $\epsilon>0, \exists$ a a partition $\mathcal{P}$ such that $\operatorname{Span}(f, \mathcal{P})<\epsilon$
(End of Day 29)

## 4. Fundamental Theorem of Calculus

4.0 Goal: Given $f:[a, b] \rightarrow \mathbb{R}$, to find $\int_{a}^{b} f(x) d x$. We do this by finding $P$ such that $P(a)-P(b)=\int_{a}^{b} f(x) d x$
4.1. Definition of a primitive for $f: I \rightarrow \mathbb{R}$ continuous, $I \subset \mathbb{R}$ an interval
4.2. Example : $f(x)=x^{n}$, then $P(x)=x^{n+1} /(n+1)$
4.3. Remark:
(i) $P$ may not be unique. Hence, $a$ primitive and not the primitive
(ii) $P$ is unique upto a constant (proof later)
(iii) $I$ may not be closed (eg. $f(x)=1 / x$ and $I=(0, \infty)$ )
(iv) $I$ must be an interval as $a, b \in I$ must imply that $\int_{a}^{b} f(x) d x$ makes sense
4.4. Definition : Indefinite integral of $f$ (starting at $c \in I$ ). Note that the indefinite integral is a primitive of $f$
4.5. Examples:
(i) If $f(x)=x^{n}, I=\mathbb{R}$ and $c=0$, then $F(x)=x^{n+1} /(n+1)$
(ii) If $f(x)=x^{1 / n}, I=\mathbb{R}$, and $c=0$, then $F(x)=\frac{x^{1+1 / n}}{1+1 / n}$
(iii) If $f(x)=1 / x, I=(0, \infty)$ and $c=1$, then $F(x)=\ln (x)$
4.6. Definition: Average value of a continuous function over $[a, b]$
4.7. Mean-Value Theorem for Integrals
4.8. Fundamental Theorem of Calculus (proof later)
4.9. Examples:
(i) $F(x)=\frac{x^{1+1 / n}}{1+1 / n}$ is differentiable
(ii) $\ln (x)$

- $\ln (x)$ is differentiable and $[\ln (x)]^{\prime}=1 / x$
- $\ln (x)$ is strictly increasing on $(0, \infty)$
- $\ln (1)=0$ and $\exists N \in \mathbb{N}$ such that $\ln (N)>1$ (by comparing with the harmonic series), hence $\exists f \in(0, \infty)$ such that $\ln (f)=1$. We will show that $f=e$, as defined in Homework 7.5
(End of Day 30)
Proof of the Fundamental Theorem of Calculus
4.10. Theorem: $P$ is a primitive of $f$ if and only if $P^{\prime}=f$
4.11. Note: Any two primitives differ by a constant. Hence, we write $\int f(t) d t=P(x)+C$, and $\int_{a}^{b} f(t) d t=\left.P(x)\right|_{a} ^{b}$
4.12. Examples:
(i) $\int t^{n} d t=\frac{x^{n+1}}{n+1}+C$
(ii) $\int t^{1 / n} d t=\frac{x^{1+1 / n}}{1+1 / n}+C$. Hence, $x^{1 / n}$ is differentiable on $\mathbb{R} \backslash\{0\}$, and thus $x^{r}$ is differentiable for all $r \in \mathbb{Q}$ (See HW 4.6)
(iii) $\int \cos (t)=\sin (x)+C$
(iv) $\int \sin (t)=-\cos (x)+C$
4.13. Problems
(i) Find $\int_{1}^{2} 5 x^{2} d x$
(ii) Find $\int\left[t^{4 / 3}+5 \cos (t)\right] d t$
(iii) Find a continuous $f$ such that $\int_{0}^{x} f(t) d t=\int_{0}^{x} t^{2} f(t) d t+x^{2}$
(iv) Solve the Initial Value Problem: $f^{\prime}(x)=x^{3}$ for all $x \in \mathbb{R}$ and $f(0)=1$
(v) Find $f^{\prime}(x)$ if $f(x)=\int_{0}^{x^{2}} \frac{t^{6}}{1+t^{4}} d t$
(End of Day 31)


## 5. Logarithmic and Exponential functions

5.1. Definition of the natural logarithm function $\ln (x)$
5.2. Properties of $\ln (x)$ :
(i) $\ln (x)$ is continuous and differentiable
(ii) $[\ln (x)]^{\prime}=1 / x$ (by Fundamental Theorem of Calculus)
(iii) $\ln (1)=0$
(iv) $\ln (a b)=\ln (a)+\ln (b)$ for $a, b>0$
(v) $\ln \left(a^{n}\right)=n \ln (a)$ for all $n \in \mathbb{N}$
(vi) $\ln (1 / a)=-\ln (a)$
(vii) $\ln \left(a^{r}\right)=r \ln (a)$ for any $r \in \mathbb{Q}$
(viii) $\ln (e)=1$ where $e$ is defined in Homework 7.5
(ix) $\ln \left(e^{r}\right)=r$ for all $r \in \mathbb{Q}$
5.3. Graph of $\ln (x)$ and reflection about the line $y=x$
5.4. Definition of exponential function $\exp (x)$ as inverse of $\ln (x)$
5.5. Properties of $\exp (x)$ :
(i) $\exp (0)=1$
(ii) $\exp (1)=e$
(iii) $\exp (r)=e^{r}$ for any $r \in \mathbb{Q}$
(iv) $\exp (a+b)=\exp (a) \exp (b)$
(v) $\exp (a)^{b}=\exp (a b)$ [After Definition 5.6]
(vi) $\exp (x)$ is continuous (without proof, by inspection of the graph)
(vii) $[\exp (x)]^{\prime}=\exp (x)$
(viii) $\int \exp (t) d t=\exp (x)+C$
5.6. Definition $e^{x}=\exp (x)$ for any $x \in \mathbb{R}$. Definition of $a^{x}$ for $a>0$.
(End of Day 32)

## 6. Integration by Substitution

6.1. Remark: Chain rule for derivatives
6.2. Theorem: The substitution rule
6.3. Examples
(i) $\int \sqrt{1+2 x} d x$
(ii) $\int x^{2} \cos \left(x^{3}\right) d x$
(iii) $\int x e^{x^{2}} d x$
6.4. Remark: $\int \frac{1}{u} d u=\ln (|u|)+C$ on any interval $I$ s.t. $0 \notin I$. $\operatorname{Eg}: \int \tan (x) d x$
6.5. Trigonometric substitution
(i) $\int \frac{1}{\sqrt{1-x^{2}}} d x$
(ii) $\int \frac{1}{9+x^{2}} d x$
6.6. Substitution rule for Definite Integrals
6.7. Examples
(i) $\int_{1}^{e} \frac{\ln (x)}{x} d x$
(ii) $\int_{0}^{1}\left(1-x^{2}\right)^{n-1 / 2} d x=\int_{0}^{2 \pi} \cos ^{2 n}(\theta) d \theta$

## 7. Integration by Parts

7.1. Remark: Product rule for differentiation
7.2. Examples:
(i) $\int x \sin (x) d x$
(ii) $\int \ln (x) d x$
(iii) $\int x \sqrt{a x+b} d x$
7.3. Reduction/Recursion Formulae:
(i) $\int \cos ^{n}(\theta) d \theta$
(ii) $\int_{0}^{2 \pi} \cos ^{n}(\theta) d \theta$

### 7.4. Note: Similar formulae exist for

(i) $\int \sin ^{n}(\theta) d \theta$
(ii) $\int\left(a^{2}-x^{2}\right)^{n} d x$ with $x=a \sin (\theta)$ (without proof)

## 8. Integration by Partial Fractions

8.1. Examples
(i) $\int \frac{d x}{x^{2}-4}$
(ii) $\int \frac{d x}{\left(x^{2}+1\right)(x-3)}$
(End of Day 34)

## 9. Applications of Integration

### 9.1. Theorem: Integral test for series

9.2. Examples:
(i) Proof of Theorem III.3.3: If $p>1$, then $\sum \frac{1}{n^{p}}$ converges. Warning: Sum of series is not equal to $\int_{1}^{\infty} f(x) d x$ in general.
(ii) Alternate proof of divergence of $\sum 1 / n$
(iii) $\sum_{n=2}^{\infty} \frac{1}{n[\ln (n)]^{p}}$ converges if $p>1$
(End of Day 35)
9.3. Remark: Area between curves $f, g:[a, b] \rightarrow \mathbb{R}$ with $f \geq g$
9.4. Examples:
(i) Area enclosed by $y=e^{x}$ and $y=x$ between $x=0$ and $x=1$
(ii) Area enclosed by the curves $y=x^{2}$ and $y=2 x-x^{2}$ (by first finding points of intersection)
(iii) Area enclosed by $y=\sin (x)$ and $y=\cos (x)$ between $x=0$ and $x=\pi / 2$
9.5. Remark: Volume of a solid $S \subset \mathbb{R}^{3}$ as $V=\int_{a}^{b} A(x) d x$ where $A(x)$ is the area of the cross-section of $S$ by the plane $P_{x}$ perpendicular to the $X$-axis at ( $x, 0,0$ )
9.6. Examples:
(i) Volume of cylinder of base radius $r$ and height $h$
(ii) Volume of cone of base radius $r$ and height $h$
(iii) Volume of sphere of radius $r$
9.7. Note: (Volume of Revolution) The volume of the solid obtained by revolving a curve $y=f(x), a \leq x \leq b$ about the $X$-axis is $V=\pi \int_{a}^{b}[f(x)]^{2} d x$
(End of Day 36)
9.8. Examples: Find the volume of the solid obtained by rotating the region bounded by the given curves about the given line.
(i) Region bounded by the curves $y=\sqrt{x}, y=0, x=0, x=1$ about the $X$-axis.
(ii) Region bounded by the curves $y=x$ and $y=x^{2}$ about the $X$-axis.
(iii) Region bounded by $y=x^{3}, y=0, y=8$ about the $Y$-axis. (by slicing perpendicular to the $Y$-axis)
(iv) Region bounded by the curves $y=x$ and $y=x^{2}$ about the line $x=-1$
9.9. Examples: Other solids which are not solids of revolution
(i) Volume of a pyramid whose base is a square with side $L$ and height $h$
(End of Day 37)
Review for Final Exam - Chapters I, II, III
(End of Day 38)
Review for Final Exam - Chapter IV
(End of Day 39)

## V. Instructor Notes

The following are some of my observations after having taught this course.
0.1. The $\epsilon-\delta$ definition of the limit was very hard for the students. Perhaps this can be simplified by doing sequences before talking about limits of a function.
0.2. More time needs to be spent on suprema/infima. While the concepts seemed obvious to me, they were difficult for the students.
0.3 . Proofs of the different tests for convergence can be avoided. This will save some time, and allow us to focus on applications.
0.4 . Less time should have been spent on step functions, and more on the lower and upper Riemann sums.
0.5 . Doing the properties of the integral (3.1-3.7) in great detail is unnecessary. These properties can merely be stated and their proofs can be ignored.
0.6. Proof that a continuous function is integrable was very hard for the students to follow. This was perhaps my fault, but the difference between uniform continuity and continuity is difficult without more mathematical maturity.
0.7 . The $\log$ and $\exp$ functions need to be introduced earlier in the course. They provide an important source of examples and should not have been left till the end of the course.

