K-theory for C*-Algebras

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Contents

Introduction

Given a C*-algebra A, we associate two abelian groups $K_0(A)$ and $K_1(A)$ to A, in a functorial way. ie. Given a $*$-homomorphism $\varphi : A \to B$, we obtain induced group homomorphisms $\varphi_* : K_p(A) \to K_p(B)$. Furthermore, if φ is an isomorphism, so is φ_*. Hence, K-theory can be used to distinguish C*-algebras.

The goal of this course is to introduce K-theory to those who have seen the basics of C*-algebra theory (from [Murphy] or the equivalent). We will be following [Rørdam, Larsen, and Laustsen] almost verbatim.

I. Projections and Unitary Elements

1. Homotopy classes of Unitary elements

Definition 1.1. Let X be a topological space. We say two points $a, b \in X$ are homotopic (in symbols, $a \sim_h b$) if there is a continuous path $v : [0,1] \to X$ such that $v(0) = a$ and $v(1) = b$.

If such a path exists, we denote it by $t \mapsto v_t$ or $t \mapsto v(t)$.

Definition 1.2. Let A be a unital C*-algebra. An element $u \in A$ is called a unitary if $uu^* = u^*u = 1$. The group of unitaries is denoted by $\mathcal{U}(A)$. Write $\mathcal{U}_0(A)$ for all the elements $u \in \mathcal{U}(A)$ such that $u \sim_h 1$ in $\mathcal{U}(A)$. This is a normal subgroup of $\mathcal{U}(A)$.

Recall that if $u \in \mathcal{U}(A)$, then the spectrum $sp(u) \subset \mathbb{T}$. Given $h \in A$, we write $\exp(ih)$ for the element obtained by applying the continuous function $f(z) := \exp(iz)$ to h.

Lemma 1.3. *Let A be a unital C*-algebra.*

1.1. If $h \in A_{sa}$, then $\exp(ih) \in \mathcal{U}_0(A)$

1.2. If $u \in \mathcal{U}(A)$ such that $sp(u) \neq \mathbb{T}$, then $u \in \mathcal{U}_0(A)$

1.3. If $u, v \in \mathcal{U}(A)$ such that $\|u - v\| < 2$, then $u \sim_h v$.

Proof. 1.1. The path $t \mapsto \exp(ith)$ connects it to the identity.

1.2. If $sp(u) \neq \mathbb{T}$, then there is a continuous branch of the log function $\varphi : \mathbb{T} \to \mathbb{R}$ which satisfies $z = \exp(i\varphi(z))$. Then $h := \varphi(u)$ works.

1.3. eee $\|u - v\| < 2$, then $\|v^*u - 1\| < 2$, so $-1 \notin sp(v^*u)$. Therefore, $v^*u \in \mathcal{U}_0(A)$, so $u \sim_h v$.

□

Example 1.4. If $A = M_n(\mathbb{C})$, then $\mathcal{U}(A) = \mathcal{U}_0(A)$ because every $u \in \mathcal{U}(A)$ satisfies part (2) of the above theorem.

Example 1.5. The above proof gives us an interesting fact: If $A = \mathcal{B}(H)$, and $u \in \mathcal{U}(A)$, then there is a branch of the log function $\varphi : \mathbb{T} \to \mathbb{R}$ which, while not necessarily continuous, is at least a Borel function. Hence, $h := \varphi(u) \in A$ is a well-defined element, and clearly $u = \exp(ih)$ must hold. Hence, $u \in \mathcal{U}_0(A)$. Hence, $\mathcal{U}(A)$ is connected.

More generally, if A is a Von Neummann algebra, then $\mathcal{U}(A)$ is connected.

Proposition 1.6. *1.1. $\mathcal{U}_0(A)$ is a normal subgroup of $\mathcal{U}(A)$*

1.2. $\mathcal{U}_0(A)$ is open and closed in $\mathcal{U}(A)$

1.3. $u \in \mathcal{U}_0(A)$ iff $\exists h_1, h_2, \ldots, h_k \in A_{sa}$ such that

$$u = \exp(ih_1)\exp(ih_2)\ldots\exp(ih_k)$$

Proof. 1.1. Easy check.

1.2. If $u \in \mathcal{U}(A)$ such that $\|u - 1\| < 1$, then $u \in \mathcal{U}_0(A)$. Hence, as in the case on $GL(A)$, we can show that $\mathcal{U}_0(A)$ is open in $\mathcal{U}(A)$. Now $\mathcal{U}_0(A)$ is a subgroup, so $\mathcal{U}(A)$ is the disjoint union of its cosets. Each coset is homeomorphic to $\mathcal{U}_0(A)$, so each coset is open. Hence, $\mathcal{U}_0(A)$, being the complement of an open set, must also be closed in $\mathcal{U}(A)$.

1.3. Let F denote the set of finite products as above. This set is open because if $u \in F$ and $\|v - u\| < 2$, then $v \in F$ as in the previous lemma. Once again, F is a subgroup of $\mathcal{U}_0(A)$, so it must also be closed. Since $\mathcal{U}_0(A)$ is connected, it follows that $F = \mathcal{U}_0(A)$.

□

Corollary 1.7. *Let $\varphi : A \to B$ be a unital surjective $*$-homomorphism, then*

1.1. $\varphi(\mathcal{U}_0(A)) = \mathcal{U}_0(B)$

1.2. Let $u_1, u_2 \in \mathcal{U}(A)$ such that $u_1 \sim_h u_2$. If u_2 lifts to a unitary in $\mathcal{U}(A)$, then so does u_1.

Proof. 1.1. If $v \in \mathcal{U}_0(B)$, then write $v = \prod_{i=1}^n \exp(ik_j)$. Lift k_j to elements $h_j \in A$, and consider $t_j := (h_j + h_j^*)/2$. Then $t_j \in A_{sa}$, so $u := \prod_{i=1}^n \exp(it_j) \in \mathcal{U}_0(A)$, and clearly $\varphi(u) = v$.

1.2. Note that $u_1 u_2^* \in \mathcal{U}_0(A)$, so $\exists u \in \mathcal{U}_0(A)$ such that $\varphi(u) = u_1 u_2^*$. Now suppose $u_2 = \varphi(v)$, then $\varphi(uv) = u_1 u_2^* u_2 = u_1$.

□

Example 1.8. [Rørdam, Larsen, and Laustsen, Exercise 2.12] Consider the short exact sequence

$$0 \to C_0(\mathbb{R}^2) \to C(\mathbb{D}) \xrightarrow{\psi} C(\mathbb{T}) \to 0$$

where ψ is the restriction map. Let $v \in C(\mathbb{T})$ be the identity map, $v(z) = z$. Then there does not exist $u \in U(C(\mathbb{D}))$ such that $\psi(u) = v$.

Proof. Suppose $u \in U(C(\mathbb{D}))$ such that $\psi(u) = v$, then $u : \mathbb{D} \to \mathbb{T}$ is a continuous function such that $u|_{\mathbb{T}} = v$. Let $\iota : \mathbb{T} \to \mathbb{D}$ denote the inclusion map, then $u \circ \iota = \mathrm{id}_{\mathbb{T}}$. So the composition of maps induced on the fundamental group

$$\pi_1(\mathbb{T}, 1) \xrightarrow{\iota_*} \pi_1(\mathbb{D}, 1) \xrightarrow{u_*} \pi_1(\mathbb{T}, 1)$$

should be the identity map. But this is not possible because $\pi_1(\mathbb{T}, 1) = \mathbb{Z}$ while $\pi_1(\mathbb{D}, 1) = 0$. □

This also shows that $\mathcal{U}(C(\mathbb{T}))$ is not connected. Compare this with the earlier statement about Von Neumann Algebras.

(End of Day 1)

a. Lifting invertibles

Given a unital Banach algebra A, write $GL(A)$ for the set of invertibles in A, and $GL_0(A)$ for the set of all invertibles that are path connected to 1. Note that $GL(A)$ is open in A, and A is locally path connected, so path components in $GL(A)$ coincide with components in $GL(A)$. Hence, $GL_0(A)$ is the connected component of the identity in $GL(A)$. Hence, it is a normal subgroup of $GL(A)$

Definition 1.9. If A is a Banach algebra, and $a \in A$, we write

$$\exp(a) := \sum_{n=1}^{\infty} \frac{a^n}{n!}$$

Note that the series converges in A, and if $a, b \in A$ commute, then $\exp(a+b) = \exp(a)\exp(b)$. We write $\exp(A)$ for the set of all finite products of elements of the form $\exp(a)$. Note that $\exp(A) \subset GL(A)$

Lemma 1.10. *[Douglas, Lemma 2.13] If $\|1-a\| < 1$, then $a \in \exp(A)$. Hence, $\exp(A)$ is an open subset of $GL(A)$*

Proof. Define

$$b := \sum_{n=1}^{\infty} \frac{1}{n}(1-a)^n$$

Then the series converges absolutely, and so it converges in A, and $\exp(b) = a$ □

Theorem 1.11. *[Douglas, Theorem 2.14]*

$$GL_0(A) = \exp(A)$$

Proof. If $a \in A$, then $\exp(ta)$ defines a path from $\exp(a)$ to 1, so $\exp(A) \subset GL_0(A)$. Conversely, $\exp(A)$ is a subgroup of $GL_0(A)$ which is an open set by the previous lemma. Hence, every coset of $\exp(A)$ in $GL_0(A)$ is open, being homeomorphic to $\exp(A)$, so $\exp(A)$ is also closed in $GL_0(A)$. Since $GL_0(A)$ is connected, $\exp(A) = GL_0(A)$. □

Corollary 1.12. *If $\varphi : A \to B$ is a surjective unital $*$-homomorphism, and $b \in GL_0(B)$, then $\exists a \in GL_0(A)$ such that $\varphi(a) = b$*

Proof. Write $b = \prod_{i=1}^{n} \exp(b_i)$. Choose $a_i \in A$ such that $\varphi(a_i) = b_i$, and set $a := \prod_{i=1}^{n} \exp(a_i)$. □

Example 1.13. Let $S \in \mathcal{B}(\ell^2)$ be the right-shift operator

$$S((x_n)) := (0, x_1, x_2, \ldots)$$

Then T is the left-shift operator

$$T((x_n)) := (x_2, x_3, \ldots)$$

Hence, $TS = I$ and $ST = I - P_{e_1}$, where P_{e_1} is the projection onto the first coordinate. In particular, $ST - I \in \mathcal{K}(\ell^2)$, the compact operators.

Let $A := \mathcal{B}(\ell^2)$ and $B := \mathcal{Q}(\ell^2) := A/\mathcal{K}(\ell^2)$, the Calkin algebra, and let $\pi : A \to B$ be the quotient map. Then

$$\pi(S) \in GL(B) \text{ but } S \notin GL(A)$$

Moreover, suppose $R \in GL(A)$ such that $\pi(R) = \pi(S)$, then $S - R \in \mathcal{K}(\ell^2)$. However,

$$\text{index}(S) = \dim(\ker(S)) - \dim(\text{coker}(S)) = -1$$

If R is invertible, then

$$\text{index}(R) = \dim(\ker(R)) - \dim(\text{coker}(R)) = 0 - 0 = 0$$

But index is invariant under addition of compacts. See [Arveson, Chapter 3]. Hence, $\pi(S) \in GL(B)$ cannot be lifted to an invertible in $GL(A)$.

Given a surjective unital $*$-homomorphism $\varphi : A \to B$, we may lift elements in $GL_0(B)$ and $\mathcal{U}_0(B)$ to elements in $GL_0(A)$ and $\mathcal{U}_0(A)$ respectively. Therefore, we may lift (invertible or unitary) elements that are path connected to liftable elements. However, it is not, in general, possible to lift an arbitrary invertible or unitary.

b. Relationship between $GL(A)$ and $\mathcal{U}(A)$

Let A be a unital C*-algebra, then $\mathcal{U}(A)$ is a subgroup of $GL(A)$.

Definition 1.14. A subspace X_0 of a topological space X is said to be a retract of X if there is a continuous map $\tau : X \to X_0$ such that

1.1. $x \sim_h \tau(x)$

1.2. $\tau(x) = x$ for all $x \in X_0$.

Given $a \in A$, we write $|a| := (a^*a)^{1/2}$, and this is called the absolute value of a.

Proposition 1.15. *1.1. If $a \in GL(A)$, then $|a| \in GL(A)$ and $w(a) := a|a|^{-1} \in \mathcal{U}(A)$.*

1.2. The map $w : GL(A) \to \mathcal{U}(A)$ is a retract.

1.3. If $u, v \in \mathcal{U}(A)$ and $u \sim_h v$ in $GL(A)$, then $u \sim_h v$ in $\mathcal{U}(A)$

In order to prove this, we need a lemma

Lemma 1.16. *[Rørdam, Larsen, and Laustsen, Lemma 1.2.5] Let $K \subset \mathbb{R}$ be compact and $f : K \to \mathbb{C}$ continuous. Let A be a unital C^*-algebra and Ω_K be the set of all self-adjoint elements in A with spectrum contained in K. The induced map*

$$f : \Omega_K \to A \text{ given by } a \mapsto f(a)$$

is continuous.

Proof. Note that $a \mapsto a^n$ is continuous because multiplication is continuous by the Banach algebra identity. Hence, any polynomial is continuous. Now apply Stone-Weierstrass. $\square$

Now returning to the proof of the above proposition:

Proof. 1.1. If $a \in GL(A)$, so is a^*, so $(a^*a)^{1/2} \in GL(A)$ with inverse $(a^*a)^{-1/2}$. Now $u := a|a|^{-1}$ has the property that

$$u^*u = |a|^{-1}a^*a|a|^{-1} = 1 = uu^*$$

so $u \in \mathcal{U}(A)$

1.2. To show that ω is continuous: Note that $a \mapsto a^*$ is continuous and multiplication is continuous, so $a \mapsto a^*a$ is also continuous. The inverse map is continuous on $GL(A)$, so it suffices to show that $h \mapsto h^{1/2}$ is continuous on bounded sets of A^+. This follows from the previous lemma because a bounded set is contained in Ω_K where $K = [0, R]$ for some $R > 0$.

To see that ω is a retract: Let $a \in GL(A)$, then the path $w_t := \omega(a)(t|a| + (1-t)1_A)$ is continuous and $w_0 = \omega(a)$ and $w_1 = a$. To see that $w_t \in GL(A)$, note that $|a| \in GL(A) \cap A^+$, so $\exists \lambda \in (0, 1]$ such that $|a| \geq \lambda 1_A$. Hence,

$$t|a| + (1-t)1_A \geq \lambda 1_A$$

Hence, w_t is invertible, so $w(a) \sim_h a$ in $GL(A)$

1.3. If $u \sim_h v$ in $GL(A)$ via a path u_t, then $\omega(u_t)$ is a path in $\mathcal{U}(A)$ from u to v. $\square$

Remark 1.17. Let $X_0 \subset X$. We say that X_0 is a deformation retract of X if there is a retract $\tau : X \to X_0$ and a continuous map

$$H : [0, 1] \times X \to X$$

such that, for all $x \in X$

1.1. $H(x, 0) = x$

1.2. $H(x,1) = \tau(x)$

ie. H is a homotopy between the identity map on X and the map τ.

The above proof shows that $\mathcal{U}(A)$ is a deformation retract of $GL(A)$. A deformation retract is a special case of a homotopy equivalence, ie. the above proposition implies that the homotopy groups of $\mathcal{U}(A)$ and $GL(A)$ are isomorphic.

(End of Day 2)

c. Whitehead's Lemma

Lemma 1.18. *If $u, v \in A$, then*

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$$

in $\mathcal{U}_2(A)$. In particular,

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Proof. In $M_2(\mathbb{C})$,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

because $\mathcal{U}(M_2(\mathbb{C}))$ is connected. Hence,

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix}$$

and similarly the other claims also hold. $\square$

Corollary 1.19. *Let $\varphi : A \to B$ be a surjective unital $*$-homomorphism and $u \in \mathcal{U}(B)$, then $\exists v \in \mathcal{U}_2(A)$ such that*

$$\varphi_2(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$$

where $\varphi_2 : M_2(A) \to M_2(B)$ is the induced homomorphism.

2. Equivalence of Projections

A projection in a C*-algebra A is an element $p \in A$ such that $p = p^2 = p^*$.

Example 2.1. If $A = C(X)$, then a projection $p \in A$ must be the characteristic function of a cl-open set in X. In particular, if X is connected iff $C(X)$ has no non-trivial projections.

Definition 2.2. We say that two projections $p, q \in A$ are Murray-Von-Neumann equivalent if $\exists v \in A$ such that $p = v^*v$ and $q = vv^*$. Such a v is called a partial isometry, p its support projection, and q its range projection.

We check that this is an equivalence relation, and write $p \sim q$ for it. Furthermore, we have

$$v = qv = vp = qvp$$

Example 2.3. If $A = \mathcal{B}(H)$ and $p, q \in A$ projections, then $p \sim q$ iff $\dim(p(H)) = \dim(q(H))$.

Proof. 2.1. Consider $V = p(H), W = q(H)$. Suppose $\dim(V) = \dim(W)$, then there is an isomorphism $v : V \to W$ obtained by mapping orthonormal bases to each other. Extending this map to an element of $\mathcal{B}(H)$ by defining it to be zero on the orthogonal complements, we obtain a partial isometry such that $v^*v = p$ and $vv^* = q$.

2.2. Conversely, if $p \sim q$, then any orthonormal basis of $p(H)$ must be carried to an orthonormal basis of $q(H)$ by the partial isometry v. Then choose an orthonormal basis B of $p(H)$. We claim that $\{v(b) : b \in B\}$ is an orthonormal basis for $q(H)$. Firstly, note that

$$qv(b) = v(b) \Rightarrow v(b) \in q(H)$$

Now if $b, b' \in B$, then

$$\langle v(b), v(b') \rangle = \langle v^*v(b), b' \rangle = \langle q(b), b' \rangle = \langle b, b' \rangle = \delta_{b,b'}$$

Furthermore, if $e \in q(H)$ is such that $e \perp v(b)$ for all $b \in B$, then

$$\langle v^*(e), b \rangle = \langle e, v(b) \rangle = 0 \quad \forall b \in B$$

As before, $v^*(e) \in p(H)$ so this implies $v^*(e) = 0$, whence

$$e = q(e) = vv^*(e) = 0$$

This proves the claim.

□

Example 2.4. In particular, if $A = M_n(\mathbb{C})$, then for any two projections $p, q \in A$,

$$p \sim q \Leftrightarrow Tr(p) = Tr(q)$$

There are two more equivalence relations on projections. For any C*-algebra A, write $\widetilde{A}$ for its unitization.

Definition 2.5. We say that two projections $p, q \in A$ are unitarily equivalent (In symbols, $p \sim_u q$) if $\exists u \in \mathcal{U}(\widetilde{A})$ such that $p = uqu^*$. We say that they are homotopic (In symbols, $p \sim_h q$) if there is a path $t \mapsto p_t$ of projections connecting p to q.

Proposition 2.6. *Let A be a unital C^*-algebra, and $p, q \in A$ projections. Then TFAE:*

2.1. $p \sim_u q$

2.2. $q = upu^*$ *for some* $u \in \mathcal{U}(A)$

2.3. $p \sim q$ *and* $1_A - p \sim 1_A - q$

Proof. Write $\widetilde{A} = A + \mathbb{C}f$ where $f = (1_{\widetilde{A}} - 1_A)$, and note that $af = fa = 0$ for all $a \in A$.

(i)⇒ (ii): Suppose $q = zpz^*$ for some $z \in \mathcal{U}(\widetilde{A})$, write $z = u + \lambda f$ for some $u \in A$ and $\lambda \in \mathbb{C}$. Then

$$1_{\widetilde{A}} = zz^* = uu^* + |\lambda|^2 f^2 = uu^* + |\lambda|^2 f$$

But $1_{\widetilde{A}} = 1_A + f$, so we have that $uu^* = 1_A$ by equating terms. Similarly, $u^*u = 1_A$.

(ii) ⇒ (iii): Suppose $q = upu^*$, write

$$v := up \text{ and } w = u(1_A - p)$$

Then

$$vv^* = q, v^*v = p \text{ and } w^*w = (1_A - p), ww^* = (1_A - q)$$

(iii) ⇒ (i): Suppose $v, w \in A$ are partial isometries satisfying the above relations. Set

$$z := v + w + f$$

Then $vv^* + ww^* + ff^* = v^*v + w^*w + f^*f = 1_{\widetilde{A}}$, so by Exercise 2.6, $z \in \mathcal{U}(\widetilde{A})$. Furthermore, $wv^* = v^*w = 0$, and $fp = pf = 0$, so

$$zpz^* = vpv^* = vv^* = q$$

□

Example 2.7. Let $A = \mathcal{B}(H)$ and $p \in \mathcal{B}(H)$ be any projection whose range is infinite dimensional. Then by the earlier example, $p \sim 1_A$. However, if $(1_A - p) \neq 0$, then p cannot be unitarily equivalent to 1_A. For instance, if $p \in \mathcal{B}(\ell^2)$ is the projection whose range is $\{e_1\}^\perp$, then

$$p \sim 1_A \text{ but } p \nsim_u 1_A$$

Proposition 2.8. $p \sim_h q$ *iff* $\exists u \in \mathcal{U}_0(\widetilde{A})$ *such that* $q = upu^*$

Proof. For one direction, if $q = upu^*$ with $u \in \mathcal{U}_0(\widetilde{A})$, then there is a path u_t connecting u to $1_{\widetilde{A}}$. Therefore,

$$t \mapsto u_t p u_t^*$$

is a path of projections connecting p to q.

Conversely, suppose $p \sim_h q$ via a path p_t, then choose a partition $\{t_0, t_1, \ldots, t_n\}$ of $[0, 1]$ such that

$$\|p_{t_i} - p_{t_i+1}\| < 1/2$$

It now suffices to assume that $\|p - q\| < 1/2$ as the required property is transitive. In this case, take

$$z := pq + (1-p)(1-q) \in \widetilde{A}$$

Then $pz = pq = zq$, and $\|z - 1\| \leq 2\|p - q\| < 1$. Hence, $z \in GL(\widetilde{A})$ and $z \sim_h 1_{\widetilde{A}}$ in $GL(\widetilde{A})$. Let $u := \omega(z) \in \mathcal{U}(\widetilde{A})$, then

$$u \sim_h z \Rightarrow u \in \mathcal{U}_0(\widetilde{A})$$

and also $p = uqu^*$ (See [Rørdam, Larsen, and Laustsen, Proposition 2.2.5]) □

We have the following implications:

$$p \sim_h q \Rightarrow p \sim_u q \Rightarrow p \sim q$$

The reverse implications do not hold. But the following do hold

$$p \sim q \Rightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$p \sim_u q \Rightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$$

Example 2.9. To see that $p \sim q$ does not necessarily imply $p \sim_u q$, look at the previous example with $A = \mathcal{B}(H)$. To see that $p \sim_u q$ does not necessarily imply $p \sim_h q$, we need the following fact: There exists a C*-algebra B and a unitary $u \in \mathcal{U}(M_2(B))$ such that

$$u \nsim_h \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}$$

for any $v \in \mathcal{U}(B)$. If we assume this, then we may choose

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } q := upu^*$$

Then one can show that $p \nsim_h q$, while clearly $p \sim_u q$

Now see [Rørdam, Larsen, and Laustsen, Proposition 2.2.8].

(End of Day 3)

II. The K_0 group of a unital C*-algebra

1. Definition

a. The Grothendieck construction

Compare the following construction with the construction of the field of fractions of an integral domain. Let $(S,+)$ be an abelian semi-group. Define an equivalence relation $\sim$ on $S \times S$ by

$$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow \exists z \in S \text{ such that } x_1 + y_2 + z = x_2 + y_1 + z$$

Write $G(S) := (S \times S)/\sim$, and let $\langle x, y\rangle$ denote the equivalence class of $(x,y) \in S \times S$. Define

$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle := \langle x_1 + x_2, y_1 + y_2\rangle$$

Then $(G(S),+)$ is an abelian group where $0_G = \langle y, y\rangle$ for any $y \in S$ and $-\langle x, y\rangle = \langle y, x\rangle$.

Fix $y \in S$, and define $\gamma_S : S \to G(S)$ by

$$x \mapsto \langle x + y, y\rangle$$

This map independent of y and is additive. This construction has the following properties:

1.1. Universal Property: Given an abelian group H and an additive map $\varphi : S \to H$, there is a unique group homomorphism $\widehat{\varphi} : G(S) \to H$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[rr, "\varphi"] \arrow[dr, "\gamma_S"'] & & H \\ & G(S) \arrow[ur, "\widehat{\varphi}"'] & \end{tikzcd}$$

1.2. Functoriality: Given an additive map $\varphi : S \to T$ between abelian semigroups, there is a unique group homomorphism $G(\varphi) : G(S) \to G(T)$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "\gamma_S"'] & T \arrow[d, "\gamma_T"] \\ G(S) \arrow[r, "G(\varphi)"] & G(T) \end{tikzcd}$$

1.3. $G(S) = \{\gamma_S(x) - \gamma_S(y) : x, y \in S\}$

We won't prove these results. They are easy. However, note that they have the following consequence: Given another pair (G, ψ_S) where G is an abelian group and

$$\psi_S : S \to G$$

an additive map such that these properties hold, then, $\exists \widehat{\psi_S} : G(S) \to G$ such that

$$\widehat{\psi_S} \circ \gamma_S = \psi_S$$

Similarly, there is a map $\widehat{\gamma_S} : G \to G(S)$ such that

$$\widehat{\gamma_S} \circ \psi_S = \gamma_S$$

Hence, $\theta := \widehat{\gamma_S} \circ \widehat{\psi_S}$ is a group homomorphism from $G(S)$ to $G(S)$ and has the property that

$$\theta \circ \gamma_S = \gamma_S$$

But the image of γ_S generates $G(S)$, so $\theta = \mathrm{id}_{G(S)}$. Similarly,

$$\widehat{\psi_S} \circ \widehat{\gamma_S} = \mathrm{id}_G$$

and so $G \cong G(S)$. Thus, the pair $(G(S), \gamma_S)$ is unique. Hence,

Given an abelian semi-group $(S, +)$, if we find one pair $(G, \times)$ with properties (i), (ii) and (iii), then it must be the Grothendieck completion of $(S, +)$.

Example 1.1. If $S = \mathbb{Z}^+$, then $G(S) \cong \mathbb{Z}$

Example 1.2. If $S = \mathbb{Z}^+ \cup \{\infty\}$, where addition with ∞ is as usual, then for any $x \in S$,

$$\gamma_S(x) + \gamma_S(\infty) = \gamma_S(x + \infty) = \gamma_S(\infty)$$

But $G(S)$ has cancellation, so $\gamma_S(x) = 0_{G(S)}$. This is true for any $x \in S$, so $G(S) = \{0\}$.

Remark 1.3. The map $\gamma_S : S \to G(S)$ need not be injective as the above example shows. In fact, it is injective iff S has cancellation: ie. $x + z = y + z$ implies that $x = y$ in S. (proof later)

b. Semigroups of Projections

Fix a C*-algebra A. For each $n \in \mathbb{N}$, write $\mathcal{P}_n(A)$ for the set of all projections in $M_n(A)$, and write $\mathcal{P}_\infty(A)$ for the disjoint union. Given $p, q \in \mathcal{P}_\infty(A)$, define

$$p \oplus q := \mathrm{diag}(p, q) := \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

Furthermore, if $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$, we say $p \sim_0 q$ if $\exists v \in M_{m,n}(A)$ such that $p = v^*v$ and $q = vv^*$. We have

Proposition 1.4. *[Rørdam, Larsen, and Laustsen, Proposition 2.3.2]*

1.1. If $n = m$, then $p \sim_0 q$ iff $p \sim q$

1.2. $p \sim_0 p \oplus 0_n$ for all $n \in \mathbb{N}$

1.3. If $p \sim_0 p'$ and $q \sim_0 q'$, then $p \oplus q \sim_0 p' \oplus q'$

1.4. $p \oplus q \sim_0 q \oplus p$

1.5. If $n = m$ and $pq = 0$, then $p + q$ is a projection, and $p + q \sim_0 p \oplus q$

1.6. $p \oplus (q \oplus r) = (p \oplus q) \oplus r$

Proof. We don't prove all these statements. Here is a sample.

(ii) If $p \sim_0 p'$ and $q \sim_0 q'$, write

$$p = v^*v, p' = vv^*, q = w^*w, q' = ww^*$$

Then $u := \mathrm{diag}(v, w)$ is such that

$$p \oplus q = u^*u \text{ and } p' \oplus q' = uu^*$$

□

Definition 1.5. Define

$$\mathcal{D}(A) := \mathcal{P}_\infty(A)/\sim_0$$

Write $[p]_\mathcal{D}$ for the equivalence class of an element, and define

$$[p]_\mathcal{D} + [q]_\mathcal{D} := [p \oplus q]_\mathcal{D}$$

This is a well-defined addition on $\mathcal{D}(A)$, making it into an abelian semi-group.

Definition 1.6. The K_0 group of a unital C*-algebra A is defined as

$$K_0(A) := G(\mathcal{D}(A))$$

We write $[p]_0 := \gamma([p]_\mathcal{D})$ for any $p \in \mathcal{P}_\infty(A)$.

Proposition 1.7 (The standard picture of K_0 - the unital case)**.** *Let A be a unital C*-algebra, then*

$$K_0(A) = \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_\infty(A)\} = \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_n(A), n \in \mathbb{N}\}$$

Moreover,

1.1. $[p \oplus q]_0 = [p]_0 + [q]_0$

1.2. $[0_A] = 0$ where 0_A is the zero projection in A

1.3. If $p, q \in \mathcal{P}_n(A)$ and $p \sim_h q$ in $\mathcal{P}_n(A)$, then $[p]_0 = [q]_0$

1.4. *If $p, q \in \mathcal{P}_n(A)$ such that $p \perp q$, then $[p+q]_0 = [p]_0 + [q]_0$*

Proof. The first description of K_0 follows from Property 1.3 of the Grothendieck construction. Furthermore, if

$$g = [p]_0 - [q]_0$$

where $p \in \mathcal{P}_m(A)$ and $q \in \mathcal{P}_\ell(A)$, then let $n := \max\{m, \ell\}$ and replace p, q by $p' = p \oplus 0_{n-m}$ and $q' = q \oplus 0_{n-\ell}$ respectively. Then $p \sim_0 p'$ and $q \sim_0 q'$ so

$$g = [p']_0 - [q']_0$$

and $p', q' \in \mathcal{P}_n(A)$

1.1.
$$[p \oplus q]_0 = \gamma([p \oplus q]_{\mathcal{D}}) = \gamma([p]_{\mathcal{D}} + [q]_{\mathcal{D}}) = \gamma([p]_{\mathcal{D}}) + \gamma([q]_{\mathcal{D}}) = [p]_0 + [q]_0$$

1.2. Since $0_A \oplus 0_A \sim_0 0_A$, we have $[0_A]_0 + [0_A]_0 = [0_A]_0$ whence $[0_A]_0 = 0$

1.3. Because
$$p \sim_h q \Rightarrow p \sim q \Rightarrow p \sim_0 q \Rightarrow [p]_{\mathcal{D}} = [q]_{\mathcal{D}} \Rightarrow [p]_0 = [q]_0$$

1.4. As in 1.1 because $p+q \sim_0 p \oplus q$ by [Rørdam, Larsen, and Laustsen, Proposition 2.3.2]

□

c. Stable Equivalence of Projections

Remark 1.8. Let $(S, +)$ be an abelian semigroup and $(G(S), \gamma_S)$ the corresponding Grothendieck group. Then, for all $x, y \in S$,

$$\gamma_S(x) = \gamma_S(y) \Leftrightarrow x + z = y + z$$

for some $z \in S$

Proof. Write $\gamma_S(x) = \langle x + u, u \rangle$. If $x + z = y + z$, then

$$x + u + u + z = y + u + u + z \Rightarrow \langle x + u, u \rangle = \langle y + u, u \rangle$$

Conversely, if $\gamma_S(x) = \gamma_S(y)$, then $\exists z' \in S$ such that

$$x + u + u + z' = y + u + u + z'$$

so take $z := u + u + z'$ □

Hence, $\gamma_S : S \to G(S)$ is injective iff S has cancellation: ie. $x + z = y + z$ implies that $x = y$

Definition 1.9. We say two projections $p, q \in \mathcal{P}_\infty(A)$ are stably equivalent if $\exists r \in \mathcal{P}_\infty(A)$ such that

$$p \oplus r \sim_0 q \oplus r$$

If this happens, we write $p \sim_s q$.

Note that if A is unital, then replacing r by $r \oplus (1_n - r) \sim_0 1_n$, we see that

$$p \sim_s q \Leftrightarrow p \oplus 1_n \sim_0 q \oplus 1_n$$

for some $n \in \mathbb{N}$.

Lemma 1.10. *For any two projections $p, q \in \mathcal{P}_\infty(A)$,*

$$[p]_0 = [q]_0 \Leftrightarrow p \sim_s q$$

(End of Day 4)

d. Universal Property of K_0

Proposition 1.11 (Universal Property of K_0)**.** *Let A be a unital C^*-algebra, G an abelian group, and*

$$\nu : \mathcal{P}_\infty(A) \to G$$

be a function such that

1.1. $\nu(p \oplus q) = \nu(p) + \nu(q)$ *for all* $p, q \in \mathcal{P}_\infty(A)$

1.2. $\nu(0_A) = 0$

1.3. *If* $p, q \in \mathcal{P}_n(A)$ *and* $p \sim_h q$ *in* $\mathcal{P}_n(A)$*, then* $\nu(p) = \nu(q)$

Then $\exists$ a unique group homomorphism

$$\alpha : K_0(A) \to G$$

such that

$$\alpha([p]_0) = \nu(p)$$

for all $p \in \mathcal{P}_\infty(A)$. ie The following diagram commutes

$$\begin{tikzcd}
\mathcal{P}_\infty(A) \arrow[d, "{[\cdot]_0}"'] \arrow[dr, "\nu"] & \\
K_0(A) \arrow[r, "\alpha"] & G
\end{tikzcd}$$

Proof. If $p, q \in \mathcal{P}_\infty(A)$ are projections such that $p \sim_0 q$, then we claim that $\nu(p) = \nu(q)$. To see this, note that if $p \sim_0 q$, then choose $n \in \mathbb{N}$ such that

$$p' := p \oplus 0_{n-m} \text{ and } q' := q \oplus 0_{n-\ell}$$

are both in $\mathcal{P}_n(A)$. Then
$$p' \sim_0 q' \Rightarrow p' \sim q'$$
By earlier propositions,
$$p' \oplus 0_{3n} \sim_h q' \oplus 0_{3n}$$
so that
$$\nu(p) = \nu(p) + \underbrace{\nu(0) + \ldots + \nu(0)}_{4n-m} = \nu(p' \oplus 0_{3n}) = \nu(q' \oplus 0_{3n}) = \nu(q)$$
The result now follows from the universal property of the Grothendieck construction. $\square$

Example 1.12. A bounded linear map $\tau : A \to \mathbb{C}$ is called a trace if
$$\tau(ab) = \tau(ba) \quad \forall a, b \in A$$

1.1. Hence, if $p, q \in A$ are projections such that $p \sim q$, then $\tau(p) = \tau(q)$.
1.2. If τ is a trace on A and $n \in \mathbb{N}$, then define $\tau_n : M_n(A) \to \mathbb{C}$ by
$$\tau_n((a_{i,j})) = \sum_{i=1}^{n} \tau(a_{i,i})$$
and this is trace on $M_n(A)$ (HW)
1.3. Thus, we get an induced function
$$\tau : \mathcal{P}_\infty(A) \to \mathbb{C}$$
which satisfies the above conditions. Hence, we get an induced function
$$K_0(\tau) : K_0(A) \to \mathbb{C} \text{ such that } K_0(\tau)([p]_0) = \tau(p)$$
for any projection $p \in M_n(A)$.
1.4. A trace is said to be positive if $\tau(a) \geq 0$ for all $a \in A_+$ positive. In this case, each induced map τ_n is also positive, so $K_0(\tau)$ maps $K_0(A)$ to $\mathbb{R}$.

Example 1.13.
$$K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$$

Proof. Let $A = M_n(\mathbb{C})$, $Tr : A \to \mathbb{C}$ denote the standard trace. This is a positive trace, so it induces a map
$$K_0(Tr) : K_0(M_n(\mathbb{C})) \to \mathbb{R}$$
Furthermore, $Tr(p) \in \mathbb{Z}$ for all projections $p \in M_m(A)$, so we may restrict the range so
$$K_0(Tr) : K_0(M_n(\mathbb{C})) \to \mathbb{Z}$$

1.1. $K_0(Tr)$ is injective: If $g = [p]_0 - [q]_0$ is such that $K_0(Tr)(g) = 0$, then we may assume that $p, q \in \mathcal{P}_m(A)$ for some $m \in \mathbb{N}$. Hence,

$$Tr(p) = Tr(q)$$

By Example 2.4, this implies $p \sim q$, so $p \sim_0 q$, so $g = [p]_0 - [q]_0 = 0$.

1.2. $K_0(Tr)$ is surjective: Because if $e \in A$ denotes a rank one projection, then

$$K_0(Tr)([e]) = 1$$

$\square$

Example 1.14. If $H = \ell^2$, then $K_0(\mathcal{B}(H)) = \{0\}$

Proof. Let $A = \mathcal{B}(H)$, and define $\dim : \mathcal{P}_\infty(A) \to \mathbb{N} \cup \{\infty\}$ by

$$\dim(p) := \dim(p(H^n))$$

where we think of $p \in \mathcal{P}_n(A) \cong \mathcal{P}(\mathcal{B}(H^n))$. By earlier examples,

$$p \sim_0 q \Leftrightarrow \dim(p) = \dim(q)$$

so this is a bijection. Furthermore,

$$\dim(p \oplus q) = \dim(p) + \dim(q)$$

so it is an isomorphism of semigroups. Hence,

$$K_0(A) \cong G(\mathbb{N} \cup \{\infty\}) = \{0\}$$

$\square$

Note that the same is also true for $\mathcal{B}(H)$ if H is not separable.

Example 1.15. Let X be a connected compact Hausdorff space and $A = C(X)$. For $x \in X$, define

$$\nu_x : \mathcal{P}_\infty(A) \to \mathbb{Z} \text{ given by } p \mapsto Tr(p(x))$$

where we think of p as an element of $C(X, M_n(\mathbb{C})) \cong M_n(C(X))$. As before, this induces a map

$$K_0(\nu_x) : K_0(C(X)) \to \mathbb{Z}$$

If $\mathbf{1}$ denotes the unit in A, then $K_0(\nu_x)([\mathbf{1}]_0) = 1$, so this map is surjective.

Note that if $p \in \mathcal{P}_\infty(A)$ is fixed, then the map

$$X \to \mathbb{Z} \text{ given by } x \mapsto Tr(p(x))$$

is continuous, because $p \in C(X, M_n(\mathbb{C}))$. Since X is connected, this is constant, and so the map $K_0(\nu_x)$ is independent of x. We denote this map by

$$\dim : K_0(C(X)) \to \mathbb{Z}$$

This map is surjective, but not, in general, injective [For instance, it is not injective if $X = S^2$ or $X = \mathbb{T}^2$.] However, it is injective if X is totally disconnected: A space X is totally disconnected if it has a basis of cl-open sets. See [Rørdam, Larsen, and Laustsen, Exercise 3.4]

If A is unital, $K_0(A)$ is generated by projections in matrices over A. The universal property of $K_0(\cdot)$ is that any function on A which extends to $M_n(A)$ and respects Murray von Neumann equivalence of projections induces a map at the level of $K_0(A)$. One such function is a trace on the algebra.

2. Functoriality of K_0

a. Categories and Functors

A category C consists of a class $\mathcal{O}(C)$ of objects, and for each pair $A, B \in \mathcal{O}(C)$ a set $\mathrm{Mor}(A, B)$ of morphisms from A to B with an associative rule of composition

$$\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C) \text{ denoted by } (\varphi, \psi) \mapsto \psi \circ \varphi$$

such that, for each object A, there is a morphism $\mathrm{id}_A \in \mathrm{Mor}(A, A)$ such that

$$\varphi \circ \mathrm{id}_A = \varphi = \mathrm{id}_B \circ \varphi$$

for all $\varphi \in \mathrm{Mor}(A, B)$. We are concerned with two categories, $C^* - alg$ and Ab of C*-algebras and abelian groups respectively.

A covariant functor from a category C to a category D is a map $F : \mathcal{O}(C) \to \mathcal{O}(D)$ denoted by $A \mapsto F(A)$ and a collection of maps $\varphi \mapsto F(\varphi)$ from $\mathrm{Mor}(A, B) \to \mathrm{Mor}(F(A), F(B))$ such that

2.1. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all $A \in \mathcal{O}(C)$
2.2. $F(\psi \circ \varphi) = F(\psi) \circ F(\varphi)$ for all morphisms $\varphi \in \mathrm{Mor}(B, C)$ and $\varphi \in \mathrm{Mor}(A, B)$

A contravariant functor is similar, except the arrows are reversed: Given a morphism $\varphi \in \mathrm{Mor}(A, B)$, we get a morphism $F(\varphi) \in \mathrm{Mor}(F(B), F(A))$.

Example 2.1. 2.1. $X \mapsto \pi_1(X)$ is a covariant functor from the category of topological spaces to the category of groups.

2.2. $S \mapsto G(S)$ is a covariant functor from the category of abelian semigroups to the category of abelian groups.

2.3. $X \mapsto C(X)$ is a contravariant functor from the category of compact Hausdorff spaces to the category of unital commutative C*-algebras. Gelfand-Naimark simply states that this is an equivalence of categories.

Definition 2.2. Let $\varphi : A \to B$ be a $*$-homomorphism, then it extends to a $*$-homomorphism $\varphi_n : M_n(A) \to M_n(B)$. This induces a map $\varphi : \mathcal{P}_\infty(A) \to \mathcal{P}_\infty(B)$ so the map $\nu : \mathcal{P}_\infty(A) \to K_0(B)$ given by

$$p \mapsto [\varphi(p)]_0$$

satisfies all the conditions above, and so factors through $K_0(A)$ to give a map

$$K_0(\varphi) : K_0(A) \to K_0(B)$$

such that

$$K_0(\varphi)[p] = [\varphi(p)]_0 \quad \forall p \in \mathcal{P}_\infty(A)$$

Let $\{0\}$ denote the 0 C*-algebra, and $0_{A,B}$ denote the zero morphism $A \to B$.

Proposition 2.3. *2.1.* $K_0(id_A) = id_{K_0(A)}$

2.2. $K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$

2.3. $K_0(\{0\}) = \{0\}$

2.4. $K_0(0_{B,A}) = 0_{K_0(B),K_0(A)}$

Proof. $K_0(A)$ is generated by $[p]_0$ for $p \in \mathcal{P}_\infty(A)$, so these facts are obvious from the definition above. $\square$

(End of Day 5)

b. Homotopy Invariance

Definition 2.4. Two $*$-homomorphisms $\varphi, \psi : A \to B$ are said to be homotopic if there are $*$-homomorphisms $\varphi_t : A \to B$ for each $t \in [0,1]$ such that

2.1. $\varphi_0 = \varphi$ and $\varphi_1 = \psi$

2.2. For each $a \in A$, the map $t \mapsto \varphi_t(a)$ is continuous from $[0,1]$ to B.

Equivalently, there is a $*$-homomorphism

$$\Phi : A \to C([0,1], B)$$

such that $ev_0 \circ \Phi = \varphi$ and $ev_1 \circ \Phi = \psi$, where ev_s the evaluation map

$$C([0,1], B) \to B \text{ given by } f \mapsto f(s)$$

If this happens, we write $\varphi \sim_h \psi$.

We say that A and B are homotopy equivalent if $\exists$ $*$-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that

$$\psi \circ \varphi \sim_h \mathrm{id}_A \text{ and } \varphi \circ \psi \sim_h \mathrm{id}_A$$

If this happens, we write $A \simeq B$

Example 2.5.
Let $A = C[0,1]$ and $B = \mathbb{C}$. Define $\varphi : A \to B$ by $f \mapsto f(0)$ and $\psi : B \to A$ by $\lambda \mapsto \lambda \mathbf{1}$. Then $\varphi \circ \psi = \mathrm{id}_B$. Define $\Phi : A \to C([0,1]^2) = C([0,1], A)$ by

$$\Phi(f)(t,x) = f(tx)$$

Then $ev_0 \circ \Phi(f) \equiv f(0) = \psi \circ \varphi(f)$ and $ev_1 \circ \Phi(f) = \mathrm{id}_A(f)$. Hence,

$$\psi \circ \varphi \sim_h \mathrm{id}_A$$

Example 2.6. If X and Y are compact Hausdorff spaces, two maps $\varphi, \psi : X \to Y$ are said to be homotopic if there is a continuous function $H : [0,1] \times X \to Y$ such that

$$H(0,x) = \varphi(x) \text{ and } H(1,x) = \psi(x) \quad \forall x \in X$$

We write $\varphi \sim_h \psi$.

We say that X and Y are homotopy equivalent if $\exists$ maps $\varphi : X \to Y$ and $\psi : Y \to X$ such that

$$\psi \circ \varphi \sim_h \mathrm{id}_X \text{ and } \varphi \circ \psi \sim_h \mathrm{id}_Y$$

Given $\varphi : X \to Y$, we get an induced map $\varphi^* : C(Y) \to C(X)$ given by $f \mapsto f \circ \varphi$. We can check that

$$\varphi \sim_h \psi \Leftrightarrow \varphi^* \sim_h \psi^*$$

In particular,

$$X \simeq Y \Leftrightarrow C(X) \simeq C(Y)$$

See also: [Rørdam, Larsen, and Laustsen, Example 3.3.6, Exercise 3.13]

Proposition 2.7. *2.1. If $\varphi, \psi : A \to B$ are homotopic $*$-homomorphisms, then $K_0(\varphi) = K_0(\psi)$*

2.2. If $\varphi : A \to B$ and $\psi : B \to A$ are homotopy equivalences as above, then

$$K_0(\varphi) : K_0(A) \to K_0(B)$$

is an isomorphism with inverse $K_0(\psi)$

Proof. Since $K_0(A)$ is generated by $[p]_0$ where $p \in \mathcal{P}_\infty(A)$, it suffices to check these assertions on projections.

2.1. If φ, ψ are homotopic, then $\varphi(p) \sim_h \psi(p)$ in $\mathcal{P}_\infty(B)$ (via the path $t \mapsto \varphi_t(p)$), so

$$[\varphi(p)]_0 = [\psi(p)]_0$$

2.2. Similarly, in this case,

$$[\varphi(\psi(p))]_0 = [p]_0 = K_0(\varphi) \circ K_0(\psi)([p]_0)$$

for all $p \in \mathcal{P}_\infty(B)$.

□

3. Other points of view

a. Algebraic K-theory

(See [Rosenberg] for this section)

Definition 3.1. Let A be a unital C*-algebra. A (finitely generated) projective module over A is a (right) A-module P such that

$$P \oplus Q \cong A^n$$

for some A-module Q and $n \in \mathbb{N}$.

Let $\mathrm{Proj}(A)$ denote the isomorphism classes of all such modules. This carries a binary operation $\oplus$ of direct sum, under which the zero module 0 acts as an identity element. Furthermore,

$$(P \oplus Q) \oplus R \cong P \oplus (Q \oplus R)$$

Hence, $\mathrm{Proj}(A)$ is an abelian semi-group.

Lemma 3.2. *3.1. Let $p \in \mathcal{P}_n(A)$ be a projection, then $p(A^n)$ is a finitely generated projective A-module.*

3.2. If $p, q \in \mathcal{P}_\infty(A)$, then

$$p \sim_0 q \Leftarrow p(A^m) \cong q(A^n)$$

as A-modules

Proof. 3.1. If $p \in \mathcal{P}_n(A)$, then $P := p(A^n)$ is a right R-module such that

$$P \oplus Q \cong A^n$$

where $Q := (1-p)(A^n)$. Hence it is finitely generated and projective.

3.2. If $p \sim_0 q$ for some $p, q \in \mathcal{P}_\infty(A)$, let $v \in M_{m,n}(A)$ such that $p = v^*v$ and $q = vv^*$. The map

$$\varphi : p(A^m) \to q(A^n) \text{ given by } \underline{a} \mapsto v(\underline{a})$$

is an isomorphism.

3.3. Conversely, given an isomorphism $\varphi : p(A^m) \to q(A^n)$, we define $\widehat{\varphi} : A^m \to A^n$ by extending φ to be zero on $(1-p)(A^m)$ and including $q(A^n)$ in A^n. Now $\widehat{\varphi}$ is given by left multiplication by a matrix $a \in M_{m,n}(A)$. Similarly, we get a matrix $b \in M_{n,m}(A)$ from φ^{-1}. These matrices have the property that

$$ab = p, ba = q, a = qa = aq, b = qb = bp$$

Now set

$$z := \begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix} \in M_N(A)$$

where $N := n + m$. Then [Check!] $z^2 = I$, so z is invertible, and

$$z(p \oplus 0)z^{-1} = (0 \oplus q)$$

By [Rørdam, Larsen, and Laustsen, Proposition 2.2.5], $u(p \oplus 0)u^{-1} = (0 \oplus q)$ where $u = \omega(z)$. Hence,

$$p \sim_0 q$$

$\square$

Hence,

$$\mathcal{D}(A) \to \text{Proj}(A) \text{ given by } [p]_{\mathcal{D}} \to p(A^m)$$

is an isomorphism of abelian semigroups. In particular,

$$K_0(A) = G(\text{Proj}(A))$$

Note that this definition can be applied to any ring, where we take idempotents instead of projections.

Example 3.3. If R is a PID, then the structure theorem for modules implies that every projective module is free. Hence,

$$\text{Proj}(R) \leftrightarrow \mathbb{N} \cup \{0\}$$

so that $K_0(R) \cong \mathbb{Z}$

(End of Day 6)

b. Topological K-theory

(See [Park] for this section)

Let X be a compact Hausdorff space.

Definition 3.4. A family of vector spaces over X is a topological space V and a continuous surjective map $\pi : V \to X$ such that, for each $x \in X$,

3.1. $\pi^{-1}(x)$ is a finite dimensional vector space.

3.2. Addition and scalar multiplication on $\pi^{-1}(x)$ is continuous in the subspace topology induced from V.

We write $\zeta := (V, \pi, X)$ for such a family, π is called the projection map, and $\pi^{-1}(x) =: V_x$ is called the fiber of ζ at x.

Example 3.5. Let $V := X \times \mathbb{C}^n$, $\pi(x, v) := x$. We write $\Theta^n(X) := (V, \pi, X)$

Definition 3.6. Let V and W be families of vector spaces over X. A homomorphism of families is a continuous function

$$\gamma : V \to W$$

such that $\gamma_x : V_x \to W_x$ is a linear transformation of vector spaces for each $x \in X$.

If γ is a homeomorphism (so that each γ_x is an isomorphism), then γ is called an isomorphism of families. If this exists, we write

$$V \cong W$$

Definition 3.7. Let (V, π, X) be a family of vector spaces over X and $A \subset X$. Then

$$(\pi^{-1}(A), \pi, A)$$

is a family of vector spaces over A, and is denoted by $V|_A$

Definition 3.8. A vector bundle over X is a family of vector spaces (V, π, X) over X such that, for each $x \in X$, there is a neighbourhood U of x such that

$$V|_U \cong \Theta^n(U)$$

for some $n \in \mathbb{N}$. ie. There is a homeomorphism $h : \pi^{-1}(U) \to U \times \mathbb{C}^n$ such that the following diagram commutes

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "h"] \arrow[dr, "\pi"'] & & U \times \mathbb{C}^n \arrow[dl, "\pi_1"] \\
& U &
\end{tikzcd}$$

This property is called local triviality of the vector bundle.

We write $\mathrm{Vect}(X)$ for isomorphism classes of (locally trivial) vector bundles over X. We define an addition of vector bundles by

$$V \oplus W := \{(v, w) \in V \times W : \pi_V(v) = \pi_W(w)\}$$

This is called the Whitney sum, and it descends to give an addition on $\mathrm{Vect}(X)$. Since $\Theta^0(X) = X$ acts as the identity, this makes $\mathrm{Vect}(X)$ into an abelian semigroup.

Definition 3.9.

$$K^0(X) := G(\mathrm{Vect}(X))$$

Given a continuous function $f : X \to Y$ between two compact Hausdorff spaces, and a vector bundle (V, π, Y) over Y, we define

$$f^*(V) := \{(v, x) \in V \times X : \pi(v) = f(x)\}$$

This is a vector bundle over X, and defines a map

$$f^* : \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$$

It is also additive, so we get an induced map

$$K^0(f) : K^0(Y) \to K^0(X)$$

This ensures that the functor

$$X \mapsto K^0(X)$$

is a contravariant functor from the category of compact Hausdorff spaces to the category of abelian groups.

Definition 3.10. Let $A = C(X)$, and let $p \in \mathcal{P}_n(A)$. We may think of p as a function $p : X \to M_n(\mathbb{C})$ taking values in $\mathcal{P}_n(\mathbb{C})$. Define

$$V := \{(x, v) \in X \times \mathbb{C}^n : p(x)(v) = v\} = \{(x, v) \in X \times \mathbb{C}^n : v \in \mathrm{Im}(p(x))\}$$

Define $\pi : V \to X$ by $(x, v) \mapsto x$, then we write

$$\mathrm{Ran}(p) := (V, \pi, X)$$

Definition 3.11. Let (V, π, X) be a vector bundle over X.

3.1. A section of V is a continuous function $s : X \to V$ such that

$$\pi \circ s = \mathrm{id}_X$$

3.2. Write $\Gamma(V)$ for the set of all sections of V. Given two sections $s_1, s_2 \in \Gamma(V)$, we define

$$(s_1 + s_2)(x) := s_1(x) + s_2(x)$$

where the addition on the RHS is happening in V_x. Because vector space addition is assumed to be continuous,

$$s_1 + s_2 \in \Gamma(V)$$

3.3. Given $s \in \Gamma(V)$ and $f \in C(X)$, we define

$$(s \cdot f)(x) := s(x)f(x)$$

This is well-defined because $f(x) \in \mathbb{C}$ and $s \cdot f \in \Gamma(V)$ because scalar multiplication is also continuous in V_x. Hence, $\Gamma(V)$ is a right $C(X)$-module.

Theorem 3.12 (Swan's theorem)**.** *Let X be a compact Hausdorff space.*

3.1. If $p \in \mathcal{P}_n(C(X))$, then Ran(p) is a vector bundle over X.

3.2. If (V, π, X) is a vector bundle over X, then $\Gamma(V)$ is a finitely generated projective $C(X)$-module.

3.3. The map

$$\mathcal{D}(C(X)) \to \mathit{Vect}(X) \text{ given by } [p]_{\mathcal{D}} \to [\mathit{Ran}(p)]$$

is a well-defined isomorphism of abelian semigroups.

3.4. The map

$$\mathit{Vect}(X) \to \mathit{Proj}(C(X)) \text{ given by } [V] \mapsto [\Gamma(V)]$$

is a well-defined isomorphism of abelian semigroups.

Hence,

$$K^0(X) \cong K_0(C(X))$$

Recall that the functor

$$X \mapsto C(X)$$

is contravariant from the category of compact Hausdorff spaces and unital C*-algebras. Thus, what we have here is a composition of functors

$$K_0 \circ C \cong K^0$$

Note that C is an isomorphism of categories, so

Topological K-theory for compact Hausdorff spaces is the “same” as K-theory for unital, commutative C*-algebras.

III. The Functor K_0

1. Definition

a. Unitization of a unital C*-algebra

Let $\widetilde{A}$ denote the unitization of a C*-algebra A, then there is a split exact sequence

$$0 \to A \xrightarrow{\iota} \widetilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \to 0$$

with splitting $\lambda : \mathbb{C} \to \widetilde{A}$ given by $z \mapsto z\mathbf{1}_{\widetilde{A}}$. We get an induced map

$$K_0(\pi) : K_0(\widetilde{A}) \to K_0(\mathbb{C}) \cong \mathbb{Z}$$

Lemma 1.1. *Let A, B be unital C*-algebras. If $\varphi, \psi : A \to B$ are orthogonal $*$-homomorphisms (ie. $\varphi(x)\psi(y) = 0$ for all $x, y \in A$), then $\varphi + \psi : A \to B$ is a $*$-homomorphism, and*

$$K_0(\varphi + \psi) = K_0(\varphi) + K_0(\psi)$$

Proof. Check that $\varphi + \psi$ is a $*$-homomorphism. Furthermore, for each $n \in \mathbb{N}$, the induced homomorphisms $\varphi_n, \psi_n : M_n(A) \to M_n(B)$ are mutually orthogonal. Hence, for any $p \in \mathcal{P}_n(A)$,

$$[(\varphi + \psi)_n(p)]_0 = [\varphi_n(p) + \psi_n(p)]_0 = [\varphi_n(p)]_0 + [\psi_n(p)]_0$$

because $\varphi_n(p) \perp \psi_n(p)$. ☐

Proposition 1.2. *Let A be a unital C*-algebra, then there is a split exact sequence,*

$$0 \to K_0(A) \xrightarrow{K_0(\iota)} K_0(\widetilde{A}) \xrightarrow{K_0(\pi)} K_0(\mathbb{C}) \to 0$$

In particular,

$$K_0(A) \cong \ker(K_0(\pi))$$

Proof. Let $f := 1_{\widetilde{A}} - 1_A \in \mathcal{P}_1(\widetilde{A})$, then $\widetilde{A} = A + \mathbb{C}f$. Furthermore, $af = fa = 0$ for all $a \in A$. Define $\mu : \widetilde{A} \to A$ by

$$a + \alpha f \mapsto a$$

and $\lambda' : \mathbb{C} \to \widetilde{A}$ by

$$\alpha \mapsto \alpha f$$

Then

$$\mathrm{id}_A = \mu \circ \iota, \mathrm{id}_{\widetilde{A}} = \iota \circ \mu + \lambda' \circ \pi, \pi \circ \iota = 0 \text{ and } \pi \circ \lambda = \mathrm{id}_{\mathbb{C}}$$

also, $\iota \circ \mu$ and $\lambda' \circ \pi$ are orthogonal to each other. By the previous lemma,

$$\begin{aligned} 0 &= K_0(0) = K_0(\pi \circ \iota) = K_0(\pi) \circ K_0(\iota) \\ \mathrm{id}_{K_0(\mathbb{C})} &= K_0(\pi) \circ K_0(\lambda) \\ \mathrm{id}_{K_0(A)} &= K_0(\mu) \circ K_0(\iota) \\ \mathrm{id}_{K_0(\widetilde{A})} &= K_0(\iota) \circ K_0(\mu) + K_0(\lambda') \circ K_0(\pi) \end{aligned}$$

The third equation shows that $K_0(\iota)$ is injective. The first equation shows that

$$\mathrm{Im}(K_0(\iota)) \subset \ker(K_0(\pi))$$

Finally, if $g \in \ker(K_0(\pi))$, then the last equation shows that

$$g = K_0(\iota) \circ K_0(\mu)(g) \in \mathrm{Im}(K_0(\iota))$$

□

b. K_0 for a Non-Unital C*-algebra

Definition 1.3. If A is a (possibly non-unital) C*-algebra, we define

$$K_0(A) := \ker(K_0(\pi))$$

Note that this is a subgroup of $K_0(\widetilde{A})$, and that the definition for a unital C*-algebra agrees with this one.

Let $\varphi : A \to B$ be a $*$-homomorphism, then there is an induced $*$-homomorphism

$$\widetilde{\varphi} : \widetilde{A} \to \widetilde{B}$$

such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[d, "\varphi"] & \widetilde{A} \arrow[r, "\pi_A"] \arrow[d, "\widetilde{\varphi}"] & \mathbb{C} \arrow[d, "="] \\ B \arrow[r, "\iota_B"] & \widetilde{B} \arrow[r, "\pi_B"] & \mathbb{C} \end{tikzcd}$$

Functoriality in the unital case gives a diagram

$$\begin{tikzcd} K_0(A) \arrow[r, "K_0(\iota_A)"] & K_0(\widetilde{A}) \arrow[r, "K_0(\pi_A)"] \arrow[d, "K_0(\widetilde{\varphi})"] & K_0(\mathbb{C}) \arrow[d, "e"] \\ K_0(B) \arrow[r, "K_0(\iota_B)"] & K_0(\widetilde{B}) \arrow[r, "K_0(\pi_B)"] & K_0(\mathbb{C}) \end{tikzcd}$$

Define $\theta : K_0(A) \to K_0(B)$ by

$$\theta(g) := K_0(\widetilde{\varphi}) \circ K_0(\iota_A)(g)$$

Then $K_0(\pi_B)(\theta(g)) = e \circ K_0(\pi_A) \circ K_0(\iota_A)(g) = 0$, so

$$\theta(g) \in \ker(K_0(\pi_B)) = \mathrm{Im}(K_0(\iota_B))$$

Furthermore, $K_0(\iota_B)$ is injective, so we may define

$$K_0(\varphi) : K_0(A) \to K_0(B) \text{ such that } K_0(\varphi)(g) := K_0(\iota_B)^{-1}(\theta(g))$$

Then the following diagram commutes

$$\begin{tikzcd}
K_0(A) \arrow[r, "K_0(\iota_A)"] \arrow[d, "K_0(\varphi)"] & K_0(\widetilde{A}) \arrow[r, "K_0(\pi_A)"] \arrow[d, "K_0(\widetilde{\varphi})"] & K_0(\mathbb{C}) \arrow[d, "e"] \\
K_0(B) \arrow[r, "K_0(\iota_B)"] & K_0(\widetilde{B}) \arrow[r, "K_0(\pi_B)"] & K_0(\mathbb{C})
\end{tikzcd}$$

and $K_0(\varphi)$ is the unique map with this property. Note that if $p \in \mathcal{P}_\infty(A)$,

$$K_0(\varphi)([p]_0) = [\varphi(p)]_0$$

Proposition 1.4. *For any C*-algebras A, B, C, we have*

1.1. $K_0(id_A) = id_{K_0(A)}$
1.2. $K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$
1.3. $K_0(\{0\}) = \{0\}$
1.4. $K_0(0_{B,A}) = 0_{K_0(B),K_0(A)}$
1.5. *If $\varphi, \psi : A \to B$ are homotopic, then $K_0(\varphi) = K_0(\psi)$*
1.6. *If $A \simeq B$, then $K_0(A) \cong K_0(B)$.*

Example 1.5. Let A be a C*-algebra. The cone of A is

$$CA := \{f : [0,1] \to A : f(0) = 0\}$$

Define $\varphi_t : CA \to CA$ by $\varphi_t(f)(s) := f(st)$. Then $\varphi_0 = 0$ and $\varphi_1 = \mathrm{id}_{CA}$. Hence, $0 \simeq CA$, so that

$$K_0(CA) = \{0\}$$

2. The standard picture of K_0

Consider the split exact sequence as above

$$0 \to A \xrightarrow{\iota} \widetilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \to 0$$

Recall that

$$K_0(\widetilde{A}) = \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_\infty(\widetilde{A})\}$$

Hence,

$$K_0(A) = \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_\infty(\widetilde{A}) \text{ and } [\pi(p)]_0 = [\pi(q)]_0 \text{ in } K_0(\mathbb{C})\}$$

Define $s : \widetilde{A} \to \widetilde{A}$ by $s = \lambda \circ \pi$. ie.

$$s(a + \alpha 1_{\widetilde{A}}) = \alpha 1_{\widetilde{A}}$$

Note that $\pi(s(x)) = \pi(x)$, so $x - s(x) \in A$ for all $x \in \widetilde{A}$. Let $s_n : M_n(\widetilde{A}) \to M_n(\widetilde{A})$ be the induced map, then

$$x - s_n(x) \in M_n(A) \quad \forall x \in M_n(\widetilde{A})$$

We write $s = s_n$. An element $x \in M_n(\widetilde{A})$ is called scalar if $x = s(x)$.

Note: The scalar mapping is natural. ie. Given a $*$-homomorphism $\varphi : A \to B$, we have a commuting diagram

$$\begin{tikzcd} \widetilde{A} \arrow[r, "s"] \arrow[d, "\widetilde{\varphi}"'] & \widetilde{A} \arrow[d, "\widetilde{\varphi}"] \\ \widetilde{B} \arrow[r, "s"'] & \widetilde{B} \end{tikzcd}$$

Theorem 2.1 (Standard Picture of $K_0(A)$)**.** *For any C^*-algebra A,*

$$K_0(A) = \{[p]_0 - [s(p)]_0 : p \in \mathcal{P}_\infty(\widetilde{A})\}$$

Proof. 2.1. If $p \in \mathcal{P}_\infty(\widetilde{A})$ and $g := [p]_0 - [s(p)]_0 \in K_0(\widetilde{A})$, then

$$K_0(\pi)(g) = [\pi(p)]_0 - [\pi(s(p))]_0 = [\pi(p)]_0 - [\pi \circ \lambda \circ \pi(p)]_0 = [\pi(p)]_0 - [\pi(p)]_0 = 0$$

Hence $g \in K_0(A)$

2.2. Conversely, if $g \in K_0(A)$, write $g = [e]_0 - [f]_0$ for some $e, f \in \mathcal{P}_n(\widetilde{A})$. Define

$$p := \begin{pmatrix} e & 0 \\ 0 & 1_n - f \end{pmatrix} \text{ and } q := \begin{pmatrix} 0 & 0 \\ 0 & 1_n \end{pmatrix}$$

Then $p, q \in \mathcal{P}_{2n}(\widetilde{A})$ and

$$[p]_0 - [q]_0 = [e]_0 + [1_n - f]_0 - [1_n]_0 = [e]_0 - [f]_0 = g$$

Now $q = s(q)$ and $K_0(\pi)(g) = 0$, so

$$[s(p)]_0 - [q]_0 = [s(p)]_0 - [s(q)]_0 = K_0(s)(g) = K_0(\lambda \circ \pi)(g) = 0$$

Hence, $[q]_0 = [s(p)]_0$, so

$$g = [p]_0 - [q]_0 = [p]_0 - [s(p)]_0$$

□

(End of Day 7)

Proposition 2.2. *For any $p, q \in \mathcal{P}_\infty(\widetilde{A})$, TFAE:*

2.1. $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$

2.2. $\exists k, \ell \in \mathbb{N}$ *such that*

$$p \oplus 1_k \sim_0 q \oplus 1_\ell \text{ in } \mathcal{P}_\infty(\widetilde{A})$$

2.3. $\exists$ *scalar projections r_1 and r_2 such that*

$$p \oplus r_1 \sim_0 q \oplus r_2$$

Proof. We prove $(i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$.

(i) $\Rightarrow$ (iii): If $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$, then

$$[p \oplus s(q)]_0 = [q \oplus s(p)]_0$$

Hence,

$$p \oplus s(q) \sim_s q \oplus s(p)$$

Since $\widetilde{A}$ is unital, this implies

$$p \oplus s(q) \oplus 1_n \sim_0 q \oplus s(p) \oplus 1_n$$

This proves (iii).

(iii) $\Rightarrow$ (ii): If r_1 is a scalar projection, then we may think of $r_1 \in M_n(\mathbb{C})$. If $Tr(r_1) = k$, then $r_1 \sim_0 1_k$. Similarly, $r_2 \sim_0 1_\ell$, so (ii) follows.

(ii) $\Rightarrow$ (i): Suppose $p \oplus 1_k \sim_0 q \oplus 1_\ell$, then note that

$$[p \oplus 1_k]_0 - [s(p \oplus 1_k)]_0 = [p]_0 - [s(p)]_0$$

Therefore, replacing p by $p \oplus 1_k$ and q by $q \oplus 1_\ell$, it suffices to prove that

$$p \sim_0 q \Rightarrow [p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$$

Now suppose $v \in M_{m,n}(\widetilde{A})$ is such that $v^*v = p$ and $vv^* = q$. Consider $s(v) \in M_{m,n}(\mathbb{C})$ thought of as a subset of $M_{m,n}(\widetilde{A})$. Since s is a $*$-homomorphism,

$$s(v)^*s(v) = s(p) \text{ and } s(v)s(v)^* = s(q)$$

Hence, $s(p) \sim_0 s(q)$, so

$$[p]_0 = [q]_0 \text{ and } [s(p)]_0 = [s(q)]_0$$

which proves (i).

$\square$

The proof of the next lemma is technical, and we will omit it for now.

Lemma 2.3. *[Rørdam, Larsen, and Laustsen, Lemma 4.2.3] Let $\varphi : A \to B$ be a $*$-homomorphism.*

2.1. For any $p \in \mathcal{P}_\infty(\widetilde{A})$,

$$K_0(\varphi)([p]_0 - [s(p)]_0) = [\widetilde{\varphi}(p)]_0 - [s(\widetilde{\varphi}(p))]_0$$

2.2. Let $g \in \ker(K_0(\varphi))$, then $\exists n \in \mathbb{N}, p \in \mathcal{P}_n(\widetilde{A})$ and a unitary $u \in M_n(\widetilde{B})$ such that

$$g = [p]_0 - [s(p)]_0 \text{ and } u\widetilde{\varphi}(p)u^* = s(\widetilde{\varphi}(p))$$

2.3. If φ is surjective and $g \in \ker(K_0(\varphi))$, then $\exists p \in \mathcal{P}_\infty(\widetilde{A})$ such that

$$g = [p]_0 - [s(p)]_0 \text{ and } \widetilde{\varphi}(p) = s(\widetilde{\varphi}(p))$$

For a non-unital C*-algebra A, $K_0(A)$ is defined as a subgroup of $K_0(\widetilde{A})$. This means that any property of $K_0(A)$ must necessarily involve passing to the unitization. This is a difficulty that we will encounter frequently, but these lemmas given above, while technical, will help ease the pain.

3. Basic Properties

a. Half Exactness and Split Exactness

Given a short exact sequence

$$0 \to I \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$$

Lemma 3.1. *For $n \in \mathbb{N}$*

3.1. $\widetilde{\varphi}_n : M_n(\widetilde{I}) \to M_n(\widetilde{A})$ is injective.

3.2. If $a \in M_n(\widetilde{A})$, then $a \in Im(\widetilde{\varphi}_n)$ if and only if

$$\widetilde{\psi}_n(a) = s_n(\widetilde{\psi}_n(a))$$

Proof. 3.1. Note that $\widetilde{\varphi} : \widetilde{I} \to \widetilde{A}$ maps

$$a + \alpha 1_{\widetilde{I}} \to \varphi(a) + \alpha 1_{\widetilde{A}}$$

for $a \in I$ and $\alpha \in \mathbb{C}$. This map is injective because $\widetilde{A} = A \oplus \mathbb{C}1_{\widetilde{A}}$ as a vector space. It now follows that the induced map $M_n(\widetilde{I}) \to M_n(\widetilde{A})$ is injective.

3.2. (i) If $\widetilde{\psi}_n(a) = s_n(\widetilde{\psi}_n(a))$, then all entries of $\widetilde{\psi}_n(a)$ are scalar multiples of $1_{\widetilde{B}}$. Write

$$a = (a_{i,j} + \alpha_{i,j}1_{\widetilde{A}}) \text{ so that } \widetilde{\psi}_n(a) = (\alpha_{i,j}1_{\widetilde{B}})$$

Hence, $\psi(a_{i,j}) = 0$, so $\exists c_{i,j} \in I$ such that $\varphi(c_{i,j}) = a_{i,j}$. Define

$$c := (c_{i,j} + \alpha_{i,1}1_{\widetilde{I}})$$

then $\widetilde{\varphi}_n(c) = a$.

(ii) Conversely, suppose $a = \widetilde{\varphi}_n(c)$, for some $c \in M_n(\widetilde{I})$, then write

$$c = (c_{i,j} + \alpha_{i,1}1_{\widetilde{I}})$$

so that

$$a = (\varphi(c_{i,j}) + \alpha_{i,j}1_{\widetilde{A}})$$

But then

$$\widetilde{\psi}_n(a) = (\alpha_{i,j}1_{\widetilde{B}}) = s_n(\widetilde{\psi}_n(a))$$

$\square$

Theorem 3.2 (Half-Exactness of K_0)**.** *The sequence*

$$K_0(I) \xrightarrow{K_0(\varphi)} K_0(A) \xrightarrow{K_0(\psi)} K_0(B)$$

is exact at $K_0(A)$

Proof. 3.1. $K_0(\psi) \circ K_0(\varphi) = K_0(\psi \circ \varphi) = 0$. Hence,

$$\text{Im}(K_0(\varphi)) \subset \ker(K_0(\psi))$$

3.2. If $g \in \ker(K_0(\psi))$, then by the earlier lemma, $\exists p \in \mathcal{P}_\infty(\widetilde{A})$ such that

$$g = [p]_0 - [s(p)]_0 \text{ and } \widetilde{\psi}(p) = s(\widetilde{\psi}(p))$$

By the previous lemma, $p \in \text{Im}(\widetilde{\varphi}_n)$, so $\exists e \in M_n(\widetilde{I})$ such that

$$\widetilde{\varphi}_n(e) = p$$

Since $\widetilde{\varphi}_n$ is injective, e is a projection. Hence,

$$g = [\widetilde{\varphi}(e)] - [s(\widetilde{\varphi}(e)] = K_0(\varphi)([e]_0 - [s(e)]_0)$$

so $g \in \text{Im}(K_0(\varphi))$

$\square$

Theorem 3.3 (Split Exactness of K_0)**.** *Given a split exact sequence*

$$0 \to I \xrightarrow{\varphi} A \underset{\lambda}{\overset{\psi}{\rightleftarrows}} B \to 0$$

We get a split exact sequence of abelian groups

$$0 \to K_0(I) \xrightarrow{K_0(\varphi)} K_0(A) \underset{K_0(\lambda)}{\overset{K_0(\psi)}{\rightleftarrows}} K_0(B) \to 0$$

Proof. We have

$$\mathrm{id}_{K_0(B)} = K_0(\mathrm{id}_B) = K_0(\psi \circ \lambda) = K_0(\psi) \circ K_0(\lambda)$$

Hence, $K_0(\psi)$ is surjective. It suffices to show that $K_0(\varphi)$ is injective. So suppose $g \in \ker(K_0(\varphi))$, then by the earlier technical lemma, $\exists p \in \mathcal{P}_n(\widetilde{A})$ and a unitary $u \in M_n(\widetilde{B})$ such that

$$g = [p]_0 - [s(p)]_0 \text{ and } u\widetilde{\varphi}(p)u^* = s(\widetilde{\varphi}(p))$$

Define

$$v := \widetilde{\lambda} \circ \widetilde{\psi}(u^*)u$$

Then $v \in \mathcal{U}_n(\widetilde{A})$ and $\widetilde{\psi}(v) = 1 = s(\widetilde{\psi}(v))$. Hence, by the earlier lemma, $\exists w \in M_n(\widetilde{I})$ such that

$$\widetilde{\varphi}(w) = v$$

Since $\widetilde{\varphi}$ is injective, w is a unitary. Furthermore,

$$\begin{aligned}
\widetilde{\varphi}(wpw^*) &= v\widetilde{\varphi}(p)v^* \\
&= (\widetilde{\lambda} \circ \widetilde{\psi})(u^*)s(\widetilde{\varphi}(p))(\widetilde{\lambda} \circ \widetilde{\psi})(u) \\
&= (\widetilde{\lambda} \circ \widetilde{\psi})[u^*s(\widetilde{\varphi}(p))u] \\
&= (\widetilde{\lambda} \circ \widetilde{\psi})[\widetilde{\varphi}(p)] \\
&= s(\widetilde{\varphi}(p)) = \widetilde{\varphi}(s(p))
\end{aligned}$$

Since $\widetilde{\varphi}$ is injective, it follows that

$$wpw^* = s(p)$$

so that $g = [p]_0 - [s(p)]_0 = 0$ □

Corollary 3.4.

$$K_0(A \oplus B) \cong K_0(A) \oplus K_0(B)$$

Proof. We have a split exact sequence

$$0 \to A \to A \oplus B \rightleftarrows B \to 0$$

□

Example 3.5. For any C*-algebra A,

$$K_0(\widetilde{A}) \cong K_0(A) \oplus \mathbb{Z}$$

Example 3.6. Consider the sequence

$$0 \to C_0(0,1) \to C[0,1] \xrightarrow{\psi} \mathbb{C} \oplus \mathbb{C} \to 0$$

where $\psi(f) := (f(0), f(1))$. Then

$$K_0(C[0,1]) \cong \mathbb{Z} \text{ and } K_0(\mathbb{C} \oplus \mathbb{C}) \cong \mathbb{Z} \oplus \mathbb{Z}$$

Hence, $K_0(\psi)$ is not surjective. This shows that K_0 is not, in general, exact.

Example 3.7. Consider the exact sequence

$$0 \to \mathcal{K}(H) \xrightarrow{\iota} \mathcal{B}(H) \to \mathcal{Q}(H) \to 0$$

We know that $K_0(\mathcal{B}(H)) = \{0\}$. We will show later that

$$K_0(\mathcal{K}(H)) \cong \mathbb{Z}$$

so $K_0(\iota)$ is not injective.

b. Stability

Proposition 3.8. *(Stability of K_0) Let A be a C^*-algebra and $n \in \mathbb{N}$. Define $\lambda : A \to M_n(A)$ by*

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

Then $K_0(\lambda) : K_0(A) \to K_0(M_n(A))$ is an isomorphism.

Proof. 3.1. Suppose that A is unital, and let $B := M_n(A)$. Define $\mu : \mathcal{P}_\infty(B) \to K_0(A)$ by

$$\mu(p) := [p]_0$$

if $p \in \mathcal{P}_k(B)$. This is well-defined, and additive, and respects homotopy, so by the universal property of K_0, we get a map

$$K_0(\mu) : K_0(B) \to K_0(A)$$

If $p \in \mathcal{P}_\infty(A)$, then clearly,

$$[\mu(\lambda(p))]_0 = [p]_0$$

so $K_0(\mu) \circ K_0(\lambda) = \mathrm{id}_{K_0(A)}$. Similarly, if $p \in \mathcal{P}_\infty(B)$, then

$$K_0(\lambda)K_0(\mu)([p]_0) = K_0(\lambda)([p]_0) = [p]_0$$

3.2. Now suppose A is non-unital, consider the diagram with split exact rows

$$\begin{tikzcd} 0 \arrow[r] & K_0(A) \arrow[r] \arrow[d, "K_0(\lambda_A)"] & K_0(\widetilde{A}) \arrow[r] \arrow[d, "K_0(\lambda_{\widetilde{A}})"] & K_0(\mathbb{C}) \arrow[r] \arrow[d, "K_0(\lambda_{\mathbb{C}})"] & 0 \\ 0 \arrow[r] & K_0(B) \arrow[r] & K_0(\widetilde{B}) \arrow[r] & K_0(M_n(\mathbb{C})) \arrow[r] & 0 \end{tikzcd}$$

Since $\lambda_{\widetilde{A}}$ and $\lambda_{\mathbb{C}}$ induce isomorphism, λ_A also induces an isomorphism by a diagram chase.

□

4. Example: The Cuntz Algebra

Definition 4.1. Let $n \geq 2$ and $H = \ell^2(\mathbb{N})$. Decompose $\mathbb{N} = T_1 \sqcup T_2 \sqcup T_2 \ldots \sqcup T_n$ where

$$T_i = \{i, i+n, i+2n, \ldots\}$$

Let $P_i : H \to H$ be the natural projection onto $\ell^2(T_i) \subset H$. Then, P_i is an infinite rank projection, so $P_i \sim I_H$. Choose $s_1, s_2, \ldots, s_n \in \mathcal{B}(H)$ such that

$$s_1^* s_1 = s_2^* s_2 = \ldots = s_n^* s_n = 1 = \sum_{i=1}^n s_i s_i^*$$

(Note that these s_i are isometries). Define

$$\mathcal{O}_n := C^*(s_1, s_2, \ldots, s_n)$$

This is called the Cuntz algebra.

Theorem 4.2. *4.1. $\mathcal{O}_n$ is a simple C^*-algebra (no non-trivial closed two-sided ideals)*

4.2. (Universal Property of $\mathcal{O}_n$) Given a unital C^*-algebra A and elements $t_1, t_2, \ldots, t_n \in A$ such that

$$t_j^* t_j = 1 = \sum_{i=1}^n t_i t_i^*$$

$\exists$ a unique $*$-homomorphism $\varphi : \mathcal{O}_n \to A$ such that $\varphi(s_j) = t_j$

Lemma 4.3. *4.1. Let $u \in \mathcal{U}(\mathcal{O}_n)$, then $\exists$ a unique $*$-homomorphism $\varphi_u : \mathcal{O}_n \to \mathcal{O}_n$ such that*

$$\varphi_u(s_j) = u s_j$$

Furthermore,

$$u = \sum_{j=1}^n \varphi_u(s_j) s_j^*$$

4.2. Let $\varphi : \mathcal{O}_n \to \mathcal{O}_n$ be a unital $*$-homomorphism, then $\exists u \in \mathcal{U}(\mathcal{O}_n)$ such that $\varphi = \varphi_u$

Proof. 4.1. Follows from the universal property with $t_j = us_j$. Furthermore,

$$\sum_{j=1}^{n} \varphi_u(s_j)s_j^* = \sum_{j=1}^{n} us_js_j^* = u$$

4.2. Given φ, consider

$$u := \sum_{j=1}^{n} \varphi(s_j)s_j^*$$

Then

$$uu^* = \sum_{i,j=1}^{n} \varphi(s_i)s_i^*s_j\varphi(s_j)^*$$

But the P_i are orthogonal projections, and $s_i = P_is_i$ so $s_j^*s_i = \delta_{i,j}$. Hence,

$$uu^* = \sum_{i=1}^{n} \varphi(s_i)\varphi(s_i)^* = \varphi(1) = 1$$

Similarly, $u^*u = 1$. Finally,

$$\varphi_u(s_i) = us_i = \sum_{j=1}^{n} \varphi(s_j)s_j^*s_i = \varphi(s_i)s_i^*s_i = \varphi(s_i)$$

By uniqueness of the universal property, $\varphi_u = \varphi$.

□

Lemma 4.4. *Let $\lambda : \mathcal{O}_n \to \mathcal{O}_n$ be given by*

$$\lambda(x) = \sum_{j=1}^{n} s_jxs_j^*$$

Then

4.1. λ is an endomorphism of $\mathcal{O}_n$

4.2. If $u \in \mathcal{U}(\mathcal{O}_n)$ such that $\lambda = \varphi_u$, then $u = u^*$

Proof. 4.1. $\lambda(1) = 1$ and $\lambda(x^*) = \lambda(x)^*$. By orthogonality of the P_i

$$\lambda(x)\lambda(y) = \sum_{j=1}^{n} s_jxs_j^*s_jys_j^* = \lambda(xy)$$

since $s_j^*s_j = 1$.

4.2. If $u = \sum_{j=1}^n \lambda(s_j)s_j^*$, then $\lambda = \varphi_u$ and

$$u^* = \sum_{j=1}^n s_j \lambda(s_j^*) = \sum_{j=1}^n s_j \left[\sum_{i=1}^n s_i s_j^* s_i\right] = \sum_{j=1}^n s_j s_j s_j^* s_j = \sum_{j=1}^n s_j^2$$

But

$$\lambda(s_i)s_i = \sum_{j=1}^n s_j s_i s_j^* s_i = s_i s_i s_i^* s_i = s_i^2$$

Hence, $u = u^*$.

$\square$

Lemma 4.5. *Let A be a unital C^*-algebra and $s \in A$ an isometry. Define $\mu : A \to A$ by $\mu(a) = sas^*$. Then $K_0(\mu) = id_{K_0(A)}$*

Proof. Note that $\mu_n : M_n(A) \to M_n(A)$ is given by $\mu_n(a) = s_n a s_n^*$ where

$$s_n = \text{diag}(s, s, \ldots, s)$$

and s_n is also an isometry. Furthermore, if $p \in \mathcal{P}_n(A)$, then

$$s_n p s_n = (s_n p)(s_n p)^* \sim (s_n p)^*(s_n p) = p$$

Hence, $[\mu_n(p)]_0 = [p]_0$. $\square$

Theorem 4.6. *If $g \in K_0(\mathcal{O}_n)$, then $(n-1)g = 0$. In particular, $K_0(\mathcal{O}_2) = 0$*

Proof. Let $\lambda : \mathcal{O}_n \to \mathcal{O}_n$ as above, then $\lambda = \sum_{i=1}^n \lambda_i$ where

$$\lambda_i(x) = s_i x s_i^*$$

Then $\lambda_i(x)\lambda_j(y) = 0$ for all $x, y \in \mathcal{O}_n$, so

$$K_0(\lambda) = \sum_{i=1}^n K_0(\lambda_i)$$

By the above lemma, it follows that

$$K_0(\lambda)g = ng \quad \forall g \in K_0(\mathcal{O}_n)$$

However, $\lambda = \varphi_u$, where $u = u^*$. In particular, $u \in \mathcal{U}_0(\mathcal{O}_n)$. Let u_t be a path of unitaries from u to 1, then φ_{u_t} is a path of $*$-homomorphism from

$$\lambda = \varphi_u \text{ to } \text{id}_A = \varphi_1$$

Hence, $K_0(\lambda) = \text{id}_{K_0(\mathcal{O}_n)}$. Hence the result. $\square$

It is a fact that

4.1. $K_0(\mathcal{O}_n) \cong \mathbb{Z}_{n-1}$.

4.2. Furthermore, $K_0(\mathcal{O}_n)$ is generated by $[1]_0$.

Definition 4.7. A non-zero projection $p \in A$ is said to be properly infinite if $\exists$ projections $e, f \in A$ such that

4.1. $e \perp f$

4.2. $e \leq p, f \leq p$

4.3. $p \sim e \sim f$

A unital C*-algebra A is said to be properly infinite if 1_A is properly infinite.

Example 4.8. 4.1. $\mathcal{B}(H)$ is properly infinite iff H is infinite dimensional.

4.2. $\mathcal{O}_n$ is properly infinite

Theorem 4.9. *Let A be a properly infinite C*-algebra, then*

$$K_0(A) = \{[p]_0 : p \in \mathcal{P}(A), p \neq 0\}$$

Proof. 4.1. Since 1_A is properly infinite, $\exists s_1, s_2$ isometries such that

$$s_1 s_1^* \perp s_2 s_2^*$$

Define $t_i := s_2^{i-1} s_1$ for $i \in \mathbb{N}$, then the $\{t_i\}$ are isometries such that $t_j t_j^* \perp t_i t^*$ (Check!). For $n \in \mathbb{N}$, define

$$v_n = (t_1, t_2, \ldots, t_n) \in M_{1,n}(A)$$

Then $v_n^* v_n = 1_n$. Hence, as in Lemma 4.5, for any $p \in \mathcal{P}_n(A)$,

$$p \sim_0 v_n p v_n^*$$

Note that $v_n p v_n^*$ is a projection in A. Hence,

$$K_0(A) = \{[p]_0 - [q]_0 : p, q \in \mathcal{P}(A)\}$$

4.2. Let $p, q \in A$ projections, then set

$$r := t_1 p t_1^* + t_2(1-q)t_2^* + t_3(1 - t_1 t_1^* - t_2 t_2^*)t_3^*$$

Then $r \in \mathcal{P}(A)$ and

$$[r]_0 = [p]_0 + [1-q]_0 + [1 - t_1 t_1^* - t_2 t_2^*]_0$$

But $[1 - t_1 t_1^* - t_2 t_2^*]_0 = [1]_0$, so $[r]_0 = [p]_0 - [q]_0$. Hence,

$$K_0(A) = \{[p]_0 : p \in \mathcal{P}(A), p \neq 0\}$$

□

Definition 4.10. Let A be a simple, unital C*-algebra which is not isomorphic to $\mathbb{C}$. A is said to be purely infinite if

4.1. Every non-zero projection in A is properly infinite.

4.2. Every non-zero hereditary subalgebra has a non-zero projection.

In fact, [Rørdam, Larsen, and Laustsen, Exercise 5.7] shows that, if A is purely infinite simple, then

$$K_0(A) = \{[p]_{\mathcal{D}} : p \in \mathcal{P}(A), p \neq 0\}$$

In other words, $K_0(A)$ coincides with Murray Von Neumann equivalence classes of projections in A.

It is a fact that $\mathcal{O}_n(A)$ is purely infinite. Also, if H is infinite dimensional, then the Calkin Algebra $\mathcal{B}(H)/\mathcal{K}(H)$ is purely infinite.

(End of Day 8)

IV. The Ordered Abelian group $K_0(A)$

1. Stably Finite C*-algebras

Definition 1.1. An element $a \in A$ is said to be left-invertible if $\exists b \in A$ such that $ba = 1$. Right-invertibility is similar.

Note that a is invertible iff it is both left and right invertible.

Definition 1.2. 1.1. A projection $p \in A$ is said to be infinite if $\exists$ a projection q such that $p \sim q$ and $q < p$. If p is not infinite, then it is said to be finite.

1.2. A unital C*-algebra is said to be infinite if 1_A is infinite. A is said to be finite if 1_A is finite.

1.3. A is said to be stably finite if $M_n(A)$ is finite for all $n \in \mathbb{N}$.

1.4. A non-unital C*-algebra is said to be finite if $\widetilde{A}$ is finite.

Note: A projection $p \in A$ is finite iff pAp is a finite C*-algebra.

Lemma 1.3. *If A is a unital C*-algebra, TFAE:*

1.1. A is finite.

1.2. Every isometry is a unitary.

1.3. All projections in A are finite.

1.4. Every left-invertible element is invertible.

1.5. Every right-invertible element is invertible.

Proof. We prove $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$, and $(ii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (ii)$

(i)$\Rightarrow$ (ii) : If s is an isometry, then $1_A = s^*s \sim ss^* \leq 1$. Since A is finite, $ss^* = 1$ and s is a unitary.

(ii)$\Rightarrow$ (iii) : Suppose every isometry is a unitary, and $p, q \in A$ projections such that

$$p \sim q \text{ and } q \leq p$$

Let $v \in A$ such that $v^*v = p$ and $vv^* = q$, and let

$$s := v + (1 - p)$$

Since $pq = qp = q$, we have $v^*(1 - p) = 0 = (1 - p)v$. Hence,

$$s^*s = v^*v + (1 - p) = 1 \text{ and } vv^* = 1 - (p - q)$$

By hypothesis, s is a unitary, so $p - q = 0$.

(iii)$\Rightarrow$ (i) : If every projection is finite, then 1_A is finite.

(ii)$\Rightarrow$ (iv) : Suppose every isometry is a unitary, and $a \in A$ be left invertible. Then $\exists b \in A$ such that $ba = 1_A$. Then

$$1 = (ba)^*(ba) = a^*b^*ba \leq \|b\|^2 a^*a$$

(by [Murphy, Theorem 2.2.5]). Hence,

$$a^*a - \|b\|^{-2}1_A \geq 0$$

and so $sp(a^*a) \subset [\|b\|^{-2}, \infty)$. In particular, a^*a is invertible, so $s := a(a^*a)^{-1/2}$ exists. Observe that

$$s^*s = (a^*a)^{-1/2}a^*a(a^*a)^{-1/2} = 1$$

Thus, s is an isometry, and hence a unitary by hypothesis. In particular, s is invertible, so

$$a = s(a^*a)^{1/2}$$

is invertible too.

(iv)$\Rightarrow$ (v) : If $a \in A$ is right-invertible, then a^* is left-invertible. By hypothesis, it is invertible, and hence a is invertible too.

(v)$\Rightarrow$ (ii) : If $s^*s = 1$, then s^* is right invertible, and hence invertible. It follows by uniqueness of inverse that $ss^* = 1$.

□

Definition 1.4. A pair (G, G^+) is called an ordered abelian group if G is an Abelian group, $G^+ \subset G$ such that

1.1. $G^+ + G^+ \subset G^+$
1.2. $G^+ \cap (-G^+) = \{0\}$
1.3. $G^+ - G^+ = G$

We define an order relation on G by $x \leq y$ iff $y - x \in G^+$. This makes $(G, \leq)$ a partially ordered set such that

$$x \leq y \Rightarrow x + z \leq y + z \quad \forall z \in G$$

The converse is also true: If G is a partially ordered group satisfying this condition, we may set $G^+ = \{x \in G : x \geq 0\}$, then it satisfies the above requirements.

Definition 1.5. Define

$$K_0(A)^+ := \{[p]_0 : p \in \mathcal{P}_\infty(A)\}$$

Proposition 1.6. *1.1.* $K_0(A)^+ + K_0(A)^+ \subset K_0(A)+$

1.2. If A is unital, $K_0(A)^+ - K_0(A)^+ = K_0(A)$

1.3. If A is stably finite, then $K_0(A)^+ \cap (-K_0(A)^+) = \{0\}$

Hence, if A is unital and stably finite, then $(K_0(A), K_0(A)^+)$ is an ordered Abelian group.

Proof. 1.1. $[p]_0 + [q]_0 = [p \oplus q]_0$

1.2. Proved earlier.

1.3. Suppose A is stably finite, and $g \in K_0(A)^+ \cap (-K_0(A)^+)$, then write

$$g = [p]_0 = -[q]_0$$

Hence, $[p \oplus q]_0 = 0$, so $\exists r \in \mathcal{P}_\infty(\widetilde{A})$ such that

$$p \oplus q \oplus r \sim_0 r$$

Choose mutually orthogonal projections p', q', r' such that $p \sim_0 p', q \sim_0 q'$ and $r \sim_0 r'$ and think of them in $M_n(\widetilde{A})$ for some $n \in \mathbb{N}$. Now

$$p' + q' + r' \sim r' \text{ in } M_n(\widetilde{A})$$

But $p' + q' + r' \geq r'$ and $M_n(\widetilde{A})$ is finite, so $p' + q' = 0$. Hence, $p' = q' = 0$, so that

$$g = [p]_0 = [p']_0 = 0$$

$\square$

Definition 1.7. Let (G, G^+) be an ordered abelian group. An element $u \in G^+$ is called an order unit if, for each $x \in G, \exists n \in \mathbb{N}$ such that

$$-nu \leq x \leq nu$$

Note: Not every ordered abelian group has an order unit. For example, $C_c(\mathbb{R})$ with the pointwise order.

Proposition 1.8. *If A is unital, then $[1]_0$ is an order unit of $K_0(A)$*

Proof. If $g \in K_0(A)$, write $g = [p]_0 - [q]_0$ for some $p, q \in \mathcal{P}_n(A)$. Then

$$-n[1]_0 = -[1_n]_0 = -[q]_0 + [1_n - q]_0 \leq -[q]_0 \leq [p]_0 - [q]_0 = g$$

and

$$g \leq [p]_0 \leq [p]_0 + [1_n - p]_0 = [1_n]_0 = n[1]_0$$

$\square$

Definition 1.9. Let (G, G^+) and (H, H^+) be ordered Abelian groups. A positive group homomorphism is a map $\alpha : G \to H$ such that $\alpha(G^+) \subset H^+$. It is called an order isomorphism if it is an isomorphism such that $\alpha(G^+) = H^+$. If G and H have distinguished order units u and v respectively, α is said to be order unit preserving if $\alpha(u) = v$

Example 1.10. Let $\varphi : A \to B$ be a $*$-homomorphism, then

$$K_0(\varphi)[p]_0 = [\varphi(p)]_0$$

so $K_0(\varphi)$ is a positive homomorphism. Furthermore, if φ is unital, then $K_0(\varphi)$ preserves the order unit.

2. Traces

Recall: If $\tau : A \to \mathbb{C}$ be a positive trace, then it induces a map

$$K_0(\tau) : K_0(A) \to \mathbb{R}$$

This is a positive group homomorphism from $(K_0(A), K_0(A)^+)$ to $(\mathbb{R}, \mathbb{R}^+)$. If τ is a state, then $K_0(\tau)$ preserves the order unit.

Example 2.1. Let τ denote the usual trace on $\mathbb{C}$, then $\tau_n : M_n(\mathbb{C}) \to \mathbb{C}$ is a trace. Furthermore,

$$\tau_n(1_n) = n$$

So τ_n induces an isomorphism

$$(K_0(M_n(\mathbb{C})), K_0(M_n(\mathbb{C}))^+, [1_n]) \to (\mathbb{Z}, \mathbb{Z}^+, n)$$

Thus, $(K_0(A), K_0(A)^+, [1_A]_0)$ is a useful invariant to distinguish C*-algebras.

Definition 2.2. Let (G, G^+, u) be an ordered Abelian group with order unit u. A state on this triple is a positive group homomorphism $f : (G, G^+, u) \to (\mathbb{R}, \mathbb{R}^+, 1)$. We write $S(G)$ for the set of states on G.

Note: If $\tau : A \to \mathbb{C}$ is a tracial state, then $K_0(\tau) \in S(K_0(A))$

Theorem 2.3. *Let A be a unital, exact C*-algebra, then every state on $(K_0(A), K_0(A)^+, [1_A]_0)$ is of the form $K_0(\tau)$ for some trace τ on A.*

(End of Day 9)

In general, we define a quasi-trace to be a function $\tau : A \to \mathbb{C}$ such that

2.1. $\tau(x^*x) = \tau(xx^*) \geq 0$ for any $x \in A$
2.2. τ is linear on commutative subalgebras of A
2.3. If $x = a + ib$ where $a, b \in A_{sa}$, then $\tau(x) = \tau(a) + i\tau(b)$
2.4. For each $n \in \mathbb{N}$, the map $\tau_n : M_n(A) \to \mathbb{C}$ given by

$$\tau_n((a_{i,j}) = \sum_{i=1}^{n} \tau(a_{i,i})$$

also has these properties.

In other words, a trace is simply a linear quasi-trace. Given a quasi-trace on A, we get an induced map $K_0(\tau) : K_0(A) \to \mathbb{R}$ by the first property. The above theorem is a special case of the following facts.

Theorem 2.4. *Let A be a unital C*-algebra.*

2.1. Every state on $K_0(A)$ is of the form $K_0(\tau)$ for some quasi-trace τ.

2.2. (Haageruup) If A is exact, then every quasi-trace on A is a trace.

Definition 2.5. A trace $\tau : A \to \mathbb{C}$ is called faithful if $\tau(a) > 0$ whenever $a \in A_+$ is non-zero.

Theorem 2.6. *If A is a unital C^*-algebra that admits a faithful positive trace, then A is stably finite.*

Proof. Let $\tau : A \to \mathbb{C}$ be a faithful positive trace. Define $\tau_n : M_n(A) \to \mathbb{C}$ as above. Then if $x = (x_{i,j}) \in M_n(A)$, then (Check!)

$$\tau_n(x^*x) = \sum_{i,j=1}^{n} \tau(x_{i,j}^* x_{i,j})$$

Hence, τ_n is also a faithful positive trace on $M_n(A)$. Therefore, to show A is stably finite, it suffices to show that A is finite.

Now suppose $s \in A$ is an isometry, then

$$\tau(1) = \tau(s^*s) = \tau(ss^*) \Rightarrow \tau(1 - ss^*) = 0$$

But $ss^* \leq 1$ and τ is faithful, so $ss^* = 1$. Hence, A is finite. $\square$

We have a partial converse of the above theorem:

Theorem 2.7. *2.1. If A is unital and stably finite, then it admits a quasi-trace.*

2.2. Every unital, stably finite, separable, exact C^*-algebra admits a faithful trace.

3. Example: The Rotation Algebra

Definition 3.1. Let $\theta \in \mathbb{R}$ be fixed, and set $\omega := e^{2\pi i\theta}$. Let $H := L^2(\mathbb{T} \times \mathbb{T})$ equipped with a normalized Haar measure. Let $\zeta_0 \in H$ be the unit vector $\zeta_0(z_1, z_2) := 1$. Define $u, v \in \mathcal{B}(H)$ by

$$(u\zeta)(z_1, z_2) := z_1\zeta(z_1, z_2) \text{ and } (v\zeta)(z_1, z_2) := z_2\zeta(\omega z_1, z_2)$$

Then

$$\langle u\zeta, \eta\rangle \int_{\mathbb{T}^2} z_1\zeta(z_1, z_2)\overline{\eta(z_1, z_2)} = \int_{\mathbb{T}^2} \zeta(z_1, z_2)\overline{\overline{z_1}\eta(z_1, z_2)}$$

Hence,

$$(u^*\eta)(z_1, z_2) = \overline{z_1}\eta(z_1, z_2)$$

Similarly,

$$(v^*\eta)(z_1, z_2) = \overline{z_2}\eta(\omega^{-1}z_1, z_2)$$

Hence, u and v are unitaries. Furthermore,

$$\begin{aligned}(vu\zeta)(z_1, z_2) &= z_2(u\zeta)(\omega z_1, z_2) = z_2\omega z_1\zeta(\omega z_1, z_2)\\ (uv\zeta)(z_1, z_2) &= z_1(v\zeta)(z_1, z_2) = z_1 z_2\zeta(\omega z_1, z_2)\\ &\Rightarrow vu = \omega uv\end{aligned}$$

Define

$$A_\theta := C^*(u, v) \subset \mathcal{B}(H)$$

is called the rotation C*-algebra associated to the angle θ.

We will need the following properties:

Theorem 3.2. *3.1. If θ is irrational, then A_θ is simple, and has a unique tracial state. (see below).*

3.2. (Universal property of A_θ): If D is a unital C-algebra and $u', v' \in D$ are two unitaries such that $v'u' = \omega u'v'$, then $\exists$ a unique $*$-homomorphism $\varphi : A_\theta \to D$ such that $\varphi(u) = u'$ and $\varphi(v) = v'$.*

Note: If $\theta \in \mathbb{Z}$, then A_θ is the universal C*-algebra generated by two commuting unitaries. This is $C(\mathbb{T}^2)$. If $\theta \notin \mathbb{Z}$, A_θ is called a non-commutative two torus.

Remark 3.3. If $\theta, \theta' \in \mathbb{R}$ be irrational.

3.1. Suppose $\theta - \theta' \in \mathbb{Z}$, then $e^{2\pi i\theta} = e^{2\pi i\theta'}$, and so

$$A_\theta \cong A_{\theta'}$$

3.2. If $\theta + \theta' \in \mathbb{Z}$, then $e^{2\pi i\theta} = (e^{2\pi i\theta'})^{-1}$. Hence, there is a surjective $*$-homomorphism $\varphi : A_\theta \to A_{\theta'}$ such that

$$\varphi(u) = v' \text{ and } \varphi(v) = u'$$

Since A_θ is simple, it follows that this map is an isomorphism.

We will now (partially) show that if $A_\theta \cong A_{\theta'}$, then one of the above two conditions must hold.

Define B_θ to be those elements in A_θ of the form

$$\sum_{n,m\in\mathbb{Z}} \alpha_{n,m} u^n v^m$$

where only finitely many coefficients $\alpha_{n,m}$ are non-zero. One thinks of these as Laurent polynomials in u and v. Note that B_θ is a $*$-subalgebra of A_θ, and its closure is thus a C*-algebra containing u and v. Thus, B_θ is dense in A_θ and is called the smooth $*$-subalgebra of A_θ.

Definition 3.4. Define $\tau : A_\theta \to \mathbb{C}$ by

$$\tau(a) := \langle a\zeta_0, \zeta_0\rangle$$

Then τ is a positive linear functional on A_θ of norm 1. Furthermore,

$$\tau\left(\sum_{n,m\in\mathbb{Z}} \alpha_{n,m}u^n v^m\right) = \alpha_{0,0}$$

for elements in B_θ. Hence, it follows that if $x \in B_\theta$ of the above form, then

$$\begin{aligned}\tau(x^*x) &= \tau\left[\left(\sum_{n,m\in\mathbb{Z}} \overline{\alpha_{n,m}}v^{-m}u^{-n}\right)\left(\sum_{n,m\in\mathbb{Z}} \alpha_{n,m}u^n v^m\right)\right] \\ &= \sum_{n,m\in\mathbb{Z}} |\alpha_{n,m}|^2 = \tau(xx^*)\end{aligned}$$

Since B_θ is dense in A_θ, it follows that

$$\tau(x^*x) = \tau(xx^*) \quad \forall x \in A_\theta$$

From [Rørdam, Larsen, and Laustsen, Exercise 3.6], it follows that τ is a tracial state on A_θ.

(End of Day 10)

Lemma 3.5. *Let $\varphi : \mathbb{T} \to \mathbb{T}$ is the function $z \mapsto \omega z$. Then, for any $h : \mathbb{T} \to \mathbb{C}$ continuous,*

$$vh(u) = (h\circ\varphi)(u)v, \text{ and } v^*(h\circ\varphi)(u) = h(u)v^*$$

Proof. It suffices to prove the first statement. Note that

$$\omega^k u^k v = vu^k \quad \forall k \in \mathbb{Z}$$

Hence, for any $h : \mathbb{T} \to \mathbb{R}$ Laurent polynomial

$$(h\circ\varphi)(u)v = vh(u)$$

Now approximate any continuous $h : \mathbb{T} \to \mathbb{C}$ by Laurent polynomials. □

If $\theta = 0$, then $C(\mathbb{T}^2) = A_\theta$ has no projections because $\mathbb{T}^2$ is connected. We now assume that $\theta \in (0,1)$ is irrational, and show that, in this case, A_θ has many projections.

Lemma 3.6. *Let $f, g : \mathbb{T} \to \mathbb{R}$ be continuous functions, and define*

$$p := f(u)v^* + g(u) + vf(u) \in A_\theta$$

Then

3.1. $p = p^*$

3.2. $p = p^2$ *if and only if*

(i) $f \cdot (f \circ \varphi) = 0$

(ii) $f \cdot (g + g \circ \varphi^{-1}) = f$

(iii) $g = g^2 + f^2 + (f \circ \varphi)^2$

3.3. Furthermore,

$$\tau(p) = \int_{\mathbb{T}} g(z)dz$$

Proof. 3.1. Clearly, $p = p^*$ since f and g are real-valued.

3.2. One writes out

$$\begin{aligned} p^2 &= f(u)v^*f(u)v^* + f(u)v^*g(u) + f(u)v^*vf(u) \\ &\quad + g(u)f(u)v^* + g(u)g(u) + g(u)vf(u) \\ &\quad + vf(u)f(u)v^* + vf(u)g(u) + vf(u)vf(u) \\ &= f \cdot (f \circ \varphi^{-1})(u)v^{-2} + f \cdot (g \circ \varphi^{-1})(u)v^{-1} + f^2(u) \\ &\quad + gf(u)v^{-1} + g^2(u) + g \cdot (f \circ \varphi)(u)v \\ &\quad + (f \circ \varphi)^2(u) + (f \circ \varphi) \cdot (g \circ \varphi)(u)v + (f \circ \varphi) \cdot (f \circ \varphi \circ \varphi)(u)v^2 \end{aligned}$$

Note that

$$p = f(u)v^{-1} + g(u) + (f \circ \varphi)(u)v$$

So comparing coefficients, we get

$$\begin{aligned} f \cdot (f \circ \varphi^{-1}) &= 0 \\ f \cdot (g \circ \varphi^{-1}) + (g \cdot f) &= f \\ f^2 + g^2 + (f \circ \varphi)^2 &= g \\ g \cdot (f \circ \varphi) + (f \circ \varphi) \cdot (g \circ \varphi) &= (f \circ \varphi) \\ (f \circ \varphi) \cdot (f \circ \varphi \circ \varphi) &= 0 \end{aligned}$$

Since φ is a homeomorphism of $\mathbb{T}$, for any function $h : \mathbb{T} \to \mathbb{R}$, we have

$$h = 0 \Leftrightarrow h \circ \varphi = 0 \Leftrightarrow h \circ \varphi^{-1} = 0$$

So the first and fifth conditions collapse to one, and so do the second and fourth. These are the three conditions mentioned above.

3.3. First we assume that f and g are both Laurent polynomials. Then p is a Laurent polynomial, so we may use the expression for τ on Laurent polynomials. Now approximate f and g by Laurent polynomials, and use the fact that both sides of the equation represent continuous maps.

□

Theorem 3.7. *There exists a projection* $p \in A_\theta$ *such that* $\tau(p) = \theta$

Proof. Choose $\epsilon > 0$ such that $0 < \epsilon \leq \theta < \theta + \epsilon \leq 1$. Define

$$g(t) := \begin{cases} t/\epsilon & : 0 \leq t \leq \epsilon \\ 1 & : \epsilon \leq t \leq \theta \\ \epsilon^{-1}(\theta + \epsilon - t) & : \theta \leq t \leq \theta + \epsilon \\ 0 & : \theta + \epsilon \leq t \leq 1 \end{cases}$$

and

$$f(t) = \begin{cases} \sqrt{g(t) - g(t)^2} & : \theta \leq t \leq \theta + \epsilon \\ 0 & : \text{ otherwise} \end{cases}$$

Then both f and g define functions on $\mathbb{T}$ because $f(0) = f(1) = 0 = g(0) = g(1)$. The corresponding element p as defined above is a projection, and

$$\tau(p) = \int_{\mathbb{T}} g(z)dz = \frac{1}{2} \cdot \epsilon + (\theta - \epsilon) + \frac{1}{2} \cdot \epsilon = \theta$$

□

Theorem 3.8. *The range of the map*

$$K_0(\tau) : K_0(A_\theta) \to \mathbb{R}$$

contains $(\mathbb{Z} + \mathbb{Z}\theta)$.

Proof. Since $\tau(1) = 1$, the range of $K_0(\tau)$ contains $\mathbb{Z}$. If p_θ is the Rieffel projection from the previous theorem, then $\tau(p_\theta) = \theta$, so the range contains $\mathbb{Z}\theta$. □

Theorem 3.9 (Pimsner-Voiculescu)**.** *If $\theta \in \mathbb{R}$ is irrational, then the map $K_0(\tau)$ induces an isomorphism*

$$K_0(A_\theta) \to \mathbb{Z} + \mathbb{Z}\theta$$

In fact, if we define

$$(\mathbb{Z} + \mathbb{Z}\theta)^+ = (\mathbb{Z} + \mathbb{Z}\theta) \cap \mathbb{R}^+$$

Then this is an order isomorphism

$$(K_0(A_\theta), K_0(A_\theta)^+, [1]) \to (\mathbb{Z} + \mathbb{Z}\theta, (\mathbb{Z} + \mathbb{Z}\theta)^+, 1)$$

Corollary 3.10. *Let θ and θ' be two irrational numbers. Then $A_\theta \cong A_{\theta'}$ if and only if either $\theta - \theta'$ or $\theta + \theta'$ is an integer.*

Proof. If $\varphi : A_\theta \to A_{\theta'}$ is an isomorphism, and τ' is the trace on $A_{\theta'}$, then by uniqueness of the trace, $\tau' \circ \varphi$ must be the trace on A_θ. Hence, if $p_\theta \in A_\theta$ is the Rieffel projection, then

$$K_0(\tau')([\varphi(p_\theta)]_0) = K_0(\tau)[p_\theta]_0 = \tau(p_\theta) = \theta$$

Hence, $\theta \in \mathbb{Z} + \mathbb{Z}\theta'$, so $\exists a_1, b_1 \in \mathbb{Z}$ such that

$$\theta = a_1 + b_1\theta'$$

Similarly, $\theta' = a_2 + b_2\theta$ for some $a_2, b_2 \in \mathbb{Z}$. Hence,

$$\theta = a_1 + b_1 a_2 + b_1 b_2 \theta$$

Since $\theta \notin \mathbb{Q}$, it follows that $b_1 b_2 = 1$, so that $b_1 = b_2 = \pm 1$. Hence the result. $\square$

Remark 3.11. Let $\theta \in (0,1)$ be irrational and $n \in \mathbb{N}$, then

$$A_{n\theta} \subset A_\theta$$

Proof. Let $\alpha = n\theta$, and let $u' = u^n$ and $v' = v$, then

$$v'u' = e^{2\pi i \alpha} u'v'$$

Then $\exists$ a surjective $*$-homomorphism $\varphi : A_\alpha \to C^*(u^n, v)$ such that

$$\varphi(u) = u^n \text{ and } \varphi(v) = v$$

However, α is irrational, so A_α is simple, so φ is an isomorphism by the first isomorphism theorem. Hence,

$$A_{n\theta} \cong C^*(u^n, v) \subset A_\theta$$

$\square$

This implies (see [Rørdam, Larsen, and Laustsen, Exercise 5.8]) that, for any number $\alpha \in (\mathbb{Z} + \mathbb{Z}\theta) \cap [0,1]$, $\exists$ a projection $e \in A_\theta$ such that $\tau(e) = \alpha$.

(End of Day 11)

V. Inductive Limit C*-algebras

1. Products and sums of C*-algebras

Let $\{A_i\}_{i\in\mathbb{I}}$ be a family of C*-algebras. Define $\prod_{i\in\mathbb{I}} A_i$ to be the set of all functions

$$a : \mathbb{I} \to \bigcup_{i\in\mathbb{I}} A_i : a(i) \in A_i \quad \forall i \in \mathbb{I}$$

such that

$$\|a\| := \sup_{i\in\mathbb{I}} \|a(i)\| < \infty$$

Define

$$\mathcal{I} := \{a \in \prod A_i : a(i) = 0 \text{ for all but finitely many } i \in \mathbb{I}\}$$

and define

$$\sum_{i\in\mathbb{I}} A_i := \overline{\mathcal{I}}$$

Lemma 1.1. *1.1.* $\prod A_i$ *is a C*-algebra*

1.2. $\sum A_i$ *is a closed two-sided ideal of* $\prod A_i$

Let

$$\pi : \prod A_i \to \prod A_i / \sum A_i$$

be the quotient map.

Lemma 1.2. *Let* $\{A_n\}$ *be a sequence of algebras, and* $a \in \prod A_n$*, then*

$$\|\pi(a)\| = \limsup \|a_n\|$$

In particular, $a \in \sum A_i$ *if and only if* $\limsup \|a_n\| = 0$.

Proof. Since $\mathcal{I}$ is dense in $\sum A_n$, we have

$$\|\pi(a)\| = \inf\{\|a - b\| : b \in \mathcal{I}\}$$

If $b = (b_n) \in \mathcal{I}$, then b_n is eventually zero, so

$$\|a - b\| = \sup \|a_n - b_n\| \geq \limsup \|a_n - b_n\| = \limsup \|a_n\|$$

Hence, $\|\pi(a)\| \geq \limsup \|a_n\|$.

Conversely, for $k \in \mathbb{N}$, define $b^{(k)} \in \mathcal{I}$ by

$$b_n^{(k)} := \begin{cases} a_n & : n \leq k \\ 0 & : n > k \end{cases}$$

Then

$$\|\pi(a)\| \leq \inf_{k\in\mathbb{N}} \|a - b^{(k)}\| = \inf_{k\in\mathbb{N}} \sup_{n>k} \|a_n\| = \limsup \|a_n\|$$

$\square$

2. Inductive Limits

Let $\mathcal{C}$ be a category.

Definition 2.1. An inductive sequence in $\mathcal{C}$ is a sequence $\{A_n\}$ of objects in $\mathcal{C}$ together with morphisms $\varphi_n : A_n \to A_{n+1}$, usually written as

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \ldots$$

and denoted (A_n, φ_n). For $m > n$, define

$$\varphi_{m,n} = \varphi_{m-1} \circ \varphi_{m-2} \circ \ldots \circ \varphi_n : A_n \to A_m$$

and write $\varphi_{n,n} = \mathrm{id}_{A_n}, \varphi_{m,n} = 0$ if $m < n$. These are called the connecting maps of the sequence.

Definition 2.2. Given a sequence (A_n, φ_n) in $\mathcal{C}$, and inductive limit is a system $(A, \{\mu_n\})$ where A is an object in $\mathcal{C}$ and $\mu_n : A_n \to A$ are morphisms with the following two properties:

2.1. The following diagram commutes for each $n \in \mathbb{N}$

$$\begin{tikzcd} A_n \arrow[rr, "\varphi_n"] \arrow[dr, "\mu_n"'] & & A_{n+1} \arrow[dl, "\mu_{n+1}"] \\ & A & \end{tikzcd}$$

2.2. If $(B, \{\lambda_n\})$ is another system where B is an object in $\mathcal{C}$ and $\lambda_n : A_n \to B$ are morphisms such that $\lambda_n = \lambda_{n+1} \circ \varphi_n$ for all $n \in \mathbb{N}$, then there exists a unique morphism $\lambda : A \to B$ such that the following diagram commutes

$$\begin{tikzcd} & A_n \arrow[dl, "\mu_n"'] \arrow[dr, "\lambda_n"] & \\ A \arrow[rr, "\lambda"'] & & B \end{tikzcd}$$

Remark 2.3. 2.1. Inductive limits do not always exist. For instance, in the category of finite sets. We will show that they exist in the category of C*-algebras, of abelian groups, and of ordered abelian groups.

2.2. If an inductive limit exists, it is unique by the second property above.

Example 2.4. 2.1. Let D be a C*-algebra and $A_n \subset A_{n+1} \subset D$ be an increasing chain of subalgebras. If $\varphi_n = \iota_n : A_n \hookrightarrow A_{n+1}$, then $(A, \{j_n\})$ is an inductive limit of (A_n, ι_n), where

$$A := \overline{\bigcup_{n=1}^{\infty} A_n}$$

and $\mu_n = j_n : A_n \hookrightarrow A$ is the inclusion map because

(i) $\mu_n = \mu_{n+1} \circ \iota_n$ for all $n \in \mathbb{N}$.

(ii) If $(B, \{\lambda_n\})$ is another system as above, then define $\lambda : A \to B$ by

$$\lambda(a) = \lambda_n(a) \text{ if } a \in A_n$$

This is well-defined, because if $a \in A_n \subset A_{n+1}$, then

$$\lambda_{n+1}(a) = \lambda_{n+1}(\iota_n(a)) = \lambda_n(a)$$

Then it follows that $\lambda \circ \mu_n = \lambda_n$ for all $n \in \mathbb{N}$. Furthermore, this map λ is a $*$-homomorphism, and is unique because $\bigcup A_n$ is dense in A.

2.2. Let $A_n = M_n(\mathbb{C})$ and $\varphi_n : A_n \to A_{n+1}$ is the map

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

Let $\mathcal{K}(H)$ denote the compact operators on $H = \ell^2$, then fix an ONB $\{e_i\}$ of H. Define $p_n \in \mathcal{K}(H)$ to be the canonical rank n projection. If $x, y \in H$, define $x \otimes y \in \mathcal{K}(H)$ by

$$(x \otimes y)(z) = \langle z, x\rangle y$$

Then $p_n = \sum_{i=1}^n e_i \otimes e_i$.

(i) Define $\mu_n : M_n(\mathbb{C}) \to \mathcal{K}(H)$ by

$$\mu_n(a_{i,j}) = \sum_{i,j=1}^{n} a_{i,j} e_i \otimes e_j$$

Then μ_n is injective, and the range of μ_n is $p_n \mathcal{K}(H) p_n$.

Proof. μ_n is injective because the set $\{e_i \otimes e_j\}$ is linearly independent. As for surjectivity onto $p_n\mathcal{K}(H)p_n$, note that if $u \in p_n\mathcal{K}(H)p_n$, then

$$\begin{aligned} u &= p_n u p_n \\ &= \sum_{i,j=1}^{n} (e_i \otimes e_i) u (e_j \otimes e_j) \\ &= \sum_{i,j=1}^{n} \langle u(e_i), e_j \rangle e_i \otimes e_j \\ &= \mu_n(a_{i,j}) \end{aligned}$$

where $a_{i,j} = \langle u(e_i), e_j \rangle$. ☐

(ii) Check that $\mu_{n+1} \circ \varphi_n = \mu_n$

(iii) Finally, observe that

$$\mathcal{K}(H) = \overline{\bigcup_{n=1}^{\infty} p_n\mathcal{K}(H)p_n} = \overline{\bigcup_{n=1}^{\infty} \mu_n(M_n(\mathbb{C}))}$$

(iv) As in the previous example, we see that $(\mathcal{K}(H), \{\mu_n\})$ is an inductive limit of $(M_n(\mathbb{C}), \varphi_n)$.

Proposition 2.5 (Inductive Limits of C*-algebras)**.** *Given an inductive system (A_n, φ_n) of C*-algebras, an inductive limit $(A, \{\mu_n\})$ exists.*

Proof. Consider the quotient map

$$\pi : \prod A_n \to \prod A_n / \sum A_n =: Q$$

and let $\varphi_{m,n} : A_n \to A_m$ as above.

2.1. Define $\nu_n : A_n \to \prod_m A_m$ by

$$\nu_n(a) = (\varphi_{m,n}(a))$$

This is well-defined, because $\|\varphi_{m,n}(a)\| \leq \|a\|$ for all $m \in \mathbb{N}$. Furthermore, ν_n is clearly a $*$-homomorphism.

2.2. Let $\mu_n : A_n \to Q$ by $\mu_n = \pi \circ \nu_n$, then observe that if $a \in A_n$, then

$$c := \nu_n(a) - (\nu_{n+1} \circ \varphi_n)(a)$$

has the form $c_n = a$ and $c_m = 0$ when $m \neq n$. Hence, $c \in \sum A_i$, so that

$$\mu_n(a) - (\mu_{n+1} \circ \varphi_n)(a) = \pi(c) = 0$$

Hence, $\mu_n = \mu_{n+1} \circ \varphi$.

2.3. Thus, $\{\mu_n(A_n)\}$ is an increasing sequence of C*-subalgebras of Q. Define

$$A := \overline{\bigcup_{n=1}^{\infty} \mu_n(A_n)}$$

Then A is a C*-algebra, and $\mu_n : A_n \to A$ is a sequence of $*$-homomorphisms satisfying the first condition of Definition 2.2.

2.4. To prove the second condition, suppose $(B, \{\lambda_n\})$ is another system such that $\lambda_n = \lambda_{n+1} \circ \varphi_n$. Then

$$\lambda_m \circ \varphi_{m,n} = \lambda_n \quad \forall m > n$$

Hence, $\|\lambda_n(a)\| \leq \|\varphi_{m,n}(a)\|$. So

$$\|\lambda_n(a)\| \leq \limsup \|\varphi_{m,n}(a)\| = \|\pi(\nu_n(a))\| = \|\mu_n(a)\|$$

Hence, $\ker(\mu_n) \subset \ker(\lambda_n)$. By the first isomorphism theorem, $\exists$ a unique $*$-homomorphism,

$$\lambda_n' : \mu_n(A_n) \to B \text{ such that } \lambda_n' \circ \mu_n = \lambda_n$$

By uniqueness, $\lambda_{n+1}'|_{\mu_n(A_n)} = \lambda_n'$. Hence, we get a $*$-homomorphism

$$\lambda' : \bigcup_{n=1}^{\infty} \mu_n(A_n) \to B$$

which extends λ_n'. λ is a contraction, so it extends to a $*$-homomorphism

$$\lambda : A \to B$$

such that $\lambda \circ \mu_n = \lambda_n' \circ \mu_n = \lambda_n$. Furthermore, λ is unique with this property because

$$A = \overline{\bigcup_{n=1}^{\infty} \mu_n(A_n)}$$

$\square$

(End of Day 12)

Remark 2.6. We observe the following from the above proof:

2.1.
$$A = \overline{\bigcup_{n=1}^{\infty} \mu_n(A_n)}$$

2.2. $\|\mu_n(a)\| = \limsup_{m\to\infty} \|\varphi_{m,n}(a)\|$ for all $a \in A_n$

2.3.
$$\ker(\mu_n) = \{a \in A_n : \limsup_{m\to\infty} \|\varphi_{m,n}(a)\|\}$$

2.4. If $(B, \{\lambda_n\})$ is another system as in Definition 2.2, then $\ker(\mu_n) \subset \ker(\lambda_n)$

2.5. If each φ_n is injective, then so are the μ_n.

Lemma 2.7. *Let (A_n, φ_n) be an inductive system with inductive limit (A, μ_n). If (B, λ_n) is another system as in Definition 2.2, and $\lambda : A \to B$ the unique $*$-homomorphism guaranteed by Definition 2.2, then*

2.1. λ is injective iff $\ker(\lambda_n) \subset \ker(\mu_n)$ for all $n \in \mathbb{N}$, which is equivalent to $\ker(\lambda_n) = \ker(\mu_n)$ for all $n \in \mathbb{N}$.

2.2. λ is surjective iff $B = \overline{\bigcup_{n=1}^\infty \lambda_n(A_n)}$.

Proof. Exercise (See [Rørdam, Larsen, and Laustsen, Proposition 6.2.4]) □

Proposition 2.8. *Let (G_n, α_n) be an inductive system of abelian groups, then an inductive limit (G, β_n) exists. Moreover, one has*

2.1.
$$G = \bigcup_{n=1}^\infty \beta_n(G_n)$$

2.2.
$$\ker(\beta_n) = \bigcup_{m=n+1}^\infty \ker(\alpha_{m,n})$$

2.3. If (H, γ_n) is another system and $\gamma : G \to H$ the unique group homomorphism as in Definition 2.2, then

(i) γ is injective iff $\ker(\gamma_n) = \ker(\beta_n)$ for all $n \in \mathbb{N}$

(ii) γ is surjective iff $H = \bigcup_{n=1}^\infty \gamma_n(G_n)$

Proof. The proof is similar to the one above. We give an outline.

$$\prod G_n$$

to be the set of all infinite sequences $(g_1, g_2, \ldots)$ with $g_i \in G_i$. Define

$$\sum G_n$$

to be the set of those sequences which are eventually zero. Note that $\sum G_n$ is a subgroup of $\prod G_n$, and these are all abelian groups. Let

$$\pi : \prod G_n \to \prod G_n / \sum G_n =: Q$$

be the quotient map. Now define $\beta_n : G_n \to Q$ exactly as above so that $\beta_n = \beta_{n+1} \circ \alpha_n$, and set

$$G := \bigcup_{n=1}^{\infty} \beta_n(G_n)$$

Check that (G, β_n) is an inductive limit of the system. $\square$

Example 2.9. 2.1. Consider $G_n = \mathbb{Z}$ and $\alpha_n(1) = n + 1$. ie. We may picture the system as

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \dots$$

Define $\gamma_n : \mathbb{Z} \to \mathbb{Q}$ by

$$\gamma_n(1) = \frac{1}{n!}$$

Then γ_n is a group homomorphism such that $\gamma_n = \gamma_{n+1} \circ \alpha_n$. Hence, $(\mathbb{Q}, \{\gamma_n\})$ is a system that satisfies (i) in Definition 2.2. Let $(G, \{\beta_n\})$ be an inductive limit of this system, then there is a group homomorphism

$$\gamma : G \to \mathbb{Q} \text{ such that } \gamma \circ \alpha_n = \gamma_n$$

Since

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} \gamma_n(G_n)$$

it follows that γ is surjective. Also, since

$$\ker(\beta_n) = \bigcup_{m=n+1}^{\infty} \ker(\alpha_{m,n})$$

and each α_n is injective, it follows that β_n is injective for all n. We see that each γ_n is also injective. Hence,

$$\ker(\gamma_n) = \ker(\beta_n)$$

for all $n \in \mathbb{N}$. Hence, γ is injective as well.

2.2. Let $G_n = \mathbb{Z}$ and $\alpha_n(1) = 2$ for all $n \in \mathbb{N}$. ie. We may picture the system as

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \dots$$

Define $\gamma_n : \mathbb{Z} \to \mathbb{Q}$ by

$$\gamma_n(1) = \frac{1}{2^n}$$

Then $\gamma_n = \gamma_{n+1} \circ \alpha_n$. Hence, $(\mathbb{Q}, \{\gamma_n\})$ is a system that satisfies the first condition of Definition 2.2. Hence, if $(G, \{\beta_n\})$ is an inductive limit of the system, then there is a group homomorphism

$$\gamma : G \to \mathbb{Q} \text{ such that } \gamma \circ \alpha_n = \gamma_n$$

As in the previous example, we may check that

$$\ker(\beta_n) = \ker(\gamma_n) = \{0\}$$

so that γ is injective. However, γ is not surjective, but does surject onto

$$H = \bigcup_{n=1}^{\infty} \gamma_n(G_n) \cong \left\{\frac{m}{2^n} : m \in \mathbb{Z}, n \geq 0\right\} \cong \mathbb{Z}\left[\frac{1}{2}\right]$$

This is the inductive limit of the system.

Proposition 2.10 (Inductive Limits of ordered Abelian groups)**.** *Let (G_n, α_n) be an inductive system of ordered abelian groups where $\alpha_n : G_n \to G_{n+1}$ are positive group homomorphisms. Let (G, β_n) be an inductive limit of this system, and define*

$$G^+ = \bigcup_{n=1}^{\infty} \beta_n(G_n^+)$$

Then (G, G^+) is an ordered abelian group, β_n is a positive group homomorphism, and $(G, G^+, \{\beta_n\})$ is an inductive limit in the category of ordered abelian groups.

Proof. There are a few things that need to be checked:

2.1. $G^+ + G^+ \subset G^+$: Note that

$$\beta_n(G_n^+) = \beta_{n+1}(\alpha_n(G_n^+)) \subset \beta_{n+1}(G_{n+1}^+)$$

so $\{\beta_n(G_n^+)\}$ is an increasing sequence of subsets of G. Hence, the union of closed under addition.

2.2. Show that $G^+ \cap (-G^+) = \{0\}$: If $x \in G^+ \cap (-G^+)$, then $x \in \beta_n(G_n^+) \cap (-\beta_n(G_n^+)$ for some $n \in \mathbb{N}$. Hence, $x = \beta_n(y_1) = -\beta_n(y_2)$ for some $y_1, y_2 \in G_n^+$. Now,

$$\beta_n(y_1 + y_2) = 0$$

Since

$$\ker(\beta_n) = \bigcup_{m=n+1}^{\infty} \ker(\alpha_{m,n})$$

$\exists m \geq n$ such that $\alpha_{m,n}(y_1 + y_2) = 0$. Let $z_i = \alpha_{m,n}(y_i) \in G_m^+$, then

$$z_1 = -z_2 \in G_m^+ \cap (-G_m^+)$$

Hence, $z_1 = z_2 = 0$. Thus, $x = \beta_m(z_1) = 0$

2.3. $G^+ - G^+ = G$: If $x \in G$, then $x \in \beta_n(G_n)$ for some n. Now

$$\beta_n(G_n) = \beta_n(G_n^+ - G_n^+) \subset \beta_n(G_n^+) - \beta_n(G_n^+) \subset G^+ - G^+$$

2.4. If (H, γ_n) is a system where H is an ordered abelian group and $\gamma_n : G_n \to H$ is a positive group homomorphism, then there is a positive group homomorphism $\gamma : G \to H$ making the required diagram commute: By the universal property in the category of abelian groups, $\exists$ a group homomorphism

$$\gamma : G \to H$$

making the required diagram commute. One needs to verify that γ is positive. But

$$\gamma(G^+) = \gamma\left(\bigcup_{n=1}^{\infty} \beta_n(G_n^+)\right) = \bigcup_{n=1}^{\infty} (\gamma \circ \beta_n)(G_n^+) = \bigcup_{n=1}^{\infty} \gamma_n(G_n^+) \subset H^+$$

□

3. Continuity of K_0

Lemma 3.1. *Let p be a projection in A and $a \in A_{sa}$. Let $\delta := \|p - a\|$, then*

$$sp(a) \subset [-\delta, \delta] \cup [1 - \delta, 1 + \delta]$$

Proof. Note that $sp(p) \subset \{0, 1\}$, so suppose $t \in \mathbb{R}$ such that

$$d := \min\{|t|, |1 - t|\} > \delta$$

We WTS: $t \notin sp(a)$. Note that $(p - t1) \in GL(\widetilde{A})$ and

$$\|(p - t1)^{-1}\| = \max\{|-t|^{-1}, |1 - t|^{-1}\} = d^{-1} < \delta^{-1}$$

Hence,

$$\|(p - t1)^{-1}(a - t1) - 1\| = \|(p - t1)^{-1}(a - p)\| \leq d^{-1}\delta < 1$$

Thus, $(p - t1)^{-1}(a - t1) \in GL(\widetilde{A})$, hence

$$a - t1 \in GL(\widetilde{A})$$

so $t \notin sp(a)$ as required. □

(End of Day 13)

Lemma 3.2. *Let $p, q \in A$ be projections such that $\|p - q\| < 1$, then $p \sim_h q$*

Proof. Let $\delta := \|p - q\|/2 < 1/2$, and let $a_t := (1 - t)p + tq$, then $a_t \in A_{sa}$ and

$$sp(a_t) \subset K := [-\delta, \delta] \cup [1 - \delta, 1 + \delta]$$

Let $f : K \to \mathbb{C}$ be the map

$$f(t) = \begin{cases} 0 & : |t| \leq \delta \\ 1 & : |t-1| \leq \delta \end{cases}$$

Then f is continuous. Hence by [Rørdam, Larsen, and Laustsen, Lemma 1.2.5], the induced map

$$f : \Omega_K \to \mathbb{C}$$

is also continuous. Since $t \mapsto a_t$ is a path in Ω_K, it follows that

$$t \mapsto f(a_t)$$

is a continuous path of projections. Furthermore,

$$p = f(p) = f(a_0) \sim_h f(a_1) = f(q) = q$$

□

Lemma 3.3. *Let A be a C^*-algebra.*

3.1. Let $a \in A_{sa}$ such that $\delta = \|a - a^2\| < 1/4$, then $\exists$ a projection $p \in A$ such that $\|a - p\| \leq 2\delta$

3.2. Let $p, q \in A$ be projections and $x \in A$ such that $\|x^*x - p\| < 1/2$ and $\|xx^* - q\| < 1/2$, then $p \sim q$

Proof. 3.1. If $t \in \mathbb{R}$ such that $\min\{|t|, |1-t|\} > 2\delta$, then

$$|t - t^2| > 4\delta^2 > \frac{1}{4}$$

Hence, if $|t - t^2| \leq \delta < 1/4$, then

$$t \in [-2\delta, 2\delta] \cup [1 - 2\delta, 1 + 2\delta]$$

Since a is self-adjoint, we conclude that

$$sp(a) \subset [-2\delta, 2\delta] \cup [1 - 2\delta, 1 + 2\delta]$$

Let $p = f(a)$, where

$$f(t) = \begin{cases} 0 & : t \leq 2\delta \\ 1 & : t \geq 1 - 2\delta \end{cases}$$

Then $p = p^2 = p^*$ because $f = f^2 = \overline{f}$. Furthermore,

$$|t - f(t)| \leq 2\delta \quad \forall t \in sp(a)$$

Hence $\|a - p\| \leq 2\delta$

3.2. Let
$$\delta = \frac{1}{2}\max\{\|x^*x - p\|, \|xx^* - q\|\} < \frac{1}{4}$$
and set $\Gamma := sp(x^*x) \cup sp(xx^*)$, then by the previous lemma,
$$\Gamma \subset [-2\delta, +2\delta] \cup [1-2\delta, 1+2\delta]$$
Let f as above, and set $p_0 := f(x^*x), q_0 := f(xx^*)$, then p_0, q_0 are projections such that
$$\|p - p_0\| \leq 4\delta < 1 \text{ and } \|q - q_0\| \leq 4\delta < 1$$
Hence, $p \sim p_0$ and $q \sim q_0$ by the above lemma. We now show that $p_0 \sim q_0$.

(i) First, note that $x(x^*x)x^* = (xx^*)xx^*$. Hence, for any polynomial $p \in C(\Gamma)$,
$$xp(x^*x)x^* = p(xx^*)xx^*$$
Thus, the same is true for any $p \in C(\Gamma)$ by density. Let $g \in C(\Gamma)$ be the function
$$g(t) = \begin{cases} \sqrt{\frac{f(t)}{t}} & : t \neq 0 \\ 0 & : t = 0 \end{cases}$$
This is continuous because $f \geq 0$ and $f(t) = 0$ if $t \leq 2\delta$. Observe that
$$tg(t)^2 = f(t) \quad \forall t \in \Gamma$$
$$v := xg(x^*x)$$
Then
$$v^*v = g(x^*x)x^*xg(x^*x) = x^*xg(x^*x)^2 = f(x^*x) = p_0$$
and
$$vv^* = xg(x^*x)g(x^*x)x^* = xg(x^*x)^2x^* = g(xx^*)^2xx^* = f(xx^*) = q_0$$
Hence, $p_0 \sim q_0$ as required.

$\square$

Remark 3.4. Given an inductive sequence
$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \dots$$
of C*-algebras, let $(A, \{\mu_n\})$ be the limit of the sequence. (ie. the following diagram commutes
$$\begin{tikzcd} A_n \arrow[rr, "\varphi_n"] \arrow[dr, "\mu_n"'] & & A_{n+1} \arrow[dl, "\mu_{n+1}"] \\ & A & \end{tikzcd}$$

and A is universal with this property). Then we get an inductive sequence of Abelian groups

$$K_0(A_1) \xrightarrow{K_0(\varphi_1)} K_0(A_2) \xrightarrow{K_0(\varphi_2)} K_0(A_3) \xrightarrow{K_0(\varphi_3)} \dots$$

Let $(G, \{\beta_n\})$ be the inductive limit of this sequence. ie. the following diagram commutes

$$\begin{tikzcd} K_0(A_n) \arrow[rr, "K_0(\varphi_n)"] \arrow[dr, "\beta_n"'] & & K_0(A_{n+1}) \arrow[dl, "\beta_{n+1}"] \\ & G_0 & \end{tikzcd}$$

Theorem 3.5 (Continuity of K_0)**.** *Given an inductive system (A_n, φ_n) of C^*-algebras with inductive limit A, we have*

$$K_0(A) \cong \lim(K_0(A_n), K_0(\varphi_n))$$

In fact, there is a unique group isomorphism $\gamma : G_0 \to K_0(A)$ such that the following diagram commutes

$$\begin{tikzcd} & K_0(A_n) \arrow[dl, "\beta_n"'] \arrow[dr, "K_0(\mu_n)"] & \\ G_0 \arrow[rr, "\gamma"'] & & K_0(A) \end{tikzcd}$$

Proof. Note that the following diagram commutes

$$\begin{tikzcd} K_0(A_n) \arrow[rr, "K_0(\varphi_n)"] \arrow[dr, "K_0(\mu_n)"'] & & K_0(A_{n+1}) \arrow[dl, "K_0(\mu_{n+1})"] \\ & K_0(A) & \end{tikzcd}$$

Hence, by the universal property of the inductive limit, there is a group homomorphism

$$\gamma : G_0 \to K_0(A)$$

such that $\gamma \circ \beta_n = K_0(\mu_n)$. We WTS: γ is bijective.

3.1. γ is injective: To prove this, by Proposition 2.8 above, we need to show that

$$\ker(\beta_n) = \ker(K_0(\mu_n)) \quad \forall n \in \mathbb{N}$$

Since $\gamma \circ \beta_n = K_0(\mu_n)$ clearly, $\ker(\beta_n) \subset \ker(K_0(\mu_n))$. So suppose $g \in \ker(K_0(\mu_n)) \subset K_0(A_n)$, then $\exists$ a projection $p \in M_k(\widetilde{A_n})$ such that

$$g = [p]_0 - [s(p)]_0 \text{ and } \widetilde{\mu_n}(p) \sim \widetilde{\mu_n}(s(p)) \text{ in } M_k(\widetilde{A})$$

Hence, $\exists v \in M_k(\widetilde{A})$ such that

$$\widetilde{\mu_n}(p) = v^* v \text{ and } \widetilde{\mu_n}(s(p)) = vv^*$$

However,

$$A = \overline{\bigcup_{j=1}^\infty \mu_j(A_j)}$$

Taking unitizations, and matrices, we see that

$$M_k(\widetilde{A})) = \overline{\bigcup_{j=1}^\infty \widetilde{\mu_j}(M_k(\widetilde{A_j}))}$$

Hence, $\exists \ell \geq n$ and $x_\ell \in M_k(\widetilde{A_\ell})$ such that $\widetilde{\mu_\ell}(x_\ell)$ is close enough to v so that

$$\|\widetilde{\mu_\ell}(x_\ell^* x_\ell) - \widetilde{\mu_n}(p)\| < 1/2 \text{ and } \|\widetilde{\mu_\ell}(x_\ell x_\ell^*) - \widetilde{\mu_n}(s(p))\| < 1/2 \text{ in } M_k(\widetilde{A})$$

Now note that $\widetilde{\mu_n} = \widetilde{\mu_\ell} \circ \widetilde{\varphi_{\ell,n}}$, so

$$\|\widetilde{\mu_\ell}[x_\ell^* x_\ell - \widetilde{\varphi_{\ell,n}}(p)]\| < 1/2$$

But by Remark 2.6,

$$\|\widetilde{\mu_\ell}(a)\| = \lim_{m\to\infty} \|\widetilde{\varphi_{m,\ell}}(a)\|$$

Hence, $\exists m \geq \ell$ such that

$$\|\widetilde{\varphi_{m,\ell}}[x_\ell^* x_\ell - \widetilde{\varphi_{\ell,n}}(p)]\| < 1/2$$

So if $x_m = \widetilde{\varphi_{m,\ell}}(x_\ell)$, then

$$\|x_m^* x_m - \widetilde{\varphi_{m,n}}(p)\| < 1/2$$

Applying the same idea to the second equation above, we can arrange it so that

$$\|x_m x_m^* - \widetilde{\varphi_{m,n}}(s(p))\| < 1/2$$

(Note that in principle, we get two m's for the two equations, but the max of the two will work for both). Hence by the previous lemma,

$$\widetilde{\varphi_{m,n}}(p) \sim \widetilde{\varphi_{m,n}}(s(p)) \text{ in } M_k(\widetilde{A_m})$$

Thus,

$$K_0(\varphi_{m,n})(g) = 0$$

But $\beta_n = \beta_m \circ K_0(\varphi_{m,n})$, so that $\beta_n(g) = 0$, whence

$$\ker(K_0(\mu_n)) \subset \ker(\beta_n)$$

(End of Day 14)

3.2. γ is surjective: To prove this, we need to show that

$$K_0(A) = \bigcup_{j=1}^{\infty} K_0(\mu_j)(K_0(A_j))$$

Clearly, $\supset$ holds, so we fix $g \in K_0(A)$, and we WTS: $\exists n \in \mathbb{N}$ such that $g \in K_0(\mu_n)(K_0(A_n))$. So write

$$g = [p]_0 - [s(p)]_0$$

for some projection $p \in M_k(\widetilde{A})$. Since

$$M_k(\widetilde{A}) = \overline{\bigcup_{j=1}^{\infty} M_k(\widetilde{A_j})}$$

$\exists b_n \in M_k(\widetilde{A_n})$ such that $\|\widetilde{\mu_n}(b_n) - p\| < 1/5$. Let

$$a_n := \frac{b_n + b_n^*}{2}$$

and set $a_m := \widetilde{\varphi_{m,n}}(a_n)$, then a_n is self-adjoint, and

$$\|\widetilde{\mu_m}(a_m) - p\| = \|\widetilde{\mu_n}(a_n) - p\| \leq \frac{1}{2}\left\{\|\widetilde{\mu_n}(b_n) - p\| + \|b_n^* - p\|\right\} < \frac{1}{5}$$

By the above lemma,

$$sp(\widetilde{\mu_n}(a_n)) \subset [-1/5, +1/5] \cup [4/5, 6/5]$$

Hence, by using calculus on the function $t \mapsto t^2 - t$,

$$\|\widetilde{\mu_n}(a_n^2 - a_n)\| = \max\{|t^2 - t| : t \in sp(\widetilde{\mu_n}(a_n))\} < \frac{1}{4}$$

Once again, since

$$\|\widetilde{\mu_n}(x)\| = \lim_{m\to\infty} \|\widetilde{\varphi_{m,n}}(x)\|$$

it follows that $\exists m \geq n$ such that

$$\|a_m^2 - a_m\| < 1/4$$

By the previous lemma, $\exists q \in M_k(\widetilde{A_m})$ a projection such that $\|a_m - q\| < 1/2$. Now

$$\|\widetilde{\mu_m}(q) - p\| \leq \|q - a_m\| + \|\widetilde{\mu_m}(a_m) - p\| < 1$$

so $\widetilde{\mu_m}(q) \sim_h p$. Hence,

$$g = [p]_0 - [s(p)]_0 = [\widetilde{\mu_m}(q)]_0 - [s(\widetilde{\mu_m}(q))]_0 = K_0(\mu_m)([q]_0 - [s(q)]_0)$$

Hence,

$$g \in \bigcup_{j=1}^{\infty} K_0(\mu_j)(K_0(A_j))$$

as required.

□

Proposition 3.6. *If each $(K_0(A_n), K_0(A_n)^+)$ is an ordered abelian group, then so is $(K_0(A), K_0(A)^+)$ and*

$$K_0(A) \cong \lim K_0(A_n)$$

in the category of ordered abelian groups. ie.

$$K_0(A)^+ = \bigcup_{j=1}^{\infty} K_0(\mu_j)(K_0(A_j)^+)$$

Proof. Since μ_n is a $*$-homomorphism, $K_0(\mu_n)$ is a positive group homomorphism, so

$$K_0(\mu_n)(K_0(A_n)^+) \subset K_0(A)^+$$

Conversely, suppose $g \in K_0(A)^+$, then $g = [p]_0$ for some projection $p \in M_k(A)$. As in the proof of surjectivity above, $\exists m \in \mathbb{N}$ and a projection $q \in M_k(A_m)$ such that

$$\|\mu_m(q) - p\| < 1$$

Hence, $\mu_m(q) \sim_h p$, so

$$g = [\mu_m(q)]_0 = K_0(\mu_m)([q]_0) \in \bigcup_{j=1}^{\infty} K_0(\mu_j)(K_0(A_j)^+)$$

□

(End of Day 15)

4. Stabilized C*-algebras

In what follows, $\otimes$ refers to the minimal tensor product between two C*-algebras.

Theorem 4.1. *Let (A_n, φ_n) be an inductive system of C*-algebras, where each φ_n is injective. Let B be any C*-algebra, then*

$$(\lim(A_n, \varphi_n)) \otimes B \cong \lim(A_n \otimes B, \varphi_n \otimes id_B)$$

Proof. Let $(A, \{\mu_n\})$ be the inductive limit of (A_n, φ_n). ie. $\mu_n = \mu_{n+1} \circ \varphi_n$ holds. Note that $(A_n \otimes B, \varphi_n \otimes \mathrm{id}_B)$ is an inductive system. Let $(C, \{\lambda_n\})$ be the inductive limit of the system, so that $\lambda_n = \lambda_{n+1} \circ \varphi_n \otimes \mathrm{id}_B$ for all $n \in \mathbb{N}$. Note that

$$\mu_n \otimes \mathrm{id}_B : A_n \otimes B \to A \otimes B$$

has the property that

$$(\mu_n \otimes \mathrm{id}_B) = (\mu_{n+1} \otimes \mathrm{id}_B) \circ (\varphi_n \otimes \mathrm{id}_B)$$

Hence, by the universal property of C, there is a unique $*$-homomorphism $\lambda : C \to A \otimes B$ such that

$$\begin{tikzcd} & A_n \otimes B \arrow[dl, "\lambda_n"'] \arrow[dr, "\mu_n \otimes \mathrm{id}_B"] & \\ C \arrow[rr, "\lambda"'] & & A \otimes B \end{tikzcd}$$

4.1. λ is surjective: To show this, we need to show that

$$A \otimes B = \overline{\bigcup_{n=1}^{\infty} (\mu_n \otimes \mathrm{id}_B)(A_n \otimes B)}$$

For this, let $\epsilon > 0$ and $z \in A \otimes_{alg} B$ be given by

$$z = \sum_{i=1}^{m} a_i \otimes b_i$$

where $a_i \in A$ and $b_i \in B$. Since $A = \overline{\bigcup_{j=1}^{\infty} \mu_j(A_j)}$, $\exists n \in \mathbb{N}$ and $s_i \in A_n$ such that

$$\|\mu_n(s_i) - a_i\| < \frac{\epsilon}{m \max_i \|b_i\|}$$

Then $x := \sum_{i=1}^{m} s_i \otimes b_i \in A_n \otimes B$ is such that

$$\|(\mu_n \otimes \mathrm{id}_B)(x) - z\| < \epsilon$$

Hence,

$$z \in \overline{\bigcup_{n=1}^{\infty} (\mu_n \otimes \mathrm{id}_B)(A_n \otimes B)}$$

This is true for every $z \in A \otimes_{alg} B$, and hence for every $z \in A \otimes B$.

4.2. λ is injective: Since

$$C = \overline{\bigcup_{n=1}^{\infty} \lambda_n(A_n \otimes B)}$$

it suffices to show that λ is isometric on each $\lambda_n(A_n \otimes B)$. But

$$\lambda \circ \lambda_n = \mu_n \otimes \mathrm{id}_B$$

But μ_n is injective (see the construction of the inductive limit in Proposition 2.5), and id_B is injective, so $\mu_n \otimes \mathrm{id}_B$ is injective on $A \otimes B$ (See [Murphy, Theorem 6.5.1]). Hence, λ must be isometric on $\lambda_n(A_n \otimes B)$ as required.

□

Remark 4.2. The same result holds for $\otimes_{max}$ without the requirement that the maps φ_n be injective. To prove this, one needs two things:

4.1. The universal property of $\otimes_{max}$: Given two $*$-homomorphism $\eta : A \to C$ and $\delta : B \to C$ with commuting ranges, there is a unique $*$-homomorphism $\theta : A \otimes_{max} B \to C$ such that $\theta(a \otimes b) = \eta(a)\delta(b)$.

4.2. $\otimes_{max} B$ is an exact functor.

We write $\mathcal{K} := \mathcal{K}(\ell^2)$

Proposition 4.3. *For any C^*-algebra A, define $\varphi_n : M_n(A) \to M_{n+1}(A)$ by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Then*

$$A \otimes \mathcal{K} \cong \lim(M_n(A), \varphi_n)$$

Proof. Example 2.4(2) + Theorem 4.3. □

Definition 4.4. Let $e \in \mathcal{K}$ be the fixed projection of rank one $e_1 \otimes e_1$, and $\kappa : A \to A \otimes \mathcal{K}$ be given by $a \mapsto a \otimes e$. Then κ is an injective $*$-homomorphism, called the canonical inclusion of A into $A \otimes \mathcal{K}$

Lemma 4.5. *Let $p \in \mathcal{K}$ be any rank one projection and $\varphi : A \to A \otimes \mathcal{K}$ be given by $a \mapsto a \otimes p$, then $K_0(\varphi) = K_0(\alpha)$*

Proof. Note that $p \sim e$ and $1 - p \sim 1 - e$, so $\exists u \in \mathcal{U}(\mathcal{B}(H))$ such that $e = upu^*$. By the Borel functional calculus, $\exists h \in \mathcal{B}(H)$ self-adjoint such that $u = e^{ih}$. Hence the path $u_t := e^{ith}$ connects u to the identity. Hence, $e = upu^* \sim_h p$. Furthermore, if $\varphi_t : A \to A \otimes \mathcal{K}$ is given by

$$a \mapsto a \otimes u_t p u_t^*$$

Then φ_t is a path of $*$-homomorphisms such that $\varphi_0 = \varphi$ and $\varphi_1 = \alpha$. Hence, $K_0(\alpha) = K_0(\varphi)$. □

Theorem 4.6 (Stability of K_0)**.** *The map $\kappa : A \to A \otimes \mathcal{K}$ induces an isomorphism $K_0(\kappa) : K_0(A) \to K_0(A \otimes \mathcal{K})$*

Proof. Let $\varphi_n : M_n(A) \to M_{n+1}(A)$ and $\mu_n : M_n(A) \to A \otimes \mathcal{K}$ be the maps as above

4.1. $K_0(\kappa)$ is surjective:

$$K_0(A \otimes \mathcal{K}) = \bigcup_{j=1}^{\infty} K_0(\mu_n)(K_0(M_n(A))$$

so if $g \in K_0(A \otimes \mathcal{K}), \exists n \in \mathbb{N}$ and $g' \in K_0(M_n(A))$ such that

$$g = K_0(\mu_n)(g')$$

But $\varphi_{n,1} : A \to M_n(A)$ is the map λ_n from III.3.8. Hence, $K_0(\varphi_{n,1}) : K_0(A) \to K_0(M_n(A))$ is an isomorphism, so $\exists h \in K_0(A)$ such that $g' = K_0(\varphi_{n,1})(h)$. Hence,

$$g = K_0(\mu_n \circ \varphi_{n,1})(h) = K_0(\kappa)(h)$$

so $K_0(\kappa)$ is surjective.

4.2. $K_0(\kappa)$ is injective: If $h \in K_0(A)$ is such that $K_0(\kappa)(h) = 0$, then

$$K_0(\mu_n)K_0(\varphi_{n,1})(h) = 0 \quad \forall n \in \mathbb{N}$$

But by Proposition 2.8,

$$\ker(K_0(\mu_n)) = \bigcup_{m=n+1}^{\infty} \ker(K_0(\varphi_{m,n}))$$

hence,

$$K_0(\varphi_{m,n})(K_0(\varphi_{n,1}(h)) = 0 = K_0(\varphi_{m,1})(h) \text{ in } K_0(M_m(A))$$

But $K_0(\varphi_{m,1})$ is an isomorphism, so $h = 0$ as required.

□

This next corollary completes Example III.3.7 which showed that the functor $K_0(\cdot)$ is not exact.

Corollary 4.7. *There is an isomorphism* $\alpha : K_0(\mathcal{K}) \to \mathbb{Z}$ *such that*

$$\alpha([E]_0) = Tr(E)$$

for every projection $E \in \mathcal{K}$*. This isomorphism is denoted by* $K_0(Tr)$

Proof. Let $\kappa : \mathbb{C} \to \mathbb{C} \otimes \mathcal{K} \cong \mathcal{K}$ be the map as above, and $\alpha_1 : K_0(\mathbb{C}) \to \mathbb{Z}$ the isomorphism such that

$$\alpha_1([1]_0) = 1$$

Define $\alpha = \alpha_1 \circ K_0(\kappa)^{-1} : K_0(\mathcal{K}) \to \mathbb{Z}$. Then α is an isomorphism. Furthermore, $F := \mathcal{K}(1)$ is a one-dimensional projection in $\mathcal{K}$, and

$$\alpha([F]_0) = \alpha_1([1]_0) = 1$$

If $E \in \mathcal{K}$ is any one-dimensional projection, then $E \sim F$ in $\widetilde{\mathcal{K}(H)}$ as in Example I.2.3. Hence,

$$\alpha([E]_0) = 1$$

If E is any arbitrary n-dimensional projection, then E is a sum of orthogonal rank one projections, so

$$\alpha([E]_0) = n = Tr(E)$$

□

Remark 4.8. 4.1. The stabilization of a C*-algebra A is defined as $A \otimes \mathcal{K}$. We say that A is stable if $A \cong A \otimes \mathcal{K}$.

4.2. If A and B are two C*-algebras such that $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, then $K_0(A) \cong K_0(B)$

4.3. If $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, then we say that A and B are stably isomorphic. Stably isomorphic algebras share many interesting properties. They have the "same representation theory" in the sense that they are strongly Morita equivalent. This implies that any statement concerning only modules over A holds for any C*-algebra B stably isomorphic to A.

Theorem 4.9. *For any C*-algebra A, $A \otimes \mathcal{K}$ is stable.*

Proof. Suppose $\mathcal{K}$ were stable, then

$$(A \otimes \mathcal{K}) \otimes \mathcal{K} \cong A \otimes (\mathcal{K} \otimes \mathcal{K}) \cong A \otimes \mathcal{K}$$

would hold, so it suffices to show that $\mathcal{K} \cong \mathcal{K} \otimes \mathcal{K}$.

Let $H := \ell^2$. and observe that $\mathcal{K}(H)$ is nuclear, so the spatial tensor product may be realized as

$$\mathcal{K}(H) \otimes \mathcal{K}(H) = \overline{\text{span}}\{a \otimes b : a, b \in \mathcal{K}(H)\} =: E \subset \mathcal{B}(H \otimes H)$$

where, for $a, b \in \mathcal{K}(H)$ define $a \otimes b \in \mathcal{B}(H \otimes H)$ by

$$(a \otimes b)(x \otimes y) := a(x) \otimes b(y)$$

We claim that $E = \mathcal{K}(H \otimes H)$.

4.1. If $a = x_1 \otimes y_1, b = x_2 \otimes y_2$, then

$$a(z) = \langle z, y_1 \rangle x_1 \text{ and } b(z) = \langle z, y_2 \rangle x_2$$

So

$$(a \otimes b)(z_1 \otimes z_2) = \langle z_1, y_1 \rangle \langle z_2, y_2 \rangle x_1 \otimes x_2$$

Hence, $a \otimes b \in \mathcal{K}(H \otimes H)$. Hence, if $a, b \in \mathcal{F}(H)$ (the space of finite rank operators), then $a \otimes b \in \mathcal{K}(H \otimes H)$. Finally, if $u, v \in \mathcal{B}(H)$, then

$$\|u \otimes v\| = \|u\| \|v\|$$

so the map $\otimes : \mathcal{B}(H) \times \mathcal{B}(H) \to \mathcal{B}(H \otimes H)$ is continuous, so

$$E := \overline{\text{span}}(a \otimes b : a, b \in \mathcal{K}(H)\} \subset \mathcal{K}(H \otimes H)$$

4.2. Conversely, if $T \in \mathcal{K}(H \otimes H)$, then T is the limit of finite rank operators. Hence, to show the reverse inclusion, it suffices to show that $\mathcal{F}(H \otimes H) \subset E$. Every finite rank operator is a linear combination of rank one operators, so it suffices to consider rank one operators. So suppose

$$T = z \otimes w$$

for some $z, w \in H \otimes H$, then z and w are limits of elements in $H \odot H$. Hence, it suffices to assume that $z, w \in H \odot H$. Once again, each $z, w \in H \odot H$ is a linear combination of elementary tensors, so it suffices to assume that $z = x_1 \otimes y_1$ and $w = x_2 \otimes y_2$ for $x_i, y_i \in H$. But then

$$T(z_1 \otimes z_2) = \langle z_1, y_1 \rangle \langle z_2, y_2 \rangle x_1 \otimes x_2 = (x_1 \otimes x_2) \otimes (y_1 \otimes y_2)(z_1 \otimes z_2)$$

Hence, $T = (x_1 \otimes x_2) \otimes (y_1 \otimes y_2) \in E$. So we conclude that

$$E = \mathcal{K}(H \otimes H)$$

$\square$

(End of Day 16)

VI. Classification of AF-Algebras

1. Finite Dimensional C*-Algebras

Definition 1.1. Define $e(n, i, j) \in M_n(\mathbb{C})$ to be the matrix whose $(i, j)^{th}$ entry is 1 and other entries are zero. If

$$A = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \ldots M_{n_r}(\mathbb{C})$$

define

$$e_{i,j}^{(k)} := (0, 0, \ldots, e(n_k, i, j), 0, 0, \ldots, 0) \in A$$

These are called the matrix units of A, and they satisfy the following identities

1.1. $e_{i,j}^{(k)} e_{j,\ell}^{(k)} = e_{i,\ell}^{(k)}$

1.2. $e_{i,j}^{(k)} e_{m,n}^{\ell} = 0$ if $k \neq \ell$ or if $j \neq m$

1.3. $(e_{i,j}^{(k)})^* = e_{j,i}^{(k)}$

1.4. $A = \text{span}\{e_{i,j}^{(k)} : 1 \leq k \leq r, 1 \leq i, j \leq n_k\}$

Definition 1.2. Let B be a C*-algebra and $\{f_{i,j}^{(k)}\}$ be a set of elements in B satisfying $(i), (ii)$ and (iii) above. Then this is called a system of matrix units in B of type A.

Note: Given a system of matrix units of type A as above, there is a unique $*$-homomorphism $\varphi : A \to B$ such that $\varphi(e_{i,j}^{(k)}) = f_{i,j}^{(k)}$ for all k, i, j. Furthermore, this map is

1.1. injective if all the $f_{i,j}^{(k)}$ are non-zero.

1.2. surjective if $B = \text{span}\{f_{i,j}^{(k)}\}$

Remark 1.3. If $A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_r}(\mathbb{C})$, then

$$K_0(A) \cong \mathbb{Z}^r$$

In fact, since A is stably finite (since it is finite dimensional) and unital, $(K_0(A), K_0(A)^+, [1_A])$ is an ordered abelian group with order unit, given by

$$\begin{aligned} K_0(A) &= \mathbb{Z}[e_{1,1}^{(1)}] + \mathbb{Z}[e_{1,1}^{(2)}] + \ldots + \mathbb{Z}[e_{1,1}^{(r)}] \cong \mathbb{Z}^r \\ K_0(A)^+ &= \mathbb{Z}^+[e_{1,1}^{(1)}] + \mathbb{Z}^+[e_{1,1}^{(2)}] + \ldots + \mathbb{Z}^+[e_{1,1}^{(r)}] \cong (\mathbb{Z}^+)^r \\ [1_A]_0 &= n_1[e_{1,1}^{(1)}]_0 + n_2[e_{1,1}^{(2)}]_0 + \ldots + n_r[e_{1,1}^{(r)}]_0 \end{aligned}$$

Lemma 1.4. *Suppose that* $\{f_{i,i}^{(k)} : 1 \leq k \leq r, 1 \leq i \leq n_k\}$ *is a set of mutually orthogonal projections in a C*-algebra* B *such that*

$$f_{1,1}^{(k)} \sim f_{2,2}^{(k)} \sim \ldots \sim f_{n_k,n_k}^{(k)}$$

for $1 \leq k \leq r$. *Then there is a system of matrix units* $\{f_{i,j}^{(k)}\}$ *in* V *that extends* $\{f_{i,i}^{(k)}\}$.

Proof. Choose partial isometries $f_{1,i}^{(k)}$ such that

$$(f_{1,i}^{(k)})^* f_{1,i}^{(k)} = f_{i,i}^{(k)} \text{ and } f_{1,i}^{(k)}(f_{1,i}^{(k)})^* = f_{1,1}^{(k)}$$

and define

$$f_{i,j}^{(k)} = (f_{1,i}^{(k)})^* f_{1,j}^{(k)}$$

Then this system works [Check!] $\square$

Definition 1.5. A C*-subalgebra $D \subset A$ is called a maximal abelian subalgebra (masa) if it is abelian, and it is not properly contained in any other abelian C*-subalgebra of A.

By Zorn's lemma, every Abelian C*-subalgebra is contained in a masa.

Definition 1.6. Let $X \subset A$. Define

$$X' := \{a \in A : ax = xa \quad \forall x \in X\}$$

Note that X' is a norm-closed subalgebra of A. Furthermore, it is a C*-subalgebra if X is self-adjoint (ie. if $a \in X$, then $a^* \in X$)

Note: $B \subset A$ is Abelian iff $B \subset B'$.

Lemma 1.7. $D \subset A$ *is a masa iff* $D = D'$

Proof. Suppose $D = D'$, then D is Abelian, and if E is Abelian and contains D, then

$$D \subset E \subset E' \subset D' = D$$

so $E = D$. Hence D is a masa.

Conversely, suppose D is a masa, then $D \subset D'$ and D' is a C*-subalgebra. WTS: $D' \subset D$. Since D' and D are C*-algebras, it suffices to show that $(D')_{sa} \subset D$. So fix $a \in D'$ self-adjoint, and set

$$X := D \cup \{a\}$$

Since elements in X commute with each other,

$$X \subset X'$$

Since X is self-adjoint, X' is a C*-subalgebra of A, and so

$$C^*(X) \subset X'$$

So if $y \in C^*(X)$ and $x \in X$, then $xy = yx$. Hence,

$$X \subset C^*(X)'$$

Once again, $C^*(X)'$ is a C*-algebra, so

$$C^*(X) \subset C^*(X)'$$

It follows that $C^*(X)$ is Abelian. Since $D \subset X \subset C^*(X)$, and D is a masa, we conclude that

$$D = C^*(X)$$

In particular, $a \in D$ as required. □

Example 1.8. Let $A = M_n(\mathbb{C})$ and D denote the set of all diagonal matrices. Then D is an Abelian C*-subalgebra of A. Furthermore, if $a \in D'$, then

$$ae_{1,1} = e_{1,1}a$$

So

$$e_{1,1}(a(e_1)) = ae_{1,1}(e_1) = a(e_1)$$

Hence, $a(e_1)$ is an eigen-vector of $e_{1,1}$ with eigen-value 1. So $a(e_1) = \lambda_1 e_1$. Thus continuing, we see that a must be diagonal. Hence, $D' = D$, so D is a masa.

Lemma 1.9. *Let D be a masa in a C*-algebra A.*

1.1. If D is unital, then A is unital and $1_A = 1_D$

1.2. If p is a projection in D such that $pDp = \mathbb{C}p$, then $pAp = \mathbb{C}p$ (Note: A projection with this property is minimal, in the sense that there is no projection $q \in A$ such that $q < p$ other than $q = 0$)

Proof. 1.1. If $a \in A$, then WTS: $a = a1_D$. Let $z := a - a1_D$, then $zd = 0$ for all $d \in D$. Since D is self-adjoint, this implies $(zd^*)^* = dz^* = 0$ for all $d \in D$. Hence,

$$d(z^*z) = 0(z^*z)d \quad \forall d \in D$$

Hence, $(z^*z) \in D' = D$ since D is a masa. Hence,

$$(z^*z)(z^*z) = 0 \Rightarrow \|z\|^4 = 0 \Rightarrow z = 0$$

Hence, $a = a1_D$ for all $a \in A$. Hence,

$$1_D a = (a^*1_D)^* = (a^*)^* = a \quad \forall a \in A$$

So $1_D = 1_A$

1.2. Let $a \in pAp$, then $a = pa = ap$. So if $d \in D$, we have $pd = dp = pdp = \lambda p$ for some $\lambda \in \mathbb{C}$. Hence,

$$ad = apd = \lambda ap = \lambda a = da$$

Hence, $a \in D' = D$, so $a \in D$. In that case, $a \in pDp$. Hence, $pAp \subset pDp = \mathbb{C}p$.

$\square$

(End of Day 17)

Theorem 1.10. *Any finite dimensional C*-algebra is isomorphic to*

$$M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \ldots \oplus M_{n_r}(\mathbb{C})$$

for some positive integers $r, n_1, n_2, \ldots, n_r \in \mathbb{N}$

Proof. 1.1. Choose a masa $D \subset A$. By Gelfand, $D \cong C_0(X)$ for some space X. Since D is finite dimensional, it follows that X is finite. In particular, X is compact. Hence, D is unital, and so A is unital and $1_A = 1_D$ by the previous lemma.

1.2. Let $X = \{x_1, x_2, \ldots, x_N\}$ and let $p_i \in D$ denote the corresponding characteristic functions

$$p_i(x_j) = \delta_{i,j}$$

Then $\{p_1, p_2, \ldots, p_N\} \subset D$ are projections such that

$$p_1 + p_2 + \ldots + p_N = 1_D \text{ and } p_jDp_j = \mathbb{C}p_j$$

By the previous lemma, $p_jAp_j = \mathbb{C}p_j$ for all $1 \leq j \leq N$

1.3. Fix $1 \leq i, j \leq N$ such that $p_jAp_i \neq 0$. Choose $v \in p_jAp_i$ such that $\|v\| = 1$, then

$$v^*v \in p_iAp_i$$

is a positive element of norm 1. But $p_iAp_i = \mathbb{C}p_i$. Hence,

$$v^*v = p_i$$

Similarly, $vv^* = p_j$. Hence, we conclude

$$p_jAp_i = \{0\} \text{ or } p_i \sim p_j$$

1.4. Now suppose $p_i \sim p_j$ and $a \in p_jAp_i$, then $a = ap_i = (av^*)v$. As $av^* \in p_jAp_j = \mathbb{C}p_j$, so $av^* = \lambda p_j$ for some $\lambda \in \mathbb{C}$. Furthermore, $p_jv = v$, so

$$a = av^*v = \lambda p_jv = \lambda v$$

Hence, $a \in \mathbb{C}v$, so if $p_i \sim p_j$, then

$$p_jAp_i = \mathbb{C}v$$

1.5. Partition the set $\{p_1, p_2, \ldots, p_N\}$ into Murray von-Neumann equivalence classes. Suppose there are r equivalence equivalence classes, and that the k^{th} class has n_k elements

$$\{f_{1,1}^{(k)}, f_{2,2}^{(k)}, \ldots, f_{n_k,n_k}^{(k)}\}$$

By choice of these projections, we have

$$f_{i,i}^{(k)} A f_{j,j}^{(\ell)} = \{0\} \text{ if } k \neq \ell \text{ and } f_{i,j}^{(k)} \sim f_{j,j}^{(k)}$$

By Lemma 1.4, we can extend this collection to a system of matrix units $\{f_{i,j}^{(k)}\}$ in A.

1.6. By Step 4,

$$f_{i,i}^{(k)} A f_{j,j}^{(k)} = \mathbb{C} f_{i,j}^{(k)}$$

and by Step 2,

$$1 = \sum_{i,k} f_{i,i}^{(k)}$$

1.7. Hence if $a \in A$, then

$$\begin{aligned} a &= \left(\sum_{i,k} f_{i,i}^{(k)}\right) a \left(\sum_{i,k} f_{i,i}^{(k)}\right) = \sum_{k=1}^{r} \sum_{i,j=1}^{n_k} f_{i,i}^{(k)} a f_{j,j}^{(k)} \\ &= \sum_{k=1}^{r} \sum_{i,j=1}^{n_k} \lambda_{i,j}^{(k)} f_{i,j}^{(k)} \end{aligned}$$

for some scalars $\lambda_{i,j}^{(k)} \in \mathbb{C}$. Hence,

$$A = \text{span}\{f_{i,j}^{(k)}\}$$

Thus the system of matrix units satisfies all conditions (1) - (4). Hence, by the remark following Definition 1.2,

$$A \cong M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \ldots \oplus M_{n_r}(\mathbb{C})$$

□

2. AF-Algebras

Definition 2.1. An approximately finite dimensional (AF) algebra is an inductive limit of finite dimensional C*-algebras.

Example 2.2. 2.1. Every finite dimensional C*-algebra is AF

2.2. $\mathcal{K}(\ell^2)$ is AF.

2.3. Fix a sequence $\{n_k\}$ of integers such that $n_k \mid n_{k+1}$. Define $\varphi_k : M_{n_k}(\mathbb{C}) \to M_{n_{k+1}}(\mathbb{C})$ to be the unital map

$$a \mapsto \mathrm{diag}(\underbrace{a, a, \ldots, a}_{d_k \text{ times}})$$

where $d_k = n_{k+1}/n_k$. The inductive limit is a unital AF-algebra, called a Uniformly Hyperfinite Algebra (UHF) algebra of type $\mathfrak{N} := \{n_k\}$

2.4. If $n_k = 2^k$ for all $k \in \mathbb{N}$, then the corresponding UHF algebra of type 2^∞ is called the CAR algebra (Canonical Anticommutation relations)

Lemma 2.3. *Every AF-algebra is stably finite. Hence, $(K_0(A), K_0(A)^+)$ is an ordered abelian group.*

Proof. If A is an AF-algebra, then so is $\widetilde{A}$ and $M_k(A)$. Hence it suffices to show that A is finite when A is unital and AF. We use the characterization from Lemma IV.1.3 and show that every isometry $s \in A$ is a unitary. Suppose $s \in A$ is an isometry, then fix $\epsilon > 0$ such that

$$\epsilon(3 + 2\epsilon) < 1$$

(For instance, $\epsilon = 1/4$ works) Now, since A is an AF-algebra, $\exists$ a finite dimensional C*-subalgebra $B \subset A$ and $x \in B$ such that

$$\|s - x\| < \epsilon$$

It follows that

$$|1 - \|x\|| = |\|s\| - \|x\|| \leq \|s - x\| < \epsilon \Rightarrow \|x\| \leq 1 + \epsilon$$

$$\begin{aligned}
\|1_A - x^*x\| &= \|s^*s - x^*x\| \\
&\leq \|s^*s - s^*x\| + \|s^*x - x^*x\| \\
&\leq \|s^*\|\|s - x\| + \|s^* - x^*\|\|x\| \\
&\leq \|s - x\| + \|s - x\|(1 + \epsilon) \\
&\leq \epsilon + \epsilon(1 + \epsilon) = \epsilon^2 + 2\epsilon \leq \epsilon(3 + 2\epsilon) < 1
\end{aligned}$$

Hence, x^*x is invertible. Replacing B by $B + \mathbb{C}1_A$ (which is also finite dimensional), and using spectral permanence, we can conclude that x^*x is invertible in B. Furthermore, if $z = (x^*x)^{-1}$, then

$$z = \sum_{k=0}^{\infty}(1 - x^*x)^k \Rightarrow \|z\| \leq \sum_{k=0}^{\infty}\|1 - x^*x\|^k = \frac{1}{1 - \|1 - x^*x\|} \leq \frac{1}{1 - \epsilon^2 - 2\epsilon}$$

Hence, if $y = zx^*$, then $yx = 1_A$ and

$$\|y\| < \frac{1 + \epsilon}{1 - \epsilon^2 - 2\epsilon}$$

Now x is left-invertible in B. Since B is finite dimensional, it follows that x is right invertible in B (and hence A), and the left and right-inverses coincide. Thus, $xy = 1_A$, so

$$\|sy - 1_A\| = \|sy - xy\| \leq \|s - x\|\|y\| < \frac{\epsilon(1+\epsilon)}{1 - \epsilon^2 - 2\epsilon} < 1$$

because $\epsilon(3 + 2\epsilon) < 1$. Hence, sy is invertible, so s is right invertible as required. $\square$

a. Outline of the Classification Theorem

If A is a unital AF-algebras, we consider the triple

$$\mathcal{E}(A) := (K_0(A), K_0(A)^+, [1_A]_0)$$

If there is a unital $*$-isomorphism $\varphi : A \to B$, then we get an isomorphism of invariants

$$K_0(\varphi) : \mathcal{E}(A) \to \mathcal{E}(B)$$

(End of Day 18)

Theorem 2.4 (Elliott)**.** *Let A and B be two unital AF-algebras. Given an isomorphism $\alpha : \mathcal{E}(A) \to \mathcal{E}(B)$, there is a $*$-isomorphism $\varphi : A \to B$ such that $\alpha = K_0(\varphi)$.*

Proof. The outline of the proof is as follows:

2.1. Write both A and B as inductive limits of finite dimensional C*-algebras

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \ldots \to A$$

$$B_1 \xrightarrow{\psi_1} B_2 \xrightarrow{\psi_2} B_3 \xrightarrow{\psi_3} \ldots \to B$$

The goal of the proof is to construct an intertwining: two subsequences (A_{n_j}) and (B_{m_j}) and maps between them as below

$$\begin{tikzcd}
 & A_{n_1} \arrow[rr] \arrow[dr, "g_1"] & & A_{n_2} \arrow[rr] \arrow[dr, "g_2"] & & A_{n_3} \arrow[r] & \ldots \arrow[r] & A \\
B_{m_1} \arrow[ur, "f_1"] \arrow[rr] & & B_{m_2} \arrow[ur, "f_2"] \arrow[rr] & & B_{m_3} \arrow[ur, "f_3"] \arrow[r] & \ldots \arrow[rr] & & B
\end{tikzcd} \tag{VI.1}$$

If such an intertwining exists, then there is an isomorphism $\varphi : A \to B$. This isomorphism will have the property that $K_0(\varphi) = \alpha$ as well.

2.2. To begin with, given an isomorphism $\alpha : \mathcal{E}(A) \to \mathcal{E}(B)$, we construct an intertwining at the level of K_0 groups

$$\begin{tikzcd}
& K_0(A_{n_1}) \arrow[rr] \arrow[dr, "\beta_1"] & & K_0(A_{n_2}) \arrow[rr] \arrow[dr, "\beta_2"] & & \dots \arrow[r] & K_0(A) \arrow[d, shift left, "\alpha"] \\
K_0(B_{m_1}) \arrow[rr] \arrow[ur, "\alpha_1"] & & K_0(B_{m_2}) \arrow[rr] \arrow[ur, "\alpha_2"] & & K_0(B_{m_3}) \arrow[r] & \dots \arrow[r] & K_0(B) \arrow[u, shift left, "\alpha^{-1}"]
\end{tikzcd} \tag{VI.2}$$

This requires a lifting property of the groups $K_0(A_j)$ and $K_0(B_j)$ (which are free Abelian groups) as follows: Given an inductive limit

$$\begin{tikzcd}
K_0(A_k) \arrow[rr, "K_0(\mu_k)"] \arrow[dr, "\alpha"'] & & K_0(A) \\
& K_0(B_j) \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

Once can lift the map γ to a map $\beta : K_0(B_j) \to K_0(A_\ell)$ for some $\ell \geq k$ such that TFDC:

$$\begin{tikzcd}
K_0(A_k) \arrow[r, "K_0(\varphi_{\ell,k})"] \arrow[dr, "\alpha"'] & K_0(A_\ell) \arrow[r, "K_0(\mu_\ell)"] & K_0(A) \\
& K_0(B_j) \arrow[u, "\beta"] \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

We will apply this inductively to construct an intertwining of K_0 groups as above (Equation VI.2)

2.3. Given an intertwining of K_0 groups as above, we would like to construct $*$-homomorphisms $f_i : B_{m_i} \to A_{n_i}$ and $g_i : A_{n_i} \to B_{m_{i+1}}$ such that

$$K_0(f_i) = \alpha_i \text{ and } K_0(g_i) = \beta_i$$

For this, we need an Existence theorem:

Given finite dimensional C*-algebras A and B, and a morphism $\eta : K_0(A) \to K_0(B)$, we need to find a $*$-homomorphism $f : A \to B$ such that $K_0(f) = \eta$.

Furthermore, we would like the f_i and g_i to interact as in Equation VI.1. Hence, we need a Uniqueness theorem as well:

Given finite dimensional C*-algebras A and B and two morphisms $f, g : A \to B$. Suppose $K_0(f) = K_0(g)$, then how are f and g related to each other?

$\square$

b. Step 1: Some facts about Inductive limits

We now consider Step 1 of the outline described above - to prove that an intertwining between sequences of C*-algebras produces an isomorphism of inductive limits.

For this, we fix a sequence of C*-algebras

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \dots$$

with inductive limit $(A, \{\mu_n\})$

Lemma 2.5. *Given a subsequence* $1 \leq n_1 < n_2 < n_3 < \dots$, *set* $\psi_j := \varphi_{n_{j+1},n_j}$. *Then the inductive limit of the sequence* (A_{n_j}, ψ_j) *is* $(A, \{\mu_{n_j}\})$.

Proof. Let $(B, \{\lambda_n\})$ be the inductive limit of (A_{n_j}, ψ_j), then we have a commutative diagram

$$\begin{tikzcd} A_{n_j} \arrow[rr, "\psi_j"] \arrow[dr, "\mu_{n_j}"'] & & A_{n_{j+1}} \arrow[dl, "\mu_{n_{j+1}}"] \\ & A & \end{tikzcd}$$

Hence, by the universal property, $\exists$ a $*$-homomorphism $\lambda : B \to A$ such that TFDC:

$$\begin{tikzcd} & A_{n_j} \arrow[dl, "\lambda_j"'] \arrow[dr, "\mu_{n_j}"] & \\ B \arrow[rr, "\lambda"] & & A \end{tikzcd}$$

We wish to show that λ is bijective:

2.1. λ is injective: This happens iff $\ker(\lambda_j) \subset \ker(\mu_{n_j})$ for all $j \in \mathbb{N}$. So suppose $a \in \ker(\lambda_j) \subset A_{n_j}$, then $\|\lambda_j(a)\| = 0$, whence

$$\lim_{k\to\infty} \|\psi_{k,j}(a)\| = 0$$

where $\psi_{k,j} : A_{n_j} \to A_{n_k}$ is the connecting map. But it follows by construction that $\psi_{k,j} = \varphi_{n_k,n_j}$. But $\{\|\varphi_{n_k,n_j}(a)\|\}$ is a subsequence of $\{\|\varphi_{i,n_j}(a)\|\}$ which is a convergent sequence with

$$\|\mu_{n_j}(a)\| = \lim_{i\to\infty} \|\varphi_{i,n_j}(a)\|$$

Hence the sequence converges to zero, whence $\mu_{n_j}(a) = 0$ as required.

2.2. λ is surjective: This happens iff

$$A = \overline{\bigcup_{j=1}^{\infty} \mu_{n_j}(A_{n_j})}$$

Since $n_j \to \infty$, for any $k \in \mathbb{N}$, choose $n_j > k$, so that

$$\mu_k(A_k) = \mu_{n_j}(\varphi_{k,n_j}(A_k)) \subset \mu_{n_j}(A_{n_j})$$

Hence

$$A = \overline{\bigcup_{k=1}^{\infty} \mu_k(A_k)} \subset \overline{\bigcup_{j=1}^{\infty} \mu_{n_j}(A_{n_j})} \subset A$$

$\square$

Lemma 2.6. *Set $B_n := A_n / \ker(\mu_n)$ and let $\pi_n : A_n \to B_n$ be the quotient map. Then $\exists$ injective $*$-homomorphisms $\psi_n : B_n \to B_{n+1}$ and a $*$-homomorphism $\pi : A \to \lim B_n$ such that TFDC:*

$$\begin{tikzcd}
A_1 \arrow[r, "\varphi_1"] \arrow[d, "\pi_1"'] & A_1 \arrow[r, "\varphi_2"] \arrow[d, "\pi_2"'] & A_3 \arrow[r] & \ldots \arrow[r] & A \arrow[d, "\pi"] \\
B_1 \arrow[r, "\psi_1"] & B_2 \arrow[r, "\psi_2"] & B_3 \arrow[r] & \ldots \arrow[r] & \lim B_n
\end{tikzcd}$$

Furthermore, π is an isomorphism.

Proof. Note that $\mu_n = \mu_{n+1} \circ \varphi_n$. Hence,

$$\psi_n : B_n \to B_{n+1} \text{ given by } a + \ker(\mu_n) \mapsto \varphi_n(a) + \ker(\mu_{n+1})$$

is well-defined, and is clearly a $*$-homomorphism. Furthermore, note that

$$\psi_n(a + \ker(\mu_n)) = 0 \Rightarrow \varphi_n(a) \in \ker(\mu_{n+1}) \Rightarrow \mu_n(a) = \mu_{n+1}(\varphi_n(a)) = 0 \Rightarrow a \in \ker(\mu_n)$$

Hence each ψ_n is injective.

Now let $(B, \{\lambda_n\})$ be the inductive limit of (B_n, ψ_n). Then we have maps $\alpha_n : A_n \to B$ given by

$$\alpha_n = \lambda_n \circ \pi_n$$

and TFDC:

$$\begin{tikzcd}
A_n \arrow[rr, "\varphi_{n+1}"] \arrow[dr, "\alpha_n"'] & & A_{n+1} \arrow[dl, "\alpha_{n+1}"] \\
& B &
\end{tikzcd}$$

because if $a \in A_n$, then

$$\begin{aligned}
\alpha_{n+1} \circ \varphi_{n+1}(a) &= \lambda_{n+1} \circ \pi_{n+1} \circ \varphi_{n+1}(a) \\
&= \lambda_{n+1}(\varphi_{n+1}(a) + \ker(\mu_{n+2})) \\
&= \lambda_{n+1} \circ \psi_{n+1}(\pi_n(a)) \\
&= \lambda_n \circ \pi_n(a) = \alpha_n(a)
\end{aligned}$$

Hence by the universal property, we get a map $\pi : A \to B$ such that TFDC:

$$\begin{tikzcd} & A_n \arrow[dl, "\mu_n"'] \arrow[dr, "\alpha_n"] & \\ A \arrow[rr, "\pi"] & & B \end{tikzcd}$$

We check that π is bijective:

2.1. π is injective: As before, we need to check if

$$\ker(\alpha_n) \subset \ker(\mu_n)$$

So suppose $a \in A_n$ is such that $\alpha_n(a) = 0$, then

$$0 = \|\lambda_n(\pi_n(a))\| = \lim_{m\to\infty} \|\psi_{n,m}(\pi_n(a))\|$$

Now note that each ψ_n is injective (see above), and so isometric. Hence it follows that $\pi_n(a) = 0$, whence $a \in \ker(\mu_n)$ as required.

2.2. π is surjective: It suffices to show that

$$B = \overline{\bigcup_{n=1}^{\infty} \alpha_n(A_n)}$$

But each π_n is surjective, so

$$\alpha_n(A_n) = \lambda_n(\pi_n(A_n)) = \lambda_n(B_n)$$

and we know that

$$B = \overline{\bigcup_{n=1}^{\infty} \lambda_n(B_n)}$$

□

Lemma 2.7. *Suppose each map $\varphi_n : A_n \to A_{n+1}$ is injective, then $\mu_n : A_n \to A$ is also injective. Suppose further that A is unital, then $\exists n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, A_n is unital and the maps $\varphi_n : A_n \to A_{n+1}$ and $\mu_n : A_n \to A$ are unital.*

Proof. Note that each φ_n is isometric. So if $\mu_n(a) = 0$, then

$$0 = \lim_{m\to\infty} \|\varphi_{m,n}(a)\| = \|a\| \Rightarrow a = 0$$

Hence each μ_n is injective. Now suppose A is unital. Since

$$1_A \in A = \overline{\bigcup_{n=1}^{\infty} \mu_n(A_n)}$$

$\exists n_0 \in \mathbb{N}$ and $a \in A_{n_0}$ such that $\mu_{n_0}(a) \in GL(A)$. By spectral permanence,

$$\mu_{n_0}(a) \in GL(\mu_{n_0}(A_{n_0}) + \mathbb{C}1_A)$$

so

$$\mu_{n_0}(a)^{-1} = \mu_{n_0}(b) + \lambda 1_A$$

for some $b \in A_{n_0}$. Then

$$1_A = \mu_{n_0}(a)[\mu_{n_0}(b) + \lambda 1_A] = \mu_{n_0}(ab + a)$$

Let $x = ab + a$, then for any $y \in A_{n_0}$, we have

$$\mu_{n_0}(xy) = \mu_{n_0}(x)\mu_{n_0}(y) = 1_A\mu_{n_0}(y) = \mu_{n_0}(y)$$

Since μ_{n_0} is injective, $xy = y$. Similarly, $yx = y$, so $x = 1_{A_{n_0}}$. Note that

$$\mu_{n_0}(1_{A_{n_0}}) = 1_A$$

We claim that if $n \geq n_0$

2.1. A_n is unital,

2.2. φ_n is unit-preserving.

Now if $n \geq n_0$, let $z := \varphi_{n,n_0}(1_{A_{n_0}})$, then for any $y \in A_n$, we have

$$\mu_n(zy) = \mu_n(z)\mu_n(y) = \mu_{n_0}(1_{A_{n_0}})\mu_n(y) = 1_A\mu_n(y) = \mu_n(y)$$

Hence, $zy = y$. Similarly, $yz = y$, so $z = 1_{A_n}$. Furthermore, observe that

$$\varphi_{n,n_0}(1_{A_{n_0}}) = 1_{A_n}$$

Once again, by injectivity of μ_n it follows that each φ_n is unital for $n \geq n_0$. $\square$

Lemma 2.8 (Intertwining Lemma)**.** *Given two inductive sequences of C*-algebras $(A_n, \{\varphi_n\})$ and $(B, \{\psi_n\})$ with inductive limits $(A, \{\mu_n\})$ and $(B, \{\lambda_n\})$ respectively. Suppose there are $*$-homomorphisms $\alpha_n : A_n \to B_n$ and $\beta_n : B_n \to A_{n+1}$ such that TFDC:*

$$\begin{tikzcd}
A_1 \arrow[rr, "\varphi_1"] \arrow[dr, "\alpha_1"'] & & A_2 \arrow[rr, "\varphi_2"] \arrow[dr, "\alpha_2"'] & & A_3 \arrow[r] \arrow[dr, "\alpha_3"'] & \dots & \\
& B_1 \arrow[ur, "\beta_1"'] \arrow[rr, "\psi_1"'] & & B_2 \arrow[ur, "\beta_2"'] \arrow[rr, "\psi_2"'] & & B_3 \arrow[r] & \dots
\end{tikzcd}$$

Then $\exists$ a $*$-isomorphism $\alpha : A \to B$ such that

$$\begin{tikzcd}
A_n \arrow[r, "\mu_n"] \arrow[dr, "\lambda_n \circ \alpha_n"'] & A \arrow[d, "\alpha"] \\
& B
\end{tikzcd}$$

Proof. As usual $\alpha : A \to B$ exists by the universal property of A, and it satisfies the above commuting diagram. Similarly, we get a map $\beta : B \to A$ such that TFDC:

$$\begin{tikzcd} B_n \arrow[r, "\lambda_n"] \arrow[dr, "\mu_{n+1}\circ\beta_n"'] & B \arrow[d, "\beta"] \\ & A \end{tikzcd}$$

Observe that

$$\beta \circ \alpha \circ \mu_n = \beta \circ \lambda_n \circ \alpha_n = \mu_{n+1} \circ \beta_n \circ \alpha_n = \mu_{n+1} \circ \varphi_n = \mu_n$$

Since

$$A = \overline{\bigcup_{n=1}^{\infty} \mu_n(A_n)}$$

it follows that $\beta \circ \alpha = \mathrm{id}_A$. Similarly, $\alpha \circ \beta = \mathrm{id}_B$. □

c. Step 2: Lifting maps at the level of K_0

We now consider Step 2 of the outline of Elliott's theorem from above - that of constructing an intertwining at the level of K_0 groups.

Remark 2.9. 2.1. Let G be an Abelian group. Then G is said to be projective if, whenever one has a surjective map

$$\pi : M \to N$$

of Abelian groups and a map $\varphi : G \to N$, then $\exists$ a map $\widehat{\varphi} : G \to M$ such that TFDC:

$$\begin{tikzcd} G \arrow[r, "\widehat{\varphi}"] \arrow[dr, "\varphi"'] & M \arrow[d, "\pi"] \\ & N \arrow[d] \\ & 0 \end{tikzcd}$$

2.2. $G = \mathbb{Z}$ is projective because if $\varphi : \mathbb{Z} \to N$, then $\varphi(1) \in N = \pi(M)$, so $\exists x \in M$ such that $\pi(x) = \varphi(1)$. Now simply define $\widehat{\varphi} : \mathbb{Z} \to M$ by

$$\widehat{\varphi}(1) := x$$

Similarly, any free Abelian group $G = \mathbb{Z}^m$ is projective.

2.3. Now suppose we are given an inductive system of Abelian groups

$$H_1 \xrightarrow{\alpha_1} H_2 \xrightarrow{\alpha_2} H_3 \xrightarrow{\alpha_3} \dots$$

with inductive limit $(H, \{\beta_n\})$. Suppose we are given a group homomorphism $\varphi : G \to H$, we ask whether $\exists n \in \mathbb{N}$ and a group homomorphism $\widehat{\varphi} : G \to H_n$ such that TFDC:

$$\begin{tikzcd} G \arrow[r, "\widehat{\varphi}"] \arrow[dr, "\varphi"'] & H_n \arrow[d, "\beta_n"] \\ & H \end{tikzcd}$$

Note that each β_n is not necessarily surjective, but

$$H = \bigcup_{n=1}^{\infty} \beta_n(H_n)$$

2.4. $G = \mathbb{Z}$ satisfies this condition: If $\varphi : \mathbb{Z} \to H$, then $\varphi(1) \in H$, so $\exists n \in \mathbb{N}$ and $x \in H_n$ such that $\varphi(1) = \beta_n(x)$. Now define $\widehat{\varphi} : \mathbb{Z} \to H_n$ such that

$$\widehat{\varphi}(1) = x$$

Similarly, $G = \mathbb{Z}^m$ also satisfies this condition.

2.5. This kind of lifting property is sometimes called semi-projectivity (this is not standard usage!).

(End of Day 19)

Lemma 2.10 (Semi-Projectivity Lemma). *Let*

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \ldots$$

be a sequence of finite dimensional C^*-algebras with inductive limit $(A, \{\mu_n\})$. Let B be a finite dimensional C^*-algebra and assume that there are positive group homomorphisms α and γ as below

$$\begin{tikzcd} K_0(A_1) \arrow[rr, "K_0(\mu_1)"] \arrow[dr, "\alpha"'] & & K_0(A) \\ & K_0(B) \arrow[ur, "\gamma"'] & \end{tikzcd}$$

Then $\exists n \in \mathbb{N}$ and a positive homomorphism $\beta : K_0(B) \to K_0(A_n)$ such that TFDC:

$$\begin{tikzcd} K_0(A_1) \arrow[r, "K_0(\varphi_{n,1})"] \arrow[dr, "\alpha"'] & K_0(A_n) \arrow[r, "K_0(\mu_n)"] & K_0(A) \\ & K_0(B) \arrow[u, "\beta"] \arrow[ur, "\gamma"'] & \end{tikzcd}$$

If each φ_n is unit preserving and if $\alpha([1_{A_1}]_0) = [1_B]_0$, then $\beta([1_B]_0) = [1_{A_n}]_0$.

Proof. 2.1. Let $\{e_{i,j}^{(k)}\}$ be the matrix units of B and set $x_k := \gamma([e_{1,1}^{(k)}]_0) \in K_0(A)^+$. By continuity of $K_0(A)$,

$$K_0(A)^+ = \bigcup_{n=1}^{\infty} K_0(\mu_n)(K_0(A_n)^+)$$

Hence, $\exists m \in \mathbb{N}$ and $y_1, y_2, \ldots, y_r \in K_0(A_m)^+$ such that

$$x_k = K_0(\mu_m(y_k) \quad \forall 1 \leq k \leq r$$

By Remark 1.3,

$$K_0(B) \cong \mathbb{Z}[e_{1,1}^{(1)}]_0 \oplus \mathbb{Z}[e_{1,1}^{(2)}]_0 \oplus \ldots \mathbb{Z}[e_{1,1}^{(r)}]_0$$

So, as in the previous remark, $\exists$ a group homomorphism $\beta' : K_0(B) \to K_0(A_m)$ such that

$$\beta'([e_{1,1}^{(k)}]_0) = y_k \quad \forall 1 \leq k \leq r$$

2.2. Suppose $g \in K_0(B)^+$, then $\exists m_i \in \mathbb{N}$ such that

$$g = m_1[e_{1,1}^{(1)}]_0 + m_2[e_{1,1}^{(2)}]_0 + \ldots + m_r[e_{1,1}^{(r)}]_0$$

Hence, $\beta'(g) \in K_0(A_m)^+$. Hence, β is positive.

2.3. Furthermore,

$$(K_0(\mu_m) \circ \beta')[e_{1,1}^{(k)}]_0 = K_0(\mu_m)(y_k) = x_k = \gamma([e_{1,1}^{(k)}]_0)$$

Hence, $K_0(\mu_m) \circ \beta' = \gamma$

2.4. To ensure that $\beta \circ \alpha = K_0(\varphi_{n,1})$ still requires some work: Note that $K_0(A_1)$ is a finitely generated abelian group, so choose generators $\{g_1, g_2, \ldots, g_s\}$. Note that

$$K_0(\mu_m) \circ \beta' \circ \alpha(g_i) = \gamma \circ \alpha(g_i) = K_0(\mu_1)(g_i) = K_0(\mu_m) \circ K_0(\varphi_{m,1})(g_i)$$

Hence,

$$h_i := \beta' \circ \alpha(g_i) - K_0(\varphi_{m,1})(g_i) \in \ker(K_0(\mu_m))$$

But

$$\ker(K_0(\mu_m)) = \bigcup_{n>m} \ker(K_0(\varphi_{n,m}))$$

Hence, $\exists n > m$ such that $h_i \in \ker(K_0(\varphi_{n,m}))$ for all $1 \leq i \leq s$. Define

$$\beta := K_0(\varphi_{n,m}) \circ \beta'$$

Then

$$[\beta \circ \alpha - K_0(\varphi_{n,1})](g_i) = K_0(\varphi_{n,m})(h_i) = 0 \quad \forall 1 \leq i \leq s$$

Since $K_0(A_1)$ is generated by the $\{g_i\}$ it follows that

$$\beta \circ \alpha = K_0(\varphi_{n,1})$$

2.5. Finally, note that

$$\gamma = K_0(\mu_m) \circ \beta' = K_0(\mu_n) \circ K_0(\varphi_{n,m}) \circ \beta' = K_0(\mu_n) \circ \beta$$

as required.

2.6. Now for the final claim: If each φ_n is unit preserving and $\alpha([1_{A_1}]_0) = [1_B]_0$, then

$$\beta([1_B]_0) = \beta \circ \alpha([1_{A_1}]_0) = K_0(\varphi_{n,1})([1_{A_1}]_0) = [1_{A_n}]_0$$

□

d. Step 3: Existence and Uniqueness of maps between finite dimensional C*-algebras

We begin with Step 3 of the outline described above: To construct maps out of finite dimensional C*-algebras from maps at the level of K-theory, and to determine to what extent these maps are unique.

Definition 2.11. A C*-algebra A is said to have the cancellation property if, for any two projections $p, q \in \mathcal{P}_\infty(A)$, we have

$$[p]_0 = [q]_0 \Rightarrow p \sim_0 q$$

Example 2.12. 2.1.

2.2. $M_n(\mathbb{C})$ has cancellation

Proof. Let $A = M_n(\mathbb{C})$. If $p, q \in \mathcal{P}_\infty(A)$ are such that $[p]_0 = [q]_0$, then choose $p', q' \in M_k(A)$ such that $p \sim_0 p', q \sim_0 q'$. Then by Example II.1.13,

$$Tr(p') = Tr(q')$$

Then it follows that $p' \sim q'$. Hence, $p \sim_0 q$ □

2.3. If A, B have cancellation, so does $A \oplus B$. Hence every finite dimensional C*-algebra has cancellation.

Proof. Obvious. □

2.4. Let (A_n, φ_n) be an inductive sequence of C*-algebras with inductive limit $(A, \{\mu_n\})$. Suppose each A_n has cancellation, then so does A. Hence, every AF-algebra has cancellation.

Proof. Let $p, q \in \mathcal{P}_\infty(A)$ such that $[p]_0 = [q]_0$. Assume WLOG that $p, q \in M_k(A)$. As in the proof of Theorem 3.5, $\exists m \in \mathbb{N}$ and $p' \in M_k(A_m)$ such that $\mu_m(p') \sim p$. Similarly, $\exists q' \in M_k(A_m)$ such that $\mu_m(q') \sim q$ (Note that in principle there might be two different integers m and ℓ, but we may choose the max of them). Hence,

$$[\mu_m(p')]_0 = [\mu_m(q')]_0$$

Since

$$\ker(K_0(\mu_m)) = \bigcup_{n=m+1}^{\infty} \ker(K_0(\varphi_{n,m}))$$

it follows that $\exists n \geq m$ such that

$$[\varphi_{n,m}(p')]_0 = [\varphi_{n,m}(q')]_0 \text{ in } K_0(A_n)$$

Since A_n has cancellation,

$$\varphi_{n,m}(p') \sim_0 \varphi_{n,m}(q')$$

But then

$$p \sim \mu_m(p') = \mu_n(\varphi_{n,m}(p')) \sim_0 \mu_n(\varphi_{n,m}(q')) = \mu_m(q') \sim q$$

Hence, A has cancellation. $\square$

2.5. $\mathcal{B}(H)$ does not have cancellation because for any two projections $p, q \in \mathcal{B}(H)$, we have $[p]_0 = [q]_0$, but it is not true that $p \sim q$ in general (See Example I.2.3, and II.1.14)

Lemma 2.13. *Let B be a unital C^*-algebra with cancellation. Let $g_1, g_2, \ldots, g_n \in K_0(B)^+$ satisfy*

$$\sum g_i \leq [1_B]_0$$

Then $\exists$ mutually orthogonal projections $p_1, p_2, \ldots, p_n$ in B such that $[p_j]_0 = g_j$ for all $1 \leq j \leq n$

Proof. We proceed by induction.

2.1. If $n = 1$: $0 \leq g_1 \leq [1_B]_0$ and $[1_B]_0 - g_1 \geq 0$ so $\exists e, f \in \mathcal{P}_n(B)$ such that

$$g = [e]_0 \text{ and } [1_B] - g = [f]_0$$

Then since B has cancellation

$$[e \oplus f]_0 = [1_B]_0 \Rightarrow e \oplus f \sim_0 1_B$$

So $\exists v \in M_{1,2n}(B)$ such that

$$v^*v = e \oplus f \text{ and } vv^* = 1_B$$

Define $q := v(e \oplus 0_n)v^*$, then $q \in \mathcal{P}(B), q \leq 1_B$ and if $w = v(e \oplus 0_n)$, then

$$ww^* = q \text{ and } w^*w = (e \oplus 0_n)v^*v(e \oplus 0_n) = (e \oplus 0_n)(e \oplus f)(e \oplus 0_n) = (e \oplus 0_n)$$

Hence, $g = [e]_0 = [e \oplus 0_n]_0 = [q]_0$

2.2. If $n \geq 2$: We have

$$\sum_{i=1}^{n-1} g_i + g_n \leq [1_B]_0 \Rightarrow \sum_{i=1}^{n-1} g_i \leq [1_B]_0$$

By induction hypothesis, we may choose mutually orthogonal projections $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}(B)$ such that $g_i = [p_i]_0$. Set

$$p := p_1 + p_2 + \ldots p_{n-1}$$

Then $0 \leq g_n \leq [1_B] - [p]_0$, then $\exists e \in \mathcal{P}_m(B)$ such that $g_n = [e]_0$. Then choose $f \in \mathcal{P}_k(B)$ such that $[f]_0 = [1_B]_0 - [p]_0 - [e]_0$, then

$$[e \oplus f]_0 = [1_B]_0 - [p]_0 = [1_B - p]_0$$

Since B has cancellation, $e \oplus f \sim_0 1_B - p$, so $\exists v \in M_{1,k+m}(B)$ such that

$$e \oplus f = v^* v \text{ and } 1_B - p = vv^*$$

Put $p_n := v(e \oplus 0_k)v^*$, then $p_n \in \mathcal{P}(B)$ and if $w = v(e \oplus 0_k)$, then

$$ww^* = p_n \text{ and } w^* w = (e \oplus 0_k)v^* v(e \oplus 0_k) = (e \oplus 0_k)$$

As before, $[p_n]_0 = [e]_0 = g_n$. Furthermore,

$$(1_B - p)p_n = vv^* v(e \oplus 0_k)v^* = v(e \oplus f)(e \oplus 0_k)v^* = v(e \oplus 0_k)v^* = p_n$$

Hence, $p_n \leq 1_B - p$. Hence,

$$\sum_{i=1}^{n} p_i \leq 1_B$$

By Exercise 2.4 (below), it follows that the $\{p_i\}$ are mutually orthogonal.

$\square$

(End of Day 20)

We now prove the existence part of Step 3 of the outline of Elliott's theorem described above.

Lemma 2.14 (Existence Theorem)**.** *Let A be a finite dimensional C*-algebra, and B a unital C*-algebra with cancellation. Let $\alpha : K_0(A) \to K_0(B)$ be a group homomorphism such that*

$$\alpha([1_A]_0) \leq [1_B]_0$$

Then $\exists$ a $*$-homomorphism $\varphi : A \to B$ such that $K_0(\varphi) = \alpha$. Furthermore, if $\alpha([1_A]_0) = [1_B]_0$, then $\varphi(1_A) = 1_B$ must hold.

Proof. Write

$$A = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \dots M_{n_r}(\mathbb{C})$$

and let $\{e_{i,j}^{(k)}\}$ be a system of matrix units of A and $g_{i,k} := \alpha([e_{i,i}^{(k)}]_0)$. Then

$$\sum_{i,k} g_{i,k} = \alpha(\sum_{i,k}[e_{i,i}^{(k)}]_0) = \alpha([1_A]_0) \leq [1_B]_0$$

So by the previous lemma, $\exists$ mutually orthogonal projections $f_{i,i}^{(k)} \in B$ such that

$$\alpha([e_{i,i}^{(k)}]_0) = [f_{i,i}^{(k)}]_0$$

Since B has cancellation, for each $1 \leq k \leq r$ and $1 \leq i, j \leq n_k$, we have

$$e_{i,i}^{(k)} \sim e_{j,j}^{(k)} \Rightarrow [e_{i,i}^{(k)}]_0 = [e_{j,j}^{(k)}]_0 \Rightarrow [f_{i,i}^{(k)}]_0 = [f_{j,j}^{(k)}]_0 \Rightarrow f_{i,i}^{(k)} \sim f_{j,j}^{(k)}$$

By Lemma 1.4, the system $\{f_{i,i}^{(k)}\}$ extends to a system of matrix units $\{f_{i,j}^{(k)}\}$ in B of type A. By the note following Definition 1.2, we obtain a $*$-homomorphism $\varphi : A \to B$ such that

$$\varphi(e_{i,j}^{(k)}) = f_{i,j}^{(k)}$$

Note that by construction

$$K_0(\varphi)([e_{1,1}^{(k)}]_0) = [f_{1,1}^{(k)}]_0 = \alpha([e_{1,1}^{(k)}]_0)$$

By Remark 1.3, $K_0(A)$ is generated by the elements $\{[e_{1,1}^{(k)}]_0\}$. Hence, $K_0(\varphi) = \alpha$.

Now suppose $\alpha([1_A]_0) = [1_B]_0$. Put

$$p := \sum_{i,k} f_{i,i}^{(k)}$$

Then $p \in \mathcal{P}(B)$ and $\varphi(1_A) = p$. Hence,

$$[1_B - p]_0 = [1_B]_0 - [p]_0 = \alpha([1_A]_0) - K_0(\varphi)([1_A]_0) = 0$$

Since B has cancellation,

$$1_B - p \sim_0 0 \Rightarrow 1_B - p = 0 \Rightarrow \varphi(1_A) = p = 1_B$$

so φ is unital. □

Definition 2.15. 2.1. Let B be a unital C*-algebra, and $u \in \mathcal{U}(B)$. Define Ad u : $B \to B$ by $b \mapsto ubu^*$. Note that Ad u is an automorphism of B.

2.2. Let $\varphi, \psi : A \to B$ be two $*$-homomorphisms. We say that φ and ψ are unitarily equivalent (In symbols, $\varphi \sim_u \psi$) if $\exists u \in \mathcal{U}(B)$ such that $\varphi = \text{Ad } u \circ \psi$.

Remark 2.16. 2.1. If $\varphi \sim_u \psi$, then $K_0(\varphi) = K_0(\psi)$.

2.2. The converse is not true: Let $A = \mathcal{O}_2$, then we know that $K_0(\mathcal{O}_2) = 0$ (Theorem III.4.6). Hence, $K_0(0) = K_0(\mathrm{id}_{\mathcal{O}_2})$. However, $\mathrm{id}_{\mathcal{O}_2}$ is not unitarily equivalent to 0.

We now prove the uniqueness part of Step 3 of the outline above.

Lemma 2.17 (Uniqueness Theorem)**.** *Let A be a finite dimensional C^*-algebra and B be a unital C^*-algebra with cancellation. Let $\varphi, \psi : A \to B$ be two $*$-homomorphisms such that*

$$K_0(\varphi) = K_0(\psi)$$

Then $\varphi \sim_u \psi$.

Proof. Consider the matrix units $\{e_{i,j}^{(k)}\}$ of A. Then

$$[\varphi(e_{1,1}^{(k)})]_0 = K_0(\varphi)[e_{1,1}^{(k)}]_0 = K_0(\psi)[e_{1,1}^{(k)}]_0 = [\psi(e_{1,1}^{(k)}]_0 \text{ and}$$
$$[1_B - \varphi(1_A)]_0 = [1_B]_0 - K_0(\varphi)([1_A]_0) = [1_B]_0 - K_0(\psi)([1_A]_0) = [1_B - \psi(1_A)]_0$$

Since B has cancellation, $\exists$ partial isometries $v_1, v_2, \ldots, v_r$ and $w \in B$ such that

$$v_k v_k^* = \varphi(e_{1,1}^{(k)}), \text{ and } v_k v_k^* = \psi(e_{1,1}^{(k)}) \quad \forall 1 \leq k \leq r$$
$$w^* w = 1_B - \varphi(1_A) \text{ and } w w^* = 1_B - \psi(1_A)$$

Define

$$w_{i,k} := \psi(e_{i,1}^{(k)}) v_k \varphi(e_{1,i}^{(k)})$$

Then

$$\begin{aligned} w_{i,k}^* w_{i,k} &= \varphi(e_{1,i}^{(k)})^* v_k^* \psi(e_{i,1}^{(k)})^* \psi(e_{i,1}^{(k)}) v_k \varphi(e_{1,i}^{(k)}) \\ &= \varphi(e_{i,1}^{(k)}) v_k^* \psi(e_{1,i}^{(k)} e_{i,1}^{(k)}) v_k \varphi(e_{1,i}^{(k)}) \\ &= \varphi(e_{i,1}^{(k)}) v_k^* \psi(e_{1,1}^{(k)}) v_k \varphi(e_{1,i}^{(k)}) \\ &= \varphi(e_{i,1}^{(k)}) v_k^* v_k v_k^* v_k \varphi(e_{1,i}^{(k)}) \\ &= \varphi(e_{i,1}^{(k)}) \varphi(e_{1,1}^{(k)}) \varphi(e_{1,i}^{(k)}) \\ &= \varphi(e_{i,i}^{(k)}) \end{aligned}$$

Similarly,

$$w_{i,k} w_{i,k}^* = \psi(e_{i,i}^{(k)})$$

Hence,

$$w w^* + \sum_{i,k} w_{i,k}^* w_{i,k} = 1_B - \psi(1_A) + \sum_{i,k} \psi(e_{i,i}^{(k)}) = 1_B$$

Similarly,

$$w^* w + \sum_{i,k} w_{i,k} w_{i,k}^* = 1_B$$

Hence it follows from Exercise 2.6 (See below - See also Lemma III.4.3) that

$$u := w + \sum_{k=1}^{r} \sum_{i=1}^{n_k} w_{i,k}$$

is a unitary. Moreover, we claim that

$$u\varphi(e_{s,t}^{(m)}) = \psi(e_{s,t}^{(m)})u \quad \forall s, t, m$$

Note that

$$\begin{aligned} u\varphi(e_{s,t}^{(m)}) &= w\varphi(e_{s,t}^{(m)}) + \left[\sum_{i,k} \psi(e_{i,1}^{(k)}) v_k \varphi(e_{1,i}^{(k)}) \right] \varphi(s,t)^{(m)} \\ &= w\varphi(e_{s,t}^{(m)}) + \psi(e_{s,1}^{(m)}) v_m \varphi(e_{1,s}^{(m)}) \varphi(e_{s,t}^{(m)}) \\ &= w\varphi(e_{s,t}^{(m)}) + \psi(e_{s,1}^{(m)}) v_m \varphi(e_{1,t}^{(m)}) \end{aligned}$$

and

$$\begin{aligned} \psi(e_{s,t}^{(m)})u &= \psi(e_{s,t}^{(m)})w + \psi(e_{s,t}^{(m)}) \left[\sum_{i,k} \psi(e_{i,1}^{(k)}) v_k \varphi(e_{1,i}^{(k)}) \right] \\ &= \psi(e_{s,t}^{(m)})w + \psi(e_{s,t}^{(m)}) \psi(e_{t,1}^{(m)}) v_m \varphi(e_{1,t}^{(m)}) \\ &= \psi(e_{s,t}^{(m)})w + \psi(e_{s,1}^{(m)}) v_m \varphi(e_{1,t}^{(m)}) \end{aligned}$$

Finally, observe that $w = (1_B - \psi(1_A))w = w(1_B - \varphi(1_A))$. Hence,

$$w\varphi(e_{s,t}^{(m)}) = w(\varphi(e_{s,t}^{(m)}) - \varphi(e_{s,t}^{(m)}) = 0 = \psi(e_{s,t}^{(m)})w$$

Hence,

$$u\varphi(e_{s,t}^{(m)})u^* = \psi(e_{s,t}^{(m)})$$

for all s, t, m. Hence, $\psi \sim_u \varphi$ as required. □

e. Elliott's Classification Theorem

Theorem 2.18 (Elliott)**.** *Let A and B be two unital AF-algebras. Suppose*

$$\alpha : K_0(A) \to K_0(B)$$

is a positive group isomorphism such that $\alpha([1_A]_0) = [1_B]_0$. Then $\exists$ a unital $*$-isomorphism

$$\varphi : A \to B$$

such that $K_0(\varphi) = \alpha$.

In other words,

$$A \cong B \Leftrightarrow (K_0(A), K_0(A)^+, [1_A]_0) \cong (K_0(B), K_0(B)^+, [1_B]_0)$$

Proof. 2.1. **Step 1a:** Write $(A, \{\mu_n\})$ as an inductive limit of

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \xrightarrow{\ldots}$$

where each A_n is finite dimensional. By applying Lemma 2.6, we may assume that each map $f_n : A_n \to A_{n+1}$ is injective. Since A is unital, by Lemma 2.7, we may further assume that each f_n is unital. Similarly, we obtain a sequence of finite dimensional C*-algebras

$$B_1 \xrightarrow{g_1} B_2 \xrightarrow{g_2} B_3 \xrightarrow{g_3} \xrightarrow{\ldots}$$

where $g_n : B_n \to B_{n+1}$ is unital, and whose inductive limit is $(B, \{\lambda_n\})$.

2.2. **Step 2:** Let $B_0 = \mathbb{C}$ and set $g_0 : B_0 \to B_1$ be the unique unital map. Similarly, let $\psi_0 : B_0 \to A_1, \lambda_0 : B_0 \to B$ be the unique unital map. Set $\beta_0 := K_0(\psi_0)$. Then we get

$$\begin{tikzcd}
K_0(B_0) \arrow[rr, "\alpha\circ K_0(\mu_1)\circ\beta_0 = K_0(\lambda_0)"] \arrow[dr, "\beta_0"'] & & K_0(B) \\
& K_0(A_1) \arrow[ur, "\alpha\circ K_0(\mu_1)"'] &
\end{tikzcd}$$

By the Semi-Projectivity Lemma (2.10), $\exists m_1 \in \mathbb{N}$ and $\alpha_1 : K_0(A_1) \to K_0(B_{m_1})$ such that TFDC:

$$\begin{tikzcd}
K_0(B_0) \arrow[r] \arrow[dr] & K_0(B_{m_1}) \arrow[r, "K_0(\lambda_n)"] & K_0(B) \\
& K_0(A_1) \arrow[u, "\alpha_1"] \arrow[ur] &
\end{tikzcd}$$

Now consider the diagram

$$\begin{tikzcd}
K_0(A_1) \arrow[rr, "K_0(\mu_1)"] \arrow[dr, "\alpha_1"'] & & K_0(A) \\
& K_0(B_{m_1}) \arrow[ur, "\alpha^{-1}\circ K_0(\lambda_{m_1})"'] &
\end{tikzcd}$$

By the Semi-Projectivity Lemma 2.10, we get $n_2 \in \mathbb{N}$ and a map $\beta_1 : K_0(B_{m_1}) \to K_0(A_{n_2})$ with a corresponding commuting diagram. Thus proceeding, we get maps and diagrams as below

$$\begin{tikzcd}
& K_0(A_1) \arrow[rr, "K_0(f_{n_2,1})"] \arrow[dr, "\alpha_1"] & & K_0(A_{n_2}) \arrow[r] & \ldots \arrow[r] & K_0(A) \arrow[d, shift left, "\alpha^{-1}"] \\
K_0(B_0) \arrow[ur, "\beta_0"] \arrow[rr, "K_0(g_{m_1,0})"'] & & K_0(B_{m_1}) \arrow[ur, "\beta_1"'] \arrow[rr, "K_0(g_{m_2,m_1})"'] & & \ldots \arrow[r] & K_0(B) \arrow[u, shift left, "\alpha"]
\end{tikzcd}$$

2.3. **Step 1b:** Consider the subsequences

$$A_{n_1} \xrightarrow{f_{n_2,n_1}} A_{n_2} \xrightarrow{f_{n_3,n_2}} A_3 \to \ldots$$

and

$$B_{m_1} \xrightarrow{g_{m_2,m_1}} B_{m_2} \xrightarrow{g_{m_3,m_2}} B_3 \to \dots$$

Their inductive limits are A and B respectively by Lemma 2.5. To simplify notation, we assume $n_j = m_j = j$ so that $f_{n_{j+1},n_j} = f_j$ and $g_{n_{j+1},j} = g_j$

2.4. **Step 3:** For each $j \in \mathbb{N}$, by the Existence Theorem (Lemma 2.14), $\exists$ maps $\varphi' : A_j \to B_j$ and $\psi'_j : B_j \to A_{j+1}$ such that

$$K_0(\varphi'_j) = \alpha_j \text{ and } K_0(\psi'_j) = \beta_j$$

Note that

$$K_0(f_j) = \beta_j \circ \alpha_j = K_0(\psi'_j \circ \varphi'_j)$$
$$K_0(g_j) = \alpha_{j+1} \circ \beta_j = K_0(\varphi'_{j+1} \circ \psi'_j)$$

We define unitaries $u_j \in \mathcal{U}(A_{j+1})$ and $v_j \in \mathcal{U}(B_j)$ inductively as follows:

(i) Set $v_1 = 1_B$. By the Uniqueness Theorem (Lemma 2.17), $\exists u_1 \in \mathcal{U}(A_2)$ such that

$$f_1 = \text{Ad } u_1 \circ \psi'_1 \circ \varphi'_1$$

Set $\psi_1 := \text{Ad } u_1 \circ \psi'_1$ and $\varphi_1 := \varphi'_1$.

(ii) Note that

$$K_0(g_1) = K_0(\varphi'_2 \circ \psi'_1) = K_0(\varphi'_2 \circ \psi_1)$$

Hence, by Lemma 2.17, $\exists v_2 \in \mathcal{U}(B_2)$ such that

$$g_1 = \text{Ad } v_2 \circ \varphi'_2 \circ \psi_1$$

Set $\varphi_2 := \text{Ad } v_2 \circ \varphi'_2$.

(iii) Thus proceeding, we obtain unitaries $u_j \in \mathcal{U}(A_{j+1})$ and $v_j \in \mathcal{U}(B_j)$ such that, if

$$\varphi_j := \text{Ad } v_j \circ \varphi'_j \text{ and } \psi_j := \text{Ad } u_j \circ \psi'_j$$

Then

$$K_0(\varphi'_j) = K_0(\varphi_j) \text{ and } K_0(\psi'_j) = K_0(\psi_j)$$

and furthermore, TFDC:

$$\begin{tikzcd}
 & A_1 \arrow[rr] \arrow[dr, "\varphi_1"] & & A_2 \arrow[rr] \arrow[dr, "\varphi_2"] & & A_3 \arrow[r] & \dots \arrow[r] & A \\
B_0 \arrow[ur, "\psi_0"] \arrow[rr] & & B_1 \arrow[ur, "\psi_1"] \arrow[rr] & & B_2 \arrow[ur, "\psi_2"] \arrow[r] & \dots \arrow[rr] & & B
\end{tikzcd}$$

By the Intertwining Lemma 2.8, there is an isomorphism $\varphi : A \to B$ and $\psi = \varphi^{-1} : B \to A$.

2.5. To show that $K_0(\varphi) = \alpha$, note that we have two commuting diagrams

$$\begin{tikzcd} K_0(A_j) \arrow[r, "K_0(\mu_j)"] \arrow[d, "K_0(\varphi_j)=\alpha_j"'] & K_0(A) \arrow[d, "K_0(\varphi)"] \\ K_0(B_j) \arrow[r, "K_0(\lambda_j)"'] & K_0(B) \end{tikzcd} \qquad \begin{tikzcd} K_0(A_j) \arrow[r, "K_0(\mu_j)"] \arrow[d, "\alpha_j"'] & K_0(A) \arrow[d, "\alpha"] \\ K_0(B_j) \arrow[r, "K_0(\lambda_j)"'] & K_0(B) \end{tikzcd}$$

Hence, $\alpha = K_0(\varphi)$ on $K_0(\mu_j)(K_0(A_j))$. By continuity if K_0,

$$K_0(A) = \bigcup_{j=1}^{\infty} K_0(\mu_j)(K_0(A_j))$$

Hence, $\alpha = K_0(\varphi)$ on all of $K_0(A)$.

$\square$

VII. The Functor K_1

1. Definition

Definition 1.1. Let A be a unital C*-algebra. Define

$$\mathcal{U}_n(A) := \mathcal{U}(M_n(A)) \text{ and } \mathcal{U}_\infty(A) := \bigcup_{n=1}^{\infty} \mathcal{U}_n(A)$$

Define a binary operation $\oplus$ on $\mathcal{U}_\infty(A)$ by

$$u \oplus v := \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

and an equivalence relation $\sim_1$ on $\mathcal{U}_\infty(A)$ by: If $u \in \mathcal{U}_n(A), v \in \mathcal{U}_m(A)$, then we say $u \sim_1 v$ iff $\exists k \geq \max\{m, n\}$ such that

$$u \oplus 1_{k-n} \sim_h v \oplus 1_{k-m}$$

where 1_r is the unit in $M_r(A)$.

Lemma 1.2. *Let A be a unital C*-algebra. Then, for all $u, v \in \mathcal{U}_\infty(A)$*

1.1. $\sim_1$ *is an equivalence relation on* $\mathcal{U}_\infty(A)$

1.2. $u \sim_1 u \oplus 1_n$ *for all and* $n \in \mathbb{N}$

1.3. $u \oplus v \sim_1 v \oplus u$

1.4. *If* $u \sim_1 u'$ *and* $v \sim_1 v'$, *then* $u \oplus v \sim_1 u' \oplus v'$

1.5. *If* $u, v \in \mathcal{U}_n(A)$ *for some* $n \in \mathbb{N}$, *then* $uv \sim_1 vu \sim_1 u \oplus v$

1.6. $(u \oplus v) \oplus w = u \oplus (v \oplus w)$

Proof. (i), (ii) and (vi) are trivial, and (v) follows from Whitehead's lemma. Now consider (iii): Let $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$, and set

$$z = \begin{pmatrix} 0 & 1_m \\ 1_n & 0 \end{pmatrix} \in \mathcal{U}_{n+m}(A)$$

Then by (v),

$$v \oplus u = z(u \oplus v)z^* \sim_1 z^*z(u \oplus v)$$

To prove (iv): It suffices to prove:

1.1. $(u \oplus 1_k) \oplus (v \oplus 1_\ell) \sim_1 (u \oplus v)$: This follows from (ii), (iii) and (vi).
1.2. If $u \sim_h u'$ and $v \sim_h v'$, then $(u \oplus v) \sim_h (u' \oplus v')$. But this follows by simply taking the two paths u_t and v_t and considering $u_t \oplus v_t$.

□

Definition 1.3. Let A be a C*-algebra, then define

$$K_1(A) := \mathcal{U}_\infty(\widetilde{A})/\sim_1$$

Write $[u]_1$ for the class if $u \in \mathcal{U}_\infty(A)$ in $K_1(A)$. Define an addition on $K_1(A)$ by

$$[u]_1 + [v]_1 := [u \oplus v]_1$$

The operation is well-defined by the previous lemma. It is also commutative, associative, and has a zero element $[1]_1 = [1_n]_1$. Also, if $u \in \mathcal{U}_\infty(A)$, then

$$[u]_1 + [u^*]_1 = [u \oplus u^*]_1 = [uu^* \oplus 1_n]_1 = [1_{2n}]_1 = [1]_1$$

Hence, $[u^*]_1 = -[u]_1$. Hence, $K_1(A)$ is an Abelian group.

The next proposition follows by definition or by the previous lemma.

Proposition 1.4 (Standard picture of K_1)**.** *Let A be a C*-algebra, then*

$$K_1(A) = \{[u]_1 : u \in \mathcal{U}_\infty(\widetilde{A})\}$$

The map $[\cdot]_1 : \mathcal{U}_\infty(A) \to K_1(A)$ has the following properties:

1.1. $[u \oplus v]_1 = [u]_1 + [v]_1$
1.2. $[1]_1 = 0$
1.3. If $u, v \in \mathcal{U}_n(\widetilde{A})$ and $u \sim_h v$, then $[u]_1 = [v]_1$
1.4. If $u, v \in \mathcal{U}_n(\widetilde{A})$, then $[uv]_1 = [vu]_1 = [u]_1 + [v]_1$
1.5. If $u, v \in \mathcal{U}_\infty(\widetilde{A})$, then $[u]_1 = [v]_1$ if and only if $u \sim_1 v$.

Proposition 1.5 (Universal Property of K_1)**.** *Let A be a C*-algebra and G an Abelian group. Let $\nu : \mathcal{U}_\infty(\widetilde{A}) \to G$ be a map satisfying*

1.1. $\nu(u \oplus v) = \nu(u) + \nu(v)$
1.2. $\nu(1) = 0$
1.3. If $u, v \in \mathcal{U}_n(\widetilde{A})$ such that $u \sim_h v$, then $\nu(u) = \nu(v)$

Then $\exists$ a unique homomorphism $\alpha : K_1(A) \to G$ such that

$$\alpha([u]_1) = \nu(u) \quad \forall u \in \mathcal{U}_\infty(A)$$

Proof. Suppose $u, v \in \mathcal{U}_\infty(\widetilde{A})$ are such that $u \sim_1 v$, then $\exists k \in \mathbb{N}$ such that

$$u \oplus 1_{k-n} \sim_h v \oplus 1_{k-m}$$

By properties (i) and (ii), $\nu(1_r) = 0$ for all $r \in \mathbb{N}$. Hence,

$$\nu(u) = \nu(u) + \nu(1_{k-n}) = \nu(u \oplus 1_{k-n}) = \nu(v \oplus 1_{k-m}) = \nu(v) + \nu(1_{k-m}) = \nu(v)$$

Thus, the map $\alpha : K_1(A) \to G$ as desired exists. Uniqueness follows from the fact that $[\cdot]_1 : \mathcal{U}_\infty(\widetilde{A}) \to K_1(A)$ is surjective. $\square$

Definition 1.6. Let A be a unital C*-algebra, and let $f := 1_{\widetilde{A}} - 1_A \in \widetilde{A}$, then

$$\widetilde{A} = A + \mathbb{C}f$$

Define $\mu : \widetilde{A} \to A$ by $a + \alpha f \mapsto a$. Then μ is a unital $*$-homomorphism, which we extend to a map $\mu : M_n(\widetilde{A}) \to M_n(A)$ as usual. This gives a map

$$\mu : \mathcal{U}_\infty(\widetilde{A}) \to \mathcal{U}_\infty(A)$$

Proposition 1.7. *Let A be a unital C*-algebra, then there is an isomorphism*

$$\rho : K_1(A) \to \mathcal{U}_\infty(A)/\sim_1$$

such that TFDC:

$$\begin{tikzcd}
\mathcal{U}_\infty(\widetilde{A}) \arrow[r, "\mu"] \arrow[d, "{[\cdot]_1}"] & \mathcal{U}_\infty(A) \arrow[d] \\
K_1(A) \arrow[r, "\rho"] & \mathcal{U}_\infty(A)/\sim_1
\end{tikzcd}$$

Proof. 1.1. If $u, v \in \mathcal{U}_n(\widetilde{A})$ such that $u \sim_h v$, then $\mu(u) \sim_h \mu(v)$.

1.2. Conversely, suppose $\mu(u) \sim_h \mu(v)$, we write

$$u = \mu(u) + u_0 \text{ and } v = \mu(v) + v_0$$

where $u_0, v_0 \in \mathcal{U}_n(\mathbb{C}f)$. Now we know that

$$u_0 \sim_h v_0 \text{ in } \mathcal{U}_n(\mathbb{C}f)$$

Since $\mu(u) \sim_h \mu(v)$, we may add the paths to obtain a path $a_t + b_t$ from u to v. Note that $a_t + b_t \in \mathcal{U}_n(\widetilde{A})$ because

$$a_t b_t = a_t b_t^* = a_t^* b_t = a_t^* b_t^* = 0$$

$\square$

Hence, in what follows, if A is unital, we will simply identify

$$K_1(A) = \mathcal{U}_\infty(A)/\sim_1$$

In particular, it follows that for any C*-algebra A,

$$K_1(A) \cong K_1(\widetilde{A})$$

Example 1.8. 1.1. $K_1(\mathbb{C}) = 0 = K_1(M_n(\mathbb{C}))$

Proof. By the previous proposition,

$$K_1(M_n(C)) \cong \mathcal{U}_\infty(M_n(\mathbb{C}))/\sim_1$$

However, any two unitaries in $M_k(M_n((\mathbb{C}))$ are connected, so $K_1(M_n(\mathbb{C})) = 0$ $\square$

1.2. $K_1(\mathcal{B}(H)) = 0$ if H is infinite dimensional as well.

Proof. Recall that if $u \in \mathcal{U}_n(\mathcal{B}(H)) \cong \mathcal{U}(\mathcal{B}(H^n))$ is any unitary, then by the Borel functional calculus,

$$u = e^{ih}$$

for some $h \in \mathcal{B}(H)_{sa}$. It follows that $u \sim_h 1_n$. Hence the result. $\square$

(End of Day 21)

2. Functoriality of K_1

Definition 2.1. Let $\varphi : A \to B$ be a $*$-homomorphism, then $\widetilde{\varphi} : \widetilde{A} \to \widetilde{B}$ is a unital $*$-homomorphism, which extends to a unital $*$-homomorphism $\widetilde{\varphi} : M_n(\widetilde{A}) \to M_n(\widetilde{B})$. This gives a map

$$\widetilde{\varphi} : \mathcal{U}_\infty(\widetilde{A}) \to \mathcal{U}_\infty(\widetilde{B})$$

Define $\nu : \mathcal{U}_\infty(\widetilde{A}) \to K_1(B)$ by

$$\nu(u) := [\widetilde{\varphi}(u)]_1$$

Then ν satisfies the conditions of Proposition 1.5. Hence, we get an induced map

$$K_1(\varphi) : K_1(A) \to K_1(B) \text{ given by } [u]_1 \mapsto [\widetilde{\varphi}(u)]_1$$

Note: If A and B are unital and $\varphi : A \to B$ is a unital $*$-homomorphism, then (Exercise)

$$K_1(\varphi)[u]_1 = [\varphi(u)]_1$$

Proposition 2.2 (Functoriality of K_1)**.** *Let A, B, C be C*-algebras. Then*

2.1. $K_1(id_A) = id_{K_1(A)}$

2.2. $K_1(\psi \circ \varphi) = K_1(\psi) \circ K_1(\varphi)$

Hence, K_1 is a contravariant functor.

2.3. $K_1(\{0\}) = \{0\}$

2.4. $K_1(0_{B,A}) = 0_{K_1(B),K_1(A)}$

2.5. *If $\varphi, \psi : A \to B$ are homotopic $*$-homomorphisms, then $K_1(\varphi) = K_1(\psi)$*

2.6. *If A and B are homotopy equivalent C^*-algebras, then $K_1(A) \cong K_1(B)$*

Proof. 2.1. Note that $\widetilde{\mathrm{id}_A} = \mathrm{id}_{\widetilde{A}}$

2.2. Note that $\widetilde{\psi \circ \varphi} = \widetilde{\psi} \circ \widetilde{\varphi}$

2.3. Recall that $K_1(A) \cong K_1(\widetilde{A})$ for any A. In particular,

$$K_1(\{0\}) \cong K_1(\mathbb{C}) = 0$$

2.4. $0_{B,A}$ factors as $A \to \{0\} \to B$, so it follows that

$$K_1(0_{B,A}) = K_1(0_{B,\{0\}}) \circ K_1(0_{\{0\},A}) = 0$$

2.5. Let φ_t be a path of $*$-homomorphisms from φ to ψ. Then $\widetilde{\varphi}_t$ is a path from $\widetilde{\varphi}$ to $\widetilde{\psi}$. Hence if $u \in \mathcal{U}_\infty(\widetilde{A})$, we have

$$K_1(\varphi)[u]_1 = [\widetilde{\varphi}(u)]_1 = [\widetilde{\psi}(u)]_1 = K_1(\psi)[u]_1$$

2.6. This follows from part (v)

□

3. Half and Split Exactness of K_1

The proofs here are similar to that of K_0. Fix a short exact sequence

$$0 \to I \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$$

Lemma 3.1. *Let $\varphi : A \to B$ be a $*$-homomorphism and $g \in \ker(K_1(\varphi))$. Then*

3.1. *$\exists u \in \mathcal{U}_\infty(\widetilde{A})$ such that $g = [u]_1$ and $\widetilde{\varphi}(u) \sim_h 1$*

3.2. *If φ is surjective, then $\exists u \in \mathcal{U}_\infty(\widetilde{A})$ such that $g = [u]_1$ and $\widetilde{\varphi}(u) = 1$.*

Proof. 3.1. Let $v \in \mathcal{U}_\infty(\widetilde{A})$ such that $g = [v]_1$, then $[\widetilde{\varphi}(v)]_1 = 0 = [1_m]_1$, so $\exists n \geq m$ such that

$$\widetilde{\varphi}(v) \oplus 1_{m-n} \sim_h 1_m \oplus 1_{m-n} = 1_n$$

so take $u = v \oplus 1_{m-n}$

3.2. By part (i), $\exists v \in \mathcal{U}_\infty(\widetilde{A})$ such that $g = [v]_1$ and $\widetilde{\varphi}(u) \sim_h 1_n$. Since φ is surjective, so is $\widetilde{\varphi}$, so $\exists w \in \mathcal{U}_n(\widetilde{A})$ such that $\widetilde{\varphi}(w) = \widetilde{\varphi}(v)$ and $w \sim_h 1$. Then $u := w^*v$ has the property that $\widetilde{\varphi}(u) = 1$ and $g = [v]_1 = [w^*]_1 + [v]_1 = [u]_1$.

□

Recall the following facts we proved earlier (See Lemma III.3.1): If A is a C*-algebra, we have a split exact sequence

$$0 \to A \xrightarrow{\iota} \widetilde{A} \xrightarrow{\pi} \mathbb{C} \to 0$$

and a map $\lambda : \mathbb{C} \to \widetilde{A}$ that splits π. We define $s : \widetilde{A} \to \widetilde{A}$ by

$$s = \lambda \circ \pi$$

so that $s(a + \alpha 1) = \alpha 1$ for all $a \in A, \alpha \in \mathbb{C}$. This induces a map

$$s_n : M_n(\widetilde{A}) \to M_n(\widetilde{A})$$

whose image consists of all matrices with scalar entries.

Lemma 3.2. *For any $n \in \mathbb{N}$*

3.1. $\widetilde{\varphi}_n : M_n(\widetilde{I}) \to M_n(\widetilde{A})$ is injective.

3.2. An element $a \in M_n(\widetilde{A})$ belongs to the image of $\widetilde{\varphi}_n$ iff $\widetilde{\psi}_n(a) = s_n(\widetilde{\psi}_n(a))$

Proposition 3.3 (Half-Exactness of K_1)**.** *Given a short exact sequence*

$$0 \to I \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$$

the sequence

$$K_1(I) \xrightarrow{K_1(\varphi)} K_1(A) \xrightarrow{K_1(\psi)} K_1(B)$$

is exact.

Proof. Since K_1 is a covariant functor, $K_1(\psi) \circ K_1(\varphi) = 0$, so it suffices to show that $\ker(K_1(\psi)) \subset \mathrm{Im}(K_1(\varphi)$, so fix $g \in \ker(K_1(\psi))$, then by the previous lemma, $\exists u \in \mathcal{U}_\infty(\widetilde{A})$ such that $g = [u]_1$ and $\widetilde{\psi}(u) = 1$. In particular,

$$s_n(\widetilde{\psi_n}(u)) = \widetilde{\psi_n}(u))$$

Hence $\exists v \in M_n(\widetilde{I})$ such that $u = \widetilde{\varphi}(v)$. Since u is a unitary and $\widetilde{\varphi}$ is injective, it follows that v is also a unitary. Hence,

$$g = [u]_1 = [\widetilde{\varphi}(v)]_1 \in \mathrm{Im}(K_1(\varphi))$$

□

Remark 3.4. We will show that, for any C*-algebra A,

$$K_1(A) \cong K_0(SA)$$

where $SA = C_0(0,1) \otimes A$. The next three results will follow from that fact along with the corresponding facts for K_0

Proposition 3.5 (Split Exactness of K_1)**.** *Given a split exact sequence*

$$0 \to I \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$$

with splitting $\lambda : B \to A$, then the following sequence is also split exact

$$0 \to K_1(I) \xrightarrow{K_1(\varphi)} K_1(A) \xrightarrow{K_1(\psi)} K_1(B) \to 0$$

with splitting $K_1(\lambda) : K_1(B) \to K_1(A)$

Proposition 3.6 (Direct sums)**.**

$$K_1(A \oplus B) \cong K_1(A) \oplus K_1(B)$$

Proposition 3.7 (Continuity of K_1)**.** *Let*

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \to \dots$$

be a sequence of C-algebras with inductive limit $(A, \{\mu_n\})$. Let $(G, \{\beta_n\})$ be the inductive limit of the sequence*

$$K_1(A_1) \xrightarrow{K_1(\varphi_1)} K_1(A_2) \xrightarrow{K_1(\varphi_2)} A_3 \to \dots$$

Then there is a group isomorphism $\gamma : G \to K_1(A)$ such that

$$\gamma \circ \beta_n = K_1(\mu_n) \quad \forall n \in \mathbb{N}$$

Furthermore,

3.1.

$$K_1(A) = \bigcup_{n=1}^{\infty} K_1(\mu_n)(K_1(A_n))$$

3.2. For each $n \in \mathbb{N}$,

$$\ker(K_1(\mu_n)) = \bigcup_{m=n+1}^{\infty} \ker(K_1(\varphi_{m,n}))$$

Proposition 3.8 (Stability of K_1)**.** *Let A be a C*-algebra, $n \in \mathbb{N}$ and $\lambda_n : A \to M_n(A)$ be the map as before. Then*

$$K_1(\lambda_n) : K_1(A) \to K_1(M_n(A))$$

is an isomorphism. Furthermore, if $\kappa : A \to A \otimes \mathcal{K}$ is the map as before, then

$$K_1(\kappa) : K_1(A) \to K_1(A \otimes \mathcal{K})$$

is also an isomorphism.

Example 3.9. 3.1. Let A be any finite dimensional C*-algebra, then $K_1(A) = 0$.

3.2. If A is any AF-algebra, then $K_1(A) = 0$

3.3. In particular, $K_1(\mathcal{K}) = 0$

4. K_1 and determinants

Let A be a unital C*-algebra, then there is a group homomorphism ω such that TFDC:

$$\begin{tikzcd} \mathcal{U}(A) \arrow[d] \arrow[dr, "{[\cdot]_1}"] & \\ \mathcal{U}(A)/\mathcal{U}_0(A) \arrow[r, "\omega"'] & K_1(A) \end{tikzcd}$$

which exists because $[u]_1 = 0$ for all $u \in \mathcal{U}_0(A)$. Let $\langle u \rangle$ denote the class of u in $\mathcal{U}(A)/\mathcal{U}_0(A)$, so that

$$\omega(\langle u \rangle) = [u]_1$$

Definition 4.1. Let A be a commutative C*-algebra. For each $n \in \mathbb{N}$, define a determinant by $D : M_n(A) \to A$ by

$$D((a_{i,j}) := \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{j=1}^{n} a_{j,\sigma(j)}$$

Remark 4.2. If $A = \mathbb{C}$, this is the usual determinant. The determinant has the following properties:

4.1. $D(ab) = D(a)D(b) \quad \forall a, b \in M_n(A)$

4.2.
$$D\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = D(a)D(b)$$

4.3. $D(a^*) = D(a)^*$

4.4. $D(a) = a$ for all $a \in A$

4.5. $D : M_n(A) \to A$ is continuous for all $n \in \mathbb{N}$

4.6. If A is unital and commutative, then D maps $\mathcal{U}_\infty(A)$ to $\mathcal{U}(A)$, and if $u, v \in \mathcal{U}_n(A)$ such that $u \sim_h v$, then $D(u) \sim_h D(v)$

Proof. If u_t is a path of unitaries such that $u_0 = u$ and $u_1 = v$, then $D(u_t)$ is a path of unitaries such that $D(u_0) = D(u)$ and $D(u_1) = D(v)$. □

Hence, there is a group homomorphism

$$\Delta : K_1(A) \to \mathcal{U}(A)/\mathcal{U}_0(A) \text{ given by } [u]_1 \mapsto \langle D(u) \rangle$$

and TFDC:

$$\begin{tikzcd} & \mathcal{U}(A) \arrow[dl] \arrow[d, "{[\cdot]_1}"] \arrow[dr] & \\ \mathcal{U}(A)/\mathcal{U}_0(A) \arrow[r, "\omega"'] & K_1(A) \arrow[r, "\Delta"'] & \mathcal{U}(A)/\mathcal{U}_0(A) \end{tikzcd}$$

In particular, by property 4.4,

$$\Delta \circ \omega = \mathrm{id}_{\mathcal{U}(A)/\mathcal{U}_0(A)}$$

Proposition 4.3. *Let A be a unital commutative C^*-algebra. Then there is a split exact sequence*

$$0 \to \ker(\Delta) \xrightarrow{\iota} K_1(A) \xrightarrow{\Delta} \mathcal{U}(A)/\mathcal{U}_0(A) \to 0$$

with splitting $\omega : \mathcal{U}(A)/\mathcal{U}_0(A) \to K_1(A)$. Hence,

$$K_1(A) \cong \mathcal{U}(A)/\mathcal{U}_0(A) \oplus \ker(\Delta)$$

Corollary 4.4. *If A is a unital commutative C^*-algebra such that $\mathcal{U}(A)$ is not connected, then $K_1(A) \neq 0$*

Proof. $\omega : \mathcal{U}(A)/\mathcal{U}_0(A) \to K_1(A)$ is injective. □

(End of Day 22)

Remark 4.5. If $u, v \in \mathcal{U}(A)$ such that $u \sim_h v$, then $u^*v \in \mathcal{U}_0(A)$ so $\langle u \rangle = \langle v \rangle$. Conversely, if $\langle u \rangle = \langle v \rangle$, then $u \sim_h v$. Hence, $\mathcal{U}(A)/\mathcal{U}_0(A)$ coincides with the set of path components of $\mathcal{U}(A)$. Furthermore,

$$\|u - v\| < 2 \Rightarrow u \sim_h v$$

So $\mathcal{U}(A)$ is locally path connected. Hence,

$$\mathcal{U}(A)/\mathcal{U}_0(A) = \pi_0(\mathcal{U}(A))$$

the set of connected components of $\mathcal{U}(A)$

Definition 4.6. Let $A = C(X)$, then $\mathcal{U}(A) = C(X, \mathbb{T})$. The cohomotopy group $\pi^1(X)$ is the group

$$[X, \mathbb{T}] = \mathcal{U}(A)/\mathcal{U}_0(A)$$

of pointed homotopy classes of pointed continuous maps $f : X \to \mathbb{T}$. This is an Abelian group under the point-wise multiplication of functions.

Example 4.7. $\pi^1(\mathbb{T}) \cong \mathbb{Z}$. Hence, $K_1(C(\mathbb{T})) \neq 0$

Proof. Write $F : [0,1] \to \mathbb{C}$ by
$$F(t) := u(e^{2\pi i t})$$
Then $F(t) \neq 0$ for all $t \in [0,1]$. Hence, $F \in GL(C[0,1])$. Set
$$M := \sup_{t\in[0,1]} |F(t)|^{-1}$$
Find a partition $0 = t_0 < t_1 < t_2 \ldots < t_n = 1$ of $[0,1]$ such that
$$\sup_{t_{k-1}\leq t\leq t_k} |F(t) - F(t_{k-1})| < \frac{1}{2M}$$
Then it follows that if $t \in [t_{k-1}, t_k]$,
$$\left|1 - \frac{F(t)}{F(t_{k-1}}\right| = \frac{|F(t) - F(t_{k-1})|}{|F(t_{k-1})|} \leq \frac{1}{2M|F(t_{k-1})|} \leq \frac{1}{2} < 1$$
On the domain $\Omega := \{z \in \mathbb{C} : |1 - z| < 1\}$, let $\log(z)$ be the principal branch of the logarithm,
$$\log(z) = -\sum_{n=1}^{\infty} \frac{(1-z)^n}{n}$$
Then log is holomorphic, and satisfies $\log(1) = 0$ and
$$e^{\log(z)} = z \quad \forall z \in \Omega$$
Now define
$$G_k(t) := \log(F(t)/F(t_{k-1})) \text{ on } t_{k-1} \leq t \leq t_k$$
Then G_k is continuous and $G_k(t_{k-1}) = 0$ and $G_k(t_k) = \log(F(t_k)/F(t_{k-1}))$. Furthermore,
$$F(t) = F(t_{k-1})e^{G_k(t)} \quad \forall t \in [t_{k-1}, t_k]$$
Define $G : [0,1] \to \mathbb{C}$ as follows
$$G(t) = G_1(t) \text{ on } t_0 \leq t \leq t_1$$
and
$$G(t) = G_1(t_1) + G_2(t_2) + \ldots + G_{k-1}(t_{k-1}) + G_k(t) \text{ on } [t_{k-1}, t_k]$$
Then it follows that
$$F(t) = F(0)e^{G(t)}$$
Write $F(0) \in \mathbb{C}^\times$ as $F(0) = e^{z_0}$, we obtain $f \in C[0,1]$ as
$$f(t) = G(t) + z_0$$
which satisfies
$$F(t) = u(e^{2\pi i t}) = e^{2\pi i f(t)}$$
If $f, g : [0,1] \to \mathbb{R}$ are continuous and satisfy the above equation, then $f - g$ is a constant integer. Define $\alpha : C(\mathbb{T}, \mathbb{T}) \to \mathbb{Z}$ by
$$\alpha(u) := f(1) - f(0)$$
This is well-defined, and is called the winding number of u. α is surjective, and

4.1. $\alpha(uv) = \alpha(u) + \alpha(v)$

Proof. Write $u(e^{2\pi it}) = e^{2\pi if(t)}$ and $v(e^{2\pi it}) = e^{2\pi ig(t)}$. Then

$$uv(e^{2\pi it}) = e^{2\pi i(f(t)+g(t))}$$

Hence

$$\alpha(uv) = f(1) + g(1) - (f(0) + g(0)) = \alpha(u) + \alpha(v)$$

□

4.2. $\alpha(u^*) = -\alpha(u)$

Proof. If $u(e^{2\pi it}) = e^{2\pi if(t)}$, then $u^*(e^{2\pi it}) = e^{2\pi i(-f(t))}$ □

4.3. $u \sim_h v$ iff $\alpha(u) = \alpha(v)$

Proof. (i) Suppose $u \sim_h v$, then $w := uv^* \sim_h 1$. Hence, $\exists h_1, h_2, \ldots, h_k \in C(\mathbb{T})_{sa}$ such that

$$w = \prod_{j=1}^{k} e^{ih_j}$$

To show that $\alpha(w) = 0$, it suffices to assume that $k = 1$ by part (i), so assume

$$w = e^{ih}$$

for some $h : \mathbb{T} \to \mathbb{R}$ continuous. Then

$$f(t) := \frac{1}{2\pi} h(e^{2\pi it}) \Rightarrow w(e^{2\pi it}) = e^{2\pi if(t)}$$

Hence,

$$\alpha(w) = f(1) - f(0) = 0$$

(ii) Conversely, suppose $\alpha(u) = \alpha(v)$, then for $w := u^*v$, we have $\alpha(w) = 0$ by part (i) and (ii). So write

$$w(e^{2\pi it}) = e^{2\pi if(t)}$$

such that $f(1) - f(0) = 0$. Now recall that $\mathbb{T} = [0,1]/\sim$ via the quotient map

$$t \mapsto e^{2\pi it} \text{ from } [0,1] \to \mathbb{T}$$

Since $F(0) = F(1)$, it induces a function $h \in C(\mathbb{T})$ by the formula

$$h(e^{2\pi it}) = f(t)$$

Hence,

$$u^*v = w = \exp(2\pi ih) \in \mathcal{U}_0(A)$$

Thus, $u \sim_h v$

All these properties combine to produce an isomorphism

$$\alpha : \pi^1(\mathbb{T}) \to \mathbb{Z} \text{ given by } \langle u \rangle \mapsto \alpha(u)$$

where $\langle u \rangle$ denotes the class of u in $[\mathbb{T}, \mathbb{T}]$. □

□

Remark 4.8. In fact, we will show later that

$$\Delta : K_1(C(\mathbb{T})) \to \pi^1(\mathbb{T})$$

is an isomorphism. Hence, $K_1(C(\mathbb{T})) \cong \mathbb{Z}$

Definition 4.9. A unital C*-algebra A is said to be K_1-injective (resp. K_1-surjective/K_1-bijective) if ω is injective (resp. surjective/bijective). If A is non-unital, we require that $\widetilde{A}$ have this property.

Example 4.10. 4.1. If $A = M_n(\mathbb{C})$ or $\mathcal{B}(H)$, then $\mathcal{U}(A) = \mathcal{U}_0(A)$ and $K_1(A) = 0$, so A is K_1-bijective.

4.2. If A is a unital, commutative C*-algebra, then it is K_1-injective.

4.3. $C(\mathbb{T})$ is K_1-bijective, but $C(\mathbb{T}^3)$ is not. (to be proved later)

4.4. Also, the irrational rotation algebra A_θ is K_1-bijective [due to Rieffel]

4.5. Every purely infinite, simple, unital C*-algebra is K_1-bijective [Rørdam, Larsen, and Laustsen, Exercise 8.13]. In particular, $\mathcal{O}_n$ and $\mathcal{Q}(H)$ are K_1-bijective.

4.6. For a unital C*-algebra A, $A \otimes \mathcal{K}$ is K_1-bijective [Rørdam, Larsen, and Laustsen, Exercise 8.17]

VIII. The Index Map

Given a short exact sequence of C*-algebras

$$0 \to I \to A \to B \to 0$$

we have obtained two sequences

$$K_i(I) \to K_i(A) \to K_i(B)$$

We now wish to define a map

$$\delta : K_1(B) \to K_0(I)$$

which connects the two sequences, giving a long exact sequence

$$K_1(I) \to K_1(A) \to K_1(B) \xrightarrow{\delta} K_0(I) \to K_0(A) \to K_0(B)$$

1. The Fredholm Index

To motivate this, consider the exact sequence

$$0 \to \mathcal{K} \to \mathcal{B}(H) \xrightarrow{\pi} \mathcal{Q}(H) \to 0$$

Theorem 1.1 (Atkinson)**.** *The following conditions are equivalent for an operator $T \in \mathcal{B}(H)$:*

1.1. $\ker(T)$ *and coker*$(T) = \ker(T^*)$ *are both finite dimensional.*

1.2. $\exists S \in \mathcal{B}(H)$ *such that* $1 - ST$ *and* $1 - TS$ *are compact.*

1.3. $\pi(T)$ *is invertible in* $\mathcal{Q}(H)$

Proof. [Murphy, Theorem 1.4.15] □

Definition 1.2. 1.1. If $T \in \mathcal{B}(H)$ satisfies one, and hence all, of the above conditions, then T is said to be Fredholm. Write $\Phi(H)$ for the set of all Fredholm operators in $\mathcal{B}(H)$. Note that

$$\Phi(H) = \pi^{-1}(GL(\mathcal{Q}(H))$$

Hence, $\Phi(H)$ is an open subset of $\mathcal{B}(H)$. Furthermore, note that if $T, S \in \Phi(H)$, then $ST \in \Phi(H)$ as well.

1.2. To each $T \in \Phi(H)$, we define the index of T as

$$\text{index}(T) := \dim(\ker(T)) - \dim(\ker(T^*)) \in \mathbb{Z}$$

Example 1.3. Let $S \in \mathcal{B}(\ell^2)$ denote the left shift operator

$$S((x_n)) = (x_2, x_3, \ldots)$$

and let T denote the right shift

$$T((x_n)) = (0, x_1, x_2, \ldots)$$

Then $S = T^*, ST = I$ and $TS = I - P_{e_1}$, so T is Fredholm. Also,

$$\ker(T) = \{0\} \text{ and } \ker(T^*) = \text{span}(e_1)$$

Hence,

$$\text{index}(T) = -1$$

Theorem 1.4. *([Murphy, Section 1.4]) The map index : $\Phi(H) \to \mathbb{Z}$ has the following properties:*

1.1. index$(T + K) =$ index(T) for all $T \in \Phi(H)$ and $K \in \mathcal{K}(H)$

1.2. index$(TS) =$ index(T)index(S) for all $S, T \in \Phi(H)$

1.3. The index map is locally constant on $\Phi(H)$, and continuous in the norm.

1.4. Two Fredholm operators are homotopic in $\Phi(H)$ iff they have the same index.

Remark 1.5. If T denotes the right-shift operator, then index$(T) = -1$, and index$(T^*) = 1$. By taking powers, it follows from the previous theorem, that

$$\text{index}(T^n) = -n \text{ and } \text{index}((T^*)^m) = m$$

Hence index : $\Phi(H) \to \mathbb{Z}$ is surjective. Furthermore, observe that T^n is an isometry for each $n \in \mathbb{N}$.

Theorem 1.6. *The index map induces an isomorphism*

$$ind : K_1(\mathcal{Q}(H)) \to K_0(\mathcal{K})$$

Hence, $K_1(\mathcal{Q}(H)) \cong \mathbb{Z}$.

Proof. 1.1. Let $u \in \mathcal{U}(\mathcal{Q}(H))$, and let $T \in \mathcal{B}(H)$ such that $\pi(T) = u$. Write the polar decomposition

$$T = V|T|$$

Then V is a partial isometry. Furthermore,

$$\pi(T^*T) = u^*u = 1 \Rightarrow \pi(|T|) = \pi((T^*T)^{1/2}) = 1$$

Hence, $\pi(V) = \pi(T) = u$

1.2. Now note that (check!)

$$\ker(V) = \text{range}(1 - V^*V)$$

and similarly, $\ker(V^*) = \text{range}(1 - VV^*)$. Hence,

$$\text{index}(V) = \text{rank}(1 - V^*V) - \text{rank}(1 - VV^*)$$

1.3. Furthermore, if W is any other partial isometry such that $\pi(W) = u$, then $W - V \in \mathcal{K}(H)$, so

$$\text{index}(W) = \text{index}(V)$$

1.4. Recall that $K_0(\mathcal{K}) \cong \mathbb{Z}$ via the map $[p]_0 \mapsto \text{rank}(p) = K_0(Tr)([p]_0)$. Hence, the map

$$\nu : \mathcal{U}(\mathcal{Q}(H)) \to K_0(\mathcal{K}) \text{ given by } u \mapsto [1 - V^*V]_0 - [1 - VV^*]_0$$

is well-defined. Furthermore, if $u \in \mathcal{U}_0(\mathcal{Q}(H))$, then we may choose V to be a unitary, in which case the RHS is zero. Hence, ν descends to a map

$$\text{ind} : \mathcal{U}(\mathcal{Q}(H))/\mathcal{U}_0(\mathcal{Q}(H)) \to K_0(\mathcal{K})$$

Since $\mathcal{Q}(H)$ is purely infinite,

$$K_1(\mathcal{Q}(H)) \cong \mathcal{U}(\mathcal{Q}(H))/\mathcal{U}_0(\mathcal{Q}(H))$$

Hence, we get a map

$$\text{ind} : K_1(\mathcal{Q}(H)) \to K_0(\mathcal{K}) \text{ given by } [u]_1 \mapsto [1 - V^*V]_0 - [1 - VV^*]_0$$

where $V \in \mathcal{B}(H)$ is any partial isometry such that $\pi(V) = u$.

1.5. We claim that ind is an isomorphism. Since $K_0(Tr) : K_0(\mathcal{K}) \to \mathbb{Z}$ is an isomorphism, it suffices to show that

$$\mu := K_0(Tr) \circ \text{ind}$$

1.6. μ is a homomorphism: If $u_1, u_2 \in \mathcal{U}(\mathcal{Q}(H))$ and $V_1, V_2 \in \mathcal{B}(H)$ are partial isometries such that $\pi(V_i) = u_i$. Then

$$\pi(V_1V_2) = u_1u_2$$

Let W be any partial isometry such that $\pi(W) = u_1u_2$, then $W - V_1V_2 \in \mathcal{K}(H)$, so

$$\text{index}(W) = \text{index}(V_1V_2)$$

Hence,

$$\mu(u_1) + \mu(u_2) = \text{index}(V_1) + \text{index}(V_2) = \text{index}(V_1V_2) = \text{index}(W) = \mu(u_1u_2)$$

1.7. μ is injective: Suppose $\mu([u]_1) = 0$, then

$$\text{index}(V) = 0 = \text{index}(I) \Rightarrow V \sim_h I \text{ in } \Phi(H)$$

Hence, $u = \pi(V) \sim_h I$ in $GL(\mathcal{Q}(H))$. Since $\mathcal{U}(\mathcal{Q}(H))$ is a deformation retract of $GL(\mathcal{Q}(H))$, it follows that $u \in \mathcal{U}_0(\mathcal{Q}(H))$, whence $[u]_1 = 0$

1.8. μ is surjective: If $x \in K_0(\mathcal{K})$, then write

$$n := K_0(Tr)(x)$$

By Remark 1.5, $\exists T \in \Phi(H)$ such that $\text{index}(T) = n$. Since μ is a group homomorphism, it suffices to assume $n \leq 0$. In which case, we may choose T to be an isometry. Hence,

$$T^*T = I \text{ and } TT^* - I \in \mathcal{K}(H)$$

Hence,

$$u := \pi(T) \in \mathcal{U}(\mathcal{Q}(H))$$

and by definition

$$\mu([u]_1) = \text{index}(T) = n$$

□

(End of Day 23)

2. Definition of the Index Map

Fix a short exact sequence

$$0 \to I \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$$

Recall the following facts we proved earlier (See Lemma III.3.1): If A is a C*-algebra, we have a split exact sequence

$$0 \to A \xrightarrow{\iota} \widetilde{A} \xrightarrow{\pi} \mathbb{C} \to 0$$

and a map $\lambda : \mathbb{C} \to \widetilde{A}$ that splits π. We define $s : \widetilde{A} \to \widetilde{A}$ by

$$s = \lambda \circ \pi$$

so that $s(a + \alpha 1) = \alpha 1$ for all $a \in A, \alpha \in \mathbb{C}$. This induces a map

$$s_n : M_n(\widetilde{A}) \to M_n(\widetilde{A})$$

whose image consists of all matrices with scalar entries.

Lemma 2.1. *For any $n \in \mathbb{N}$*

2.1. $\widetilde{\varphi}_n : M_n(\widetilde{I}) \to M_n(\widetilde{A})$ is injective.

2.2. An element $a \in M_n(\widetilde{A})$ belongs to the image of $\widetilde{\varphi}_n$ iff $\widetilde{\psi}_n(a) = s_n(\widetilde{\psi}_n(a))$

In what follows, write

$$q_n := \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{P}_{2n}(\widetilde{I})$$

Lemma 2.2. *Let $u \in \mathcal{U}_n(\widetilde{B})$, then*

2.1. $\exists v \in \mathcal{U}_{2n}(\widetilde{A})$ and a projection $p \in \mathcal{P}_{2n}(\widetilde{I})$ such that

$$\widetilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \widetilde{\varphi}(p) = vq_nv^*, \quad s(p) = q_n$$

2.2. If v, p are as in (i) and $w \in \mathcal{U}_{2n}(\widetilde{A})$ and $q \in \mathcal{P}_{2n}(\widetilde{I})$ also satisfy the same equation, then

$$s(q) = q_n \text{ and } p \sim_u q \text{ in } \mathcal{P}_{2n}(\widetilde{I})$$

Proof. 2.1. By Whitehead's lemma,

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h 1_{2n} \text{ in } \mathcal{U}_{2n}(\widetilde{B})$$

Hence, v exists as required. Also,

$$\widetilde{\psi}(vq_nv^*) = q_n$$

Hence, by the previous lemma, $\exists p \in M_{2n}(\widetilde{I})$ such that

$$\widetilde{\varphi}(p) = vq_nv^*$$

Since the RHS is a projection and $\widetilde{\varphi}$ is injective, $p \in \mathcal{P}_{2n}(\widetilde{I})$. Now,

$$\widetilde{\psi}(\widetilde{\varphi}(p)) = q_n$$

Hence,

$$s(p) = q_n$$

2.2. Suppose w, q also satisfy the same equations, then the same argument as above shows that

$$s(q) = q_n$$

Note that $\widetilde{\psi}(w^*v) = 1_{2n}$. Hence, by the previous lemma, $\exists z \in M_{2n}(\widetilde{I})$ such that

$$\widetilde{\varphi}(z) = w^*v$$

Since $\widetilde{\varphi}$ is injective, it follows that z is a unitary. Furthermore,

$$\widetilde{\varphi}(zpz^*) = \widetilde{\varphi}(q)$$

Since $\widetilde{\varphi}$ is injective, we conclude that

$$zpz^* = q$$

Hence, $p \sim_u q$ in $\mathcal{P}_{2n}(\widetilde{I})$ as required.

□

Definition 2.3. Define $\mu : \mathcal{U}_\infty(\widetilde{B}) \to K_0(I)$ by

$$\mu(u) := [p]_0 - [s(p)]_0 = [vq_nv^*]_0 - [q_n]_0$$

where $v \in \mathcal{U}_{2n}(\widetilde{A})$ and $p \in \mathcal{P}_{2n}(\widetilde{I})$ satisfy the previous lemma. Then, μ is a well-defined function.

Lemma 2.4. *The map $\mu : \mathcal{U}_\infty(\widetilde{B}) \to K_0(I)$ has the following properties:*

2.1. $\mu(u_1 \oplus u_2) = \mu(u_1) + \mu(u_2)$

2.2. $\mu(1) = 0$

2.3. *If $u_1, u_2 \in \mathcal{U}_n(\widetilde{B})$ are such that $u_1 \sim_h u_2$, then $\mu(u_1) = \mu(u_2)$*

Proof. 2.1. Technical. Skipped.

2.2. If $u = 1$, then the recipe from Lemma 2.2 gives

$$v = 1_2 \text{ and } p = \text{diag}(1, 0) = s(p)$$

Hence, $\nu(1) = [p]_0 - [s(p)]_0 = 0$

2.3. If $u_1 \sim_h u_2$, choose $v_1 \in \mathcal{U}_{2n}(\widetilde{A})$ and $p_1 \in \mathcal{P}_{2n}(\widetilde{I})$ such that

$$\widetilde{\psi}(v_1) = \begin{pmatrix} u_1 & 0 \\ 0 & u_1^* \end{pmatrix}, \quad \widetilde{\varphi}(p_1) = v_1 \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v_1^*$$

Then $\mu(u_1) = [p_1]_0 - [s(p_1)]_0$. Since

$$u_1^*u_2 \sim_h 1_n \sim_h u_1u_2^*$$

there are unitaries $a, b \in M_n(\widetilde{A})$ such that

$$\widetilde{\psi}(a) = u_1^*u_2 \text{ and } \widetilde{\psi}(b) = u_1u_2^*$$

Set $v_2 := v_1\text{diag}(a, b) \in \mathcal{U}_{2n}(\widetilde{A})$, then

$$\widetilde{\psi}(v_2) = \begin{pmatrix} u_2 & 0 \\ 0 & u_2^* \end{pmatrix} \text{ and } v_1 \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v_2^* = \widetilde{\varphi}(p_1)$$

Hence, $\mu(u_2) = [p_1]_0 - [s(p_1)]_0 = \mu(u_1)$.

$\square$

Definition 2.5. The map $\mu : \mathcal{U}_\infty(\widetilde{B}) \to K_0(I)$ satisfies all the conditions of Proposition VII.1.5. Hence by the universal property of K_1, we obtain a group homomorphism

$$\delta_1 : K_1(B) \to K_0(I)$$

such that $\delta_1([u]_1) = \mu(u)$ for all $u \in \mathcal{U}_\infty(\widetilde{B})$. ie. If $u \in \mathcal{U}_n(\widetilde{B})$, choose a unitary $v \in \mathcal{U}_{2n}(\widetilde{A})$ such that $\widetilde{\psi}(v) = \text{diag}(u, u^*)$. Then

$$\delta_1([u]_1) = [vq_nv^*] - [q_n]$$

Proposition 2.6 (Naturality of the Index Map)**.** *Let*

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & A \arrow[r, "\psi"] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & 0 \\
0 \arrow[r] & I' \arrow[r, "\varphi'"] & A' \arrow[r, "\psi'"] & B' \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram with short exact rows of C^*-algebras. Let

$$\delta_1 : K_1(B) \to K_0(I) \text{ and } \delta_1' : K_1(B') \to K_0(I')$$

be the index maps associated to the two sequences. Then TFDC:

$$\begin{tikzcd}
K_1(B) \arrow[r, "\delta_1"] \arrow[d, "K_0(\beta)"'] & K_0(I) \arrow[d, "K_0(\gamma)"] \\
K_1(B') \arrow[r, "\delta_1'"'] & K_0(I')
\end{tikzcd}$$

Proof. Let $g \in K_1(B)$ and $u \in \mathcal{U}_n(\widetilde{B})$ be such that $g = [u]_1$. Then choose $v \in \mathcal{U}_{2n}(\widetilde{A})$ and $p \in \mathcal{P}_{2n}(\widetilde{I})$ such that

$$\widetilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \widetilde{\varphi}(p) = vq_nv^*$$

Set $v' := \widetilde{\alpha}(v) \in \mathcal{U}_{2n}(\widetilde{A'}$ and $p' := \widetilde{\gamma}(p) \in \mathcal{P}_{2n}(\widetilde{I'})$, then

$$\widetilde{\psi'}(v') = \widetilde{\psi' \circ \alpha}(v) = \widetilde{\beta \circ \psi}(v) = \widetilde{\beta}\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} = \begin{pmatrix} \widetilde{\beta}(u) & 0 \\ 0 & \widetilde{\beta}(u)^* \end{pmatrix}$$

and similarly,

$$\widetilde{\varphi'}(p') = v'q_n(v')^*$$

Hence by definition of the index map,

$$\delta_1'(K_0(\beta)(g)) = \delta_1'([\widetilde{\beta}(u)]_1) = [p']_0 - [s(p')]_0 = [\widetilde{\varphi}(p)]_0 - [\widetilde{\varphi}(s(p))]_0 = K_0(\varphi)(\delta_1(g))$$

Hence the result. $\square$

Consider the short exact sequence $0 \to I \to A \xrightarrow{\pi} A/I \to 0$, where A (and hence A/J) is unital.

Proposition 2.7. *Let $u \in \mathcal{U}_n(A/J)$, then there is a partial isometry $x \in M_{2n}(\widetilde{A})$ such that*

$$\delta_1([u]_1) = [1 - x^*x]_0 - [1 - xx^*]_0$$

Proof. Let $v \in \mathcal{U}_{2n}(\widetilde{A})$ and $p \in \mathcal{P}_{2n}(\widetilde{I})$ be such that

$$\widetilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \widetilde{\varphi}(p) = vq_nv^*, s(p) = q_n$$

Write $x := (1-q_n)v^*$, then

$$x^*x = v(1-q_n)v^* = 1 - vq_nv^*$$

and $xx^* = 1 - q_n$. Hence,

$$[1-x^*x]_0 - [1-xx^*]_0 = [vq_nv^*]_0 - [q_n]_0 = \delta_1([u]_1)$$

□

Proposition 2.8. *Let $u \in \mathcal{U}_n(A/J)$ and suppose u lifts to a partial isometry $x \in M_n(A)$. Then*

$$\delta_1([u]_1) = [1-x^*x]_0 - [1-xx^*]_0$$

Proof. Let

$$w := \begin{pmatrix} x & 1-xx^* \\ 1-x^*x & x^* \end{pmatrix} \in M_{2n}(A)$$

Then w is a unitary and $\pi(w) = \text{diag}(u, u^*)$. Hence,

$$\delta_1([u]_1) = [wq_nw^*]_0 - [q_n]_0$$

However,

$$x = x(x^*x) \text{ and } x^* = x^*xx^*$$

Hence,

$$\begin{aligned}
wq_nw^* &= \begin{pmatrix} x & 1-xx^* \\ 1-x^*x & x^* \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x^* & 1-x^*x \\ 1-xx^* & x \end{pmatrix} \\
&= \begin{pmatrix} x & 0 \\ 1-x^*x & 0 \end{pmatrix} \begin{pmatrix} x^* & 1-x^*x \\ 1-xx^* & x \end{pmatrix} \\
&= \begin{pmatrix} xx^* & x(1-x^*x) \\ (1-x^*x)x^* & (1-x^*x) \end{pmatrix} \\
&= \begin{pmatrix} xx^* & 0 \\ 0 & 1-x^*x \end{pmatrix} \\
\Rightarrow wq_nw^* + \text{diag}(1-xx^*, 0) &= \begin{pmatrix} 1_n & 0 \\ 0 & 1-x^*x \end{pmatrix} = q_n + \text{diag}(0, 1-x^*x)
\end{aligned}$$

$$\Rightarrow \delta_1([u]_1) = [wq_nw^*]_0 - [q_n]_0 = [1-x^*x]_0 - [1-xx^*]_0$$

□

Corollary 2.9. *For the exact sequence*

$$0 \to \mathcal{K}(H) \to \mathcal{B}(H) \to \mathcal{Q}(H) \to 0$$

The map $ind : K_1(\mathcal{Q}(H)) \to K_0(\mathcal{K}(H))$ constructed earlier coincides with δ_1

3. Exact Sequence of K-groups

Let

$$0 \to I \overset{\varphi}{\to} A \overset{\psi}{\to} B \to 0$$

be a short exact sequence of C*-algebras.

Theorem 3.1. *The sequence*

$$\begin{array}{ccccc} K_1(I) & \xrightarrow{K_1(\varphi)} & K_1(A) & \xrightarrow{K_1(\psi)} & K_1(B) \\ & & & & \downarrow{\scriptstyle \delta_1} \\ K_0(B) & \xleftarrow{K_0(\psi)} & K_0(A) & \xleftarrow{K_0(\varphi)} & K_0(I) \end{array}$$

is exact.

Proof. We show two out of four steps and omit the rest, because they are too technical.

3.1. $\delta_1 \circ K_1(\psi) = 0$: Let $u \in \mathcal{U}_n(\widetilde{A})$ and let $x := K_1(\psi)([u]_1) = [\widetilde{\psi}(u)]_1$. We WTS: $\delta_1(x) = 0$. Now set

$$v = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \text{ and } p := q_n$$

Then $p = s(p)$ and

$$\widetilde{\psi}(v) = \begin{pmatrix} \widetilde{\psi}(u) & 0 \\ 0 & \widetilde{\psi}(u^*) \end{pmatrix}, \text{ and } \widetilde{\varphi}(p) = vq_nv^*$$

Hence,

$$\delta_1(x) = [p]_0 - [s(p)]_0 = 0$$

3.2. $K_0(\varphi) \circ \delta_1 = 0$: If $[u]_1 \in K_1(B)$, then let v and p as above. Then

$$K_0(\varphi)(\delta_1([u]_1)) = [\widetilde{\varphi}(p)]_0 - [\widetilde{\varphi}(s(p))]_0$$

But by construction,

$$\widetilde{\varphi}(p) = vq_nv^* \text{ and } \widetilde{\varphi}(s(p)) = q_n$$

Hence, $\widetilde{\varphi}(p) \sim_u \widetilde{\varphi}(p)$, and so

$$K_0(\varphi) \circ \delta_1([u]_1) = 0$$

$\square$

Example 3.2 (The Toeplitz Algebra)**.** Let $H = \ell^2$ and $s \in \mathcal{B}(H)$ be the right shift operator. Then s is an isometry,

$$s^*s = 1 \text{ and } ss^* = 1 - p_{e_1}$$

Let $\mathcal{T} := C^*(S)$. This is called the Toeplitz algebra.

3.1. We claim that $\mathcal{K}(H) \subset \mathcal{T}$: First define $e_{i,j}$ to be the map

$$e_{i,j}(x) := \langle x, e_j \rangle e_i$$

and set

$$f_n := \sum_{j=1}^{n} e_{j,j}$$

to be the projection onto $\text{span}\{e_1, e_2, \ldots, e_n\}$. Observe that

$$f_1 = 1 - ss^* \in \mathcal{T} \text{ and } e_{i,j} = s^{i-1} f_1 (s^*)^{j-1}$$

Hence, $e_{i,j} \in \mathcal{T}$ for all $i, j \in \mathbb{N}$. Together, they span all finite rank operators. Hence,

$$\mathcal{K}(H) = \overline{\text{span}\{e_{i,j}\}} \subset \mathcal{T}$$

3.2. Consider the quotient map $\pi : \mathcal{T} \to \mathcal{Q}(H)$. Then $\pi(s) \in \mathcal{Q}(H)$ is a unitary and

$$[\pi(s)]_1 \neq 0 \text{ in } K_1(\mathcal{Q}(H))$$

because $\text{index}(s) = -1$. Hence, $\pi(s) \notin \mathcal{U}_0(\mathcal{Q}(H))$, so it follows that

$$sp(\pi(s)) = \mathbb{T}$$

Hence, $\pi(\mathcal{T}) = C^*(\pi(s)) \cong C(\mathbb{T})$.

3.3. Thus, we get a short exact sequence

$$0 \to \mathcal{K}(H) \to \mathcal{T} \to C(\mathbb{T}) \to 0$$

and hence a long exact sequence of K-groups with

$$\delta_1 : K_1(C(\mathbb{T})) \to K_0(\mathcal{K}(H))$$

Let $u := \pi(s) \in \mathcal{U}(C(\mathbb{T}))$, then since s is an isometry, we have

$$\delta_1([u]_1) = [1 - s^*s]_0 - [1 - ss^*]_0 = -[f_1]_0$$

Furthermore, in $K_0(\mathcal{K})$, $-[f_1]_0$ is a generator. Hence,

$$\delta_1 : K_1(C(\mathbb{T})) \to K_0(\mathcal{K}(H))$$

is surjective.

3.4. Now consider the long exact sequence of K-groups

$$K_1(\mathcal{K}) \to K_1(\mathcal{T}) \to K_1(C(\mathbb{T})) \xrightarrow{\delta_1} K_0(\mathcal{K}) \xrightarrow{K_0(\iota)} K_0(\mathcal{T}) \xrightarrow{K_0(\pi)} K_0(C(\mathbb{T}))$$

Since δ_1 is surjective,

$$\ker(K_0(\iota)) = \text{im}(\delta_1) = K_0(\mathcal{K}) \Rightarrow K_0(\iota) = 0$$

Hence, this reduces to

$$0 \to K_1(\mathcal{T}) \to K_1(C(\mathbb{T})) \xrightarrow{\delta_1} \mathbb{Z} \xrightarrow{0} K_0(\mathcal{T}) \xrightarrow{K_0(\pi)} K_0(C(\mathbb{T}))$$

where $K_0(\pi)$ is injective.

Remark 3.3. We will show that $K_0(\mathcal{T}) \cong \mathbb{Z}$ and that $K_1(\mathcal{T}) = \{0\}$.

Example 3.4. Consider the short exact sequence

$$0 \to C_0(\mathbb{R}^2) \to C(\mathbb{D}) \to C(\mathbb{T}) \to 0$$

Since $\mathbb{D}$ is contractible, $C(\mathbb{D}) \sim_h \mathbb{C}$. Hence, $K_0(C(\mathbb{D})) \cong \mathbb{Z}$ and $K_1(C(\mathbb{D})) = 0$. Hence, the long exact sequence of K-groups gives

$$K_1(C_0(\mathbb{R}^2)) \to 0 \to K_1(C(\mathbb{T})) \to K_0(C_0(\mathbb{R}^2)) \to \mathbb{Z} \xrightarrow{K_0(\psi)} K_0(C(\mathbb{T}))$$

Hence, the map

$$K_1(C(\mathbb{T})) \to K_0(C_0(\mathbb{R}^2))$$

is injective. Since $K_1(C(\mathbb{T})) \neq 0$ by Example 4.7, it follows that $K_0(C_0(\mathbb{R}^2)) \neq 0$.

4. Higher K-functors

Corollary 4.1. *For any C^*-algebra A, there is a natural isomorphism*

$$K_1(A) \cong K_0(SA)$$

Proof. Note that $SA = C_0(0,1) \otimes A$. Define the cone of A to be $CA = C_0[0,1) \otimes A$, then we have a short exact sequence

$$0 \to SA \to CA \to A \to 0$$

Since $CA \sim_h \{0\}$, we have $K_0(CA) = K_1(CA) = 0$. So by exactness, we get an isomorphism

$$\alpha_A := \delta_1 : K_1(A) \to K_0(SA)$$

To see that this is natural, note that if $\varphi : A \to B$ is a $*$-homomorphism, we get a map of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & SA \arrow[r] \arrow[d, "S\varphi"] & CA \arrow[r] \arrow[d, "C\varphi"] & A \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & SB \arrow[r] & CB \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

where $S\varphi = \mathrm{id}_{C_0(0,1)} \otimes \varphi$ and $C\varphi = \mathrm{id}_{C_0[0,1)} \otimes \varphi$. Now the naturality of α_A follows from the naturality of the index map. $\square$

Note: This completes all the proofs we had left unfinished from Chapter VII.

Definition 4.2. For each $n \geq 2$, define the functor K_n inductively as

$$K_n(A) := K_{n-1}(SA)$$

Given a $*$-homomorphism $\varphi : A \to B$, we have a $*$-homomorphism

$$S\varphi : SA \to SA$$

By induction, it follows that we get a map

$$K_n(\varphi) : K_n(A) \to K_n(B)$$

which satisfies all the required properties so that K_n is a covariant functor.

Lemma 4.3. *For each $n \geq 2$, the functor K_n is a half-exact, split exact, homotopy invariant, continuous functor from the category of C^*-algebras to the category of Abelian groups.*

Proof. Given a short exact sequence

$$0 \to I \to A \to B \to 0$$

the induced sequence

$$0 \to SI \to SA \to SB \to 0$$

Hence, the functor S from the category of C*-algebras to itself is exact, split exact and continuous (See [Rørdam, Larsen, and Laustsen, Exercise 10.2]). Furthermore, if $A \simeq B$, then $SA \simeq SB$. Hence, by induction, if K_{n-1} satisfies any of these properties, then K_n would too. □

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Definition 4.4. Given a short exact sequence

$$0 \to I \to A \to B \to 0$$

we have an induced exact sequence

$$0 \to SI \to SA \to SB \to 0$$

The index map of this sequence is a map

$$\delta_2 := K_1(SB) \to K_0(SI) \Rightarrow \delta_2 := K_2(B) \to K_1(I)$$

This allows us to extend the long exact sequence one step further

$$\ldots \to K_2(B) \xrightarrow{\delta_2} K_1(I) \to K_1(A) \to K_1(B) \xrightarrow{\delta_1} K_0(I) \to \ldots$$

Proceeding inductively, we define

$$\delta_n : K_n(B) \to K_{n-1}(I)$$

which gives a long exact sequence in K-theory.

$$\ldots \to K_n(B) \xrightarrow{\delta_n} K_{n-1}(I) \to K_{n-1}(A) \to K_{n-1}(B) \xrightarrow{\delta_{n-1}} K_{n-2}(I) \to \ldots$$

IX. Bott Periodicity

1. Cuntz' Proof of Bott Periodicity

We wish to prove the following theorem

Theorem 1.1 (Bott)**.** *For any C*-algebra A, there is a natural isomorphism*

$$\beta_A : K_n(A) \to K_{n+2}(A)$$

Consider the Toeplitz algebra $\mathcal{T}$ defined earlier. Note that

$$\mathcal{T} = C^*(s)$$

where $s \in \mathcal{B}(\ell^2)$ is the right shift operator. Then there is a short exact sequence

$$0 \to \mathcal{K}(H) \to \mathcal{T} \xrightarrow{\pi} C(\mathbb{T}) \to 0$$

We will need the following

Theorem 1.2 (Universal Property of $\mathcal{T}$)**.** *If v is an isometry in a unital C*-algebra B, then there is a unique $*$-homomorphism $\varphi : \mathcal{T} \to B$ such that $\varphi(s) = v$.*

Definition 1.3. Define $q : \mathcal{T} \to \mathbb{C}$ by

$$q := ev_1 \circ \pi$$

Then q is a $*$-homomomorphism such that $q(s) = 1$.

Theorem 1.4 (Cuntz)**.** *For any C*-algebra A and any $n \in \mathbb{N}$, the map*

$$K_n(id_A \otimes q) : K_n(A \otimes \mathcal{T}) \to K_n(A)$$

is an isomorphism.

Proof. Fix A and $n \in \mathbb{N}$. For simplicity of notation, for any C*-algebras C and D and any $*$-homomorphism $\psi : C \to D$, we write

$$C' := C \otimes A \text{ and } \psi' : C' \to D' \text{ for } \psi \otimes \mathrm{id}_A$$

Let $j : \mathbb{C} \to \mathcal{T}$ be the unique unital $*$-homomorphism. Then

$$q \circ j = \mathrm{id}_{\mathbb{C}} \Rightarrow q' \circ j' = \mathrm{id}_A$$

Hence,

$$K_n(q') \circ K_n(j') = \mathrm{id}_{K_n(A)}$$

WTS: $K_n(j') \circ K_n(q') = \mathrm{id}_{K_n(\mathcal{T}')}$.
We will show that there are maps α and β such that

- $K_n(\alpha)$ is injective and
- $K_n(\beta) + K_n(\alpha) \circ K_n(j') \circ K_n(q') = K_n(\beta) + K_n(\alpha)$

1.1. **Step 1 (Finding α):**

(i) Let $e := 1 - ss^* \in \mathcal{T}$ be the rank one projection. Let $\epsilon : \mathcal{T} \to \mathcal{K} \otimes \mathcal{T}$ be the map
$$a \mapsto e \otimes a$$
Consider $\epsilon' : \mathcal{T} \otimes A \to (\mathcal{K} \otimes \mathcal{T}) \otimes A$. There is an isomorphism
$$\gamma : (\mathcal{K} \otimes \mathcal{T}) \otimes A \to (\mathcal{T} \otimes A) \otimes \mathcal{K}$$
such that $\gamma((a \otimes b) \otimes c) = (b \otimes c) \otimes a$ for all elementary tensors. Hence,
$$\lambda := K_n(\gamma) \circ K_n(\epsilon') : K_n(\mathcal{T} \otimes A) \to K_n((\mathcal{T} \otimes A) \otimes \mathcal{K})$$
is the natural isomorphism from Theorem V.4.6 (Note that Lemma V.4.5 applies to K_n, not just K_0). Furthermore, $K_n(\gamma)$ is an isomorphism. Hence,
$$K_n(\epsilon') : K_n(\mathcal{T}') \to K_n(\mathcal{T}' \otimes \mathcal{K})$$
is an isomorphism.

(ii) Inside the algebra $\mathcal{T} \otimes \mathcal{T}$, we have a subalgebra
$$\mathcal{T} \otimes 1 := \{a \otimes 1 : a \in \mathcal{T}\}$$
and a closed ideal $\mathcal{K} \otimes A$. So define a subalgebra
$$B := \mathcal{T} \otimes 1 + \mathcal{K} \otimes \mathcal{T}$$
Let $\pi : B \to B/(\mathcal{K} \otimes \mathcal{T})$ be the quotient map and let $\theta : \mathcal{T} \to B$ be the map $a \mapsto a \otimes 1$. Define the pullback
$$C := \{(b, a) \in B \oplus \mathcal{T} : \pi(b) = \pi \circ \theta(a)\}$$

(iii) Define $\iota : \mathcal{K} \otimes \mathcal{T} \to C$ by
$$x \mapsto (x, 0)$$
Then this is a well-defined $*$-homomorphism. Also define $p : C \to \mathcal{T}$ by
$$(b, a) \mapsto a$$
Note that $\text{im}(\iota) \subset \ker(p)$ by definition. Furthermore, if $p((b, a)) = 0$, then $a = 0$, so
$$\pi(b) = 0 \Rightarrow b \in \mathcal{K} \otimes \mathcal{T} \Rightarrow (b, a) \in \text{im}(\iota)$$

Hence we have an exact sequence

$$0 \to \mathcal{K} \otimes \mathcal{T} \to C \to \mathcal{T} \to 0$$

Define $\alpha : \mathcal{T} \to C$ by

$$a \mapsto (a \otimes 1, a)$$

Then α is a well-defined $*$-homomorphism because

$$a \otimes 1 - \theta(a) = 0 \in \mathcal{K} \otimes \mathcal{T}$$

and clearly $p \circ \alpha(a) = a$. Hence, this exact sequence splits.

(iv) Hence, the exact sequence

$$0 \to (\mathcal{K} \otimes \mathcal{T})' \xrightarrow{\iota'} C' \to \mathcal{T}' \to 0$$

also splits, and so does the sequence

$$0 \to K_n(\mathcal{K} \otimes \mathcal{T}') \xrightarrow{K_n(\iota')} K_n(C') \to K_n(\mathcal{T}') \to 0$$

In particular,

$$K_n(\iota') : K_n(\mathcal{K} \otimes \mathcal{T}') \to K_n(C')$$

is injective. Define $\psi : \mathcal{T} \to C$ by $\psi = \iota \circ \epsilon$. Then

$$K_n(\psi') : K_n(\mathcal{T}') \to K_n(C')$$

is injective. We show that

$$K_n(\psi') \circ K_n(j') \circ K_n(q') = K_n(\psi')$$

1.2. **Step 2 (Finding β):**

(i) Let $v := s^2 s^*$. Then v is an isometry. Furthermore,

$$v \otimes 1 - \theta(s) = v \otimes 1 - s \otimes 1 = s((ss^* - 1) \otimes 1) \in \mathcal{K} \otimes \mathcal{T}$$

Hence, $(v \otimes 1, s) \in C$. Furthermore, this element is an isometry in C. Thus, by the universal property of $\mathcal{T}$, there is a unique $*$-homomorphism $\varphi : \mathcal{T} \to C$ such that

$$\varphi(s) = (v \otimes 1, s)$$

(ii) Recall: $\psi : \mathcal{T} \to C$ is given by

$$\psi(a) = (e \otimes a, 0)$$

Now $ev = (1 - ss^*)s^2 s^* = s^2 s^* - ss^* s^2 s^* = 0$. Hence,

$$\psi(s)\varphi(s) = ((e \otimes s)(v \otimes 1), 0) = (0, 0)$$

Similarly,
$$\psi(s)^*\varphi(s) = \varphi(s)\psi(s) = \varphi(s)\psi(s)^* = 0$$
Hence, $\varphi(x)\psi(y) = 0$ for all $x, y \in \mathcal{T}$. Therefore,
$$\varphi + \psi$$
is a $*$-homomorphism. Similarly, $\varphi \perp \psi \circ j \circ q$, and so
$$\varphi + \psi \circ j \circ q$$
is a $*$-homomorphism. We show that
$$\varphi + \psi \sim_h \varphi + \psi \circ j \circ q$$

(iii) Define
$$\begin{aligned} z_0 &= s^2(s^*)^2 \otimes 1 + es^* \otimes s + se \otimes s^* + e \otimes e \\ z_1 &= s^2(s^*)^2 \otimes 1 + es^* \otimes 1 + se \otimes 1 \end{aligned}$$
Then $z_0, z_1 \in B$. Furthermore, they are both unitaries (Check!). Define a path
$$u_t := -i \exp(i\pi(1-t)z_0/2)\exp(i\pi t z_1/2)$$
Then $t \mapsto u_t$ is a path of unitaries in B such that $u_0 = z_0$ and $u_1 = z_1$. Furthermore, there is an identification [Murphy, Remark 3.3]
$$\begin{aligned} B/(\mathcal{K} \otimes \mathcal{T}) &= (\mathcal{K} \otimes \mathcal{T} + \mathcal{T} \otimes 1)/(\mathcal{K} \otimes \mathcal{T}) \\ &\cong \mathcal{T} \otimes 1/(\mathcal{K} \otimes \mathcal{T} \cap \mathcal{T} \otimes 1) \\ &\cong \mathcal{T}/\mathcal{K} \cong C(\mathbb{T}) \end{aligned}$$
And under this isomorphism, we have a commuting diagram
$$\begin{tikzcd} \mathcal{T} \arrow[r, "\pi"] \arrow[d, "\theta"] & C(\mathbb{T}) \arrow[d, "\cong"] \\ B \arrow[r, "\pi"'] & B/\mathcal{K} \otimes \mathcal{T} \end{tikzcd}$$
Hence, it is clear that
$$\pi(z_0) = \pi(z_1) = 1$$
and so $\pi(u_t) = 1$ for all $t \in [0,1]$.

(iv) For each $t \in [0,1]$, the element
$$w_t := (u_t(s \otimes 1), s) \in C$$
is an isometry. Hence, it defines an isometry
$$w \in C[0,1] \otimes C$$

By the universal property of $\mathcal{T}$, there is a $*$-homomorphism $\varphi : \mathcal{T} \to C[0,1] \otimes C$ such that

$$\varphi(s) = w$$

This gives a $*$-homomorphism $\varphi' : \mathcal{T}' \to C[0,1] \otimes C'$ which we think of as a path of $*$-homomorphism $\varphi'_t : \mathcal{T}' \to C'$.

(v) Note that,

$$(\varphi + \psi)(s) = \varphi(s) + \psi(s) = (v \otimes 1, s) + (e \otimes s, 0) = (u_0(s \otimes 1), s)$$

Hence, $\varphi + \psi = \varphi_0$ by uniqueness. Similarly,

$$\varphi + \psi \circ j \circ q = \varphi_1$$

Tensoring with A, we get

$$\varphi' + \psi' \circ j' \circ q' = \varphi'_1 \text{ and } \varphi' + \psi' = \varphi'_0$$

1.3. **Step 3 (Completing the proof):** By homotopy invariance, we have

$$K_n(\varphi') + K_n(\psi') = K_n(\varphi') + K_n(\psi') \circ K_n(j') \circ K_n(q')$$

Hence,

$$K_n(\psi') = K_n(\psi') \circ K_n(j') \circ K_n(q')$$

Since $K_n(\psi')$ is injective, we have

$$K_n(j') \circ K_n(q') = \mathrm{id}_{K_n(\mathcal{T}')}$$

as required.

□

Example 1.5. We conclude that

$$K_0(\mathcal{T}) \cong K_0(\mathbb{C}) \cong \mathbb{Z} \text{ and } K_1(\mathcal{T}) = 0$$

This completes Example VIII.3.2.

Remark 1.6. 1.1. The reduced Toeplitz Algebra is defined as

$$\mathcal{T}_0 := \ker(q)$$

1.2. Note that $\mathcal{K}(H) \subset \mathcal{T}_0$ and

$$\pi(\mathcal{T}_0) = \{f \in C(\mathbb{T}) : f(1) = 0\}$$

Hence, we have a short exact sequence

$$0 \to \mathcal{K}(H) \to \mathcal{T}_0 \to C_0(\mathbb{R}) \to 0$$

Since all the C*-algebras are nuclear, for any C*-algebra A, we have

$$0 \to A \otimes \mathcal{K} \to A \otimes \mathcal{T}_0 \to SA \to 0$$

1.3. Furthermore, we have a split exact sequence

$$0 \to \mathcal{T}_0 \to \mathcal{T} \to \mathbb{C} \to 0$$

Once again, this induces a split exact sequence

$$0 \to A \otimes \mathcal{T}_0 \to A \otimes \mathcal{T} \to A \to 0$$

Theorem 1.7. *For any C*-algebra A, there is a natural isomorphism*

$$\beta_A : K_n(A) \to K_{n+2}(A) = K_{n+1}(SA)$$

This map is called the Bott map.

Proof. Since K_n is split exact, we have a split exact sequence

$$0 \to K_n(A \otimes \mathcal{T}_0) \to K_n(A \otimes \mathcal{T}) \xrightarrow{K_n(\mathrm{id}_A \otimes q)} K_n(A) \to 0$$

By the previous theorem, $K_n(\mathrm{id}_A \otimes q)$ is an isomorphism. Hence,

$$K_n(A \otimes \mathcal{T}_0) = 0$$

Now consider the long exact sequence in K-theory arising from the second short exact sequence from the previous remark

$$\begin{tikzcd}
K_{n+1}(A \otimes \mathcal{K}) \arrow[r] & K_{n+1}(A \otimes \mathcal{T}_0) \arrow[r] & K_{n+1}(SA) \arrow[d, "\delta_{n+1}"'] \\
K_n(SA) & K_n(A \otimes \mathcal{T}_0) \arrow[l] & K_n(A \otimes \mathcal{K}) \arrow[l]
\end{tikzcd}$$

Since $K_n(A \otimes \mathcal{T}_0) = 0 = K_{n+1}(A \otimes \mathcal{T}_0)$, it follows by exactness that

$$\delta_{n+1} : K_{n+1}(SA) \to K_n(A \otimes \mathcal{K})$$

is an isomorphism. Let $\lambda : K_0(A) \to K_0(A \otimes \mathcal{K})$ be the isomorphism from earlier, then

$$\beta_A := \lambda^{-1} \circ \delta_{n+1}$$

is an isomorphism. It is also natural because both λ and δ_{n+1} are natural. $\square$

2. The Six Term Exact Sequence

Definition 2.1. Given a short exact sequence

$$0 \to I \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$$

we define the exponential map as the map $\delta_0 : K_0(B) \to K_1(I)$ by

$$\delta_0 = \delta_2 \circ \beta_B : K_0(B) \to K_2(B) \to K_1(I)$$

where δ_2 is the index map associated to the sequence

$$0 \to SI \xrightarrow{S\varphi} SA \xrightarrow{S\psi} SB \to 0$$

Note that δ_0 is natural in the sense described earlier. Given another short exact sequence and maps as below

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & A \arrow[r, "\psi"] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & 0 \\ 0 \arrow[r] & I' \arrow[r, "\varphi'"] & A' \arrow[r, "\psi'"] & B' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram with short exact rows of C*-algebras. Then TFDC:

$$\begin{tikzcd} K_0(B) \arrow[r, "\delta_0"] \arrow[d, "K_0(\beta)"'] & K_1(I) \arrow[d, "K_1(\gamma)"] \\ K_0(B') \arrow[r, "\delta_0'"'] & K_1(I') \end{tikzcd}$$

This is because both β_B and δ_2 are natural.

Theorem 2.2. *The six term sequence*

$$\begin{tikzcd} K_0(I) \arrow[r, "K_0(\varphi)"] & K_0(A) \arrow[r, "K_0(\psi)"] & K_0(B) \arrow[d, "\delta_0"] \\ K_1(B) \arrow[u, "\delta_1"] & K_1(A) \arrow[l, "K_1(\psi)"'] & K_1(I) \arrow[l, "K_1(\varphi)"'] \end{tikzcd}$$

is exact.

Proof. It suffices to show exactness at $K_0(B)$ and $K_1(I)$. To see this, consider the diagram

$$\begin{tikzcd} K_2(A) \arrow[r, "K_2(\psi)"] & K_2(B) \arrow[r, "\delta_2"] & K_1(I) \arrow[r, "K_1(\varphi)"] & K_1(A) \\ K_0(A) \arrow[r, "K_0(\psi)"'] \arrow[u, "\beta_A"] & K_0(B) \arrow[r, "\delta_0"'] \arrow[u, "\beta_B"] & K_1(I) \arrow[r, "K_1(\varphi)"'] \arrow[u, "="] & K_1(A) \arrow[u, "="] \end{tikzcd}$$

The diagram commutes by naturality of the Bott map. The top row is exact, so the bottom row must be. □

3. Examples and Calculations

Example 3.1. For the exact sequence

$$0 \to \mathcal{K}(H) \to \mathcal{B}(H) \to \mathcal{Q}(H) \to 0$$

we had seen that the map

$$\delta_1 : K_1(\mathcal{Q}(H)) \to K_0(\mathcal{K})$$

was an isomorphism (The Fredholm index). We may now conclude, from the six-term exact sequence and the fact that $K_0(\mathcal{B}(H)) = K_1(\mathcal{B}(H)) = 0$ that

$$K_0(\mathcal{Q}(H)) \cong K_1(\mathcal{K}) = 0$$

as well.

Example 3.2. For any $n \in \mathbb{N}$

$$K_0(C_0(\mathbb{R}^n)) \cong K_n(\mathbb{C}) \cong \begin{cases} K_0(\mathbb{C}) \cong \mathbb{Z} & : n \text{ even} \\ K_1(\mathbb{C}) \cong 0 & : n \text{ odd} \end{cases}$$

and similarly,

$$K_1(C_0(\mathbb{R}^n)) \cong \begin{cases} 0 & : n \text{ even} \\ \mathbb{Z} & : n \text{ odd} \end{cases}$$

Example 3.3 ($C(\mathbb{T}^n)$)**.** 3.1. For any C*-algebra A, we have a split exact sequence

$$0 \to SA \to C(\mathbb{T}, A) \to A \to 0$$

so we get

$$K_n(C(\mathbb{T}, A)) \cong K_n(A) \oplus K_n(SA) \cong K_n(A) \oplus K_{n+1}(A)$$

3.2. If $A = \mathbb{C}$, we have

$$K_0(C(\mathbb{T})) \cong K_0(\mathbb{C}) \oplus K_1(\mathbb{C}) \cong \mathbb{Z}$$

and similarly, $K_1(C(\mathbb{T})) \cong \mathbb{Z}$. We had shown (Example VII.4.7) that there was an injective map

$$\omega : \mathcal{U}(C(\mathbb{T}))/\mathcal{U}_0(C(\mathbb{T})) \to K_1(C(\mathbb{T}))$$

and we had shown that $\mathcal{U}(C(\mathbb{T}))/\mathcal{U}_0(C(\mathbb{T})) = [\mathbb{T}, \mathbb{T}] \cong \mathbb{Z}$. We now conclude that this map is an isomorphism.

3.3. If $A = C(\mathbb{T}^{n-1})$, then $C(\mathbb{T}^n) = C(\mathbb{T}, A)$, so by induction, we have

$$K_0(C(\mathbb{T}^n)) \cong K_1(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{n-1}}$$

3.4. If $n = 3$, we observe that

$$K_1(C(\mathbb{T}^3)) \cong \mathbb{Z}^4$$

However, by [Rørdam, Larsen, and Laustsen, Exercise 8.15],

$$\mathcal{U}(C(\mathbb{T}^3))/\mathcal{U}_0(C(\mathbb{T}^3)) \cong \mathbb{Z}^3$$

so the map ω (which is always injective for commutative C*-algebras) is not surjective in this case.

3.5. Hence, if $A = C(\mathbb{T}^3)$, then $\exists x \in K_1(A)$ such that

$$x \neq [v]_1 \quad \forall v \in \mathcal{U}(A)$$

Choose $k \in \mathbb{N}$ minimal such that $\exists u \in \mathcal{U}_{2k}(A)$ such that $x = [u]_1$. Then, if $B = M_k(A)$, then we have found a unitary $u \in \mathcal{U}(M_2(B))$ such that

$$u \nsim_h \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}$$

for any $v \in \mathcal{U}(B)$. This completes Example I.2.9, where we needed such a C*-algebra to show that, for projections, $p \sim_u q$ does not necessarily imply $p \sim_h q$.

3.6. Note: In fact, we may take $k = 1$. To prove this, we need to understand homotopical stable ranks for C*-algebras.

(End of Day 25)

Example 3.4 (Dimension Drop Algebras)**.** For $n \in \mathbb{N}$, define the dimension drop algebra

$$D_n := \{f : [0,1] \to M_n(\mathbb{C}) : f(0) = 0, f(1) \in \mathbb{C}1_n\}$$

We have a short exact sequence

$$0 \to SM_n(\mathbb{C}) \xrightarrow{\iota} D_n \xrightarrow{\pi} \mathbb{C} \to 0$$

where $\pi(f) = f(1)$. So we get a six-term exact sequence

$$\begin{tikzcd} K_0(SM_n(\mathbb{C})) \arrow[r, "K_0(\iota)"] & K_0(D_n) \arrow[r, "K_0(\pi)"] & K_0(\mathbb{C}) \arrow[d, "\delta_0"] \\ K_1(\mathbb{C}) \arrow[u, "\delta_1"] & K_1(D_n) \arrow[l, "K_1(\pi)"] & K_1(SM_n(\mathbb{C})) \arrow[l, "K_1(\iota)"] \end{tikzcd}$$

Now $K_k(SM_n(\mathbb{C})) \cong K_{n+1}(\mathbb{C})$, so we get

$$\begin{tikzcd} 0 \arrow[r, "K_0(\iota)"] & K_0(D_n) \arrow[r, "K_0(\pi)"] & \mathbb{Z} \arrow[d, "\delta_0"] \\ 0 \arrow[u, "\delta_1"] & K_1(D_n) \arrow[l, "K_1(\pi)"] & \mathbb{Z} \arrow[l, "K_1(\iota)"] \end{tikzcd}$$

so it suffices to understand the map $\delta_0 : K_0(\mathbb{C}) \to K_1(SM_n(\mathbb{C}))$. There is an explicit description of the map δ_0, which allows us to compute that δ_0 is the map

$$1 \mapsto n$$

from $\mathbb{Z} \mapsto \mathbb{Z}$. Hence,

$$K_0(D_n) = 0 \text{ and } K_1(D_n) \cong \mathbb{Z}/n\mathbb{Z}$$

Hence, we have the following K-groups:

	K_0	K_1
D_n	0	$\mathbb{Z}/n\mathbb{Z}$
SD_n	$\mathbb{Z}/n\mathbb{Z}$	0
$C_0(\mathbb{R})$	0	$\mathbb{Z}$
$\mathbb{C}$	$\mathbb{Z}$	0

Corollary 3.5. *Let G_0 and G_1 be any finitely generated Abelian group, then $\exists$ a C*-algebra A such that*

$$K_0(A) \cong G_0 \text{ and } K_0(A) \cong G_1$$

Proof. Simply take direct sums of the above C*-algebras. □

X. Exercises

1. Chapter 1

2. Chapter 2

2.1. Assume A is unital by replacing it with A^+, and note that $p - q$ is a self-adjoint element.

2.2.

2.3.

2.4.

(i) $\Rightarrow$ (ii): If $pq = 0$, then $qp = q^*p^* = (pq)^* = 0$, so

$$(p+q)^2 = p^2 + q^2 + pq + qp = p + q$$

and clearly $p + q$ is self-adjoint.

(ii) $\Rightarrow$ (iii): If $p + q$ is a projection, then $p + q \leq 1$ must hold.

(iii) $\Rightarrow$ (i): If $p+q \leq 1$, then $p(p+q)p \leq p$ by [Murphy, Theorem 2.2.5]. So $p+pqp \leq p$, so $pqp = 0$. Hence, $(pq)^2 = 0$, so $pq = 0$

The second part follows by induction.

2.5.

2.6. Let $p_j := v_j^*v_j, q_j = v_jv_j^*$, then by the previous statement, the $p_i \perp p_j$ and $q_i \perp q_j$, if $i \neq j$. So if $i \neq j$, then

$$v_i^*v_j = (q_iv_i)^*q_jv_j = v_i^*q_iq_jv_j = 0$$

Hence if $u = \sum_{i=1}^n v_i$, then

$$u^*u = \sum_{i=1}^n v_i^*v_i + \sum_{i\neq j} v_i^*v_j = \sum_{i=1}^n v_i^*v_i = 1_A$$

Similarly, $uu^* = 1_A$ as well.

2.7.

2.8. If a were invertible, then $u := \omega(a)$ would be a candidate for the required unitary. Furthermore, we would need $\|a - \omega(a)\| < \epsilon$. To ensure this, let $K = [0,2]$ and Ω_K the set of all self-adjoint elements of A with spectrum contained in K. Then the square root function $h \mapsto h^{1/2}$ is continuous on Ω_K, so $\exists \delta > 0$ such that, for any $h \in \Omega_K$

$$\|h - 1\| < \delta \Rightarrow \|h^{1/2} - 1\| < \epsilon$$

In particular, if $a \in A$ such that $\|a^*a - 1\| < \delta$, then $\||a| - 1\| < \epsilon$, whence

$$\|a - \omega(a)\| = \|\omega(a)|a| - \omega(a)\| \leq \||a| - 1\| < \epsilon$$

So now we need to ensure that a is invertible, but this follows if $\delta < 1$ (which can, of course, be arranged), so that a^*a and aa^* are both invertible (so a is both left and right invertible, hence invertible).

For the second part of the problem: Fix $\epsilon > 0$, and choose $\delta_1 > 0$ such that

$$\|1 - x\| < \delta_1 \Rightarrow \|1 - x^{-1}\| < \epsilon/3$$

This is possible because the inverse map is continuous at 1. Using Ω_K as above, choose $\delta_2 > 0$ such that, for any $h \in B_{sa}$ such that $sp(h) \subset [0,2]$

$$\|1 - h\| < \delta_2 \Rightarrow \|1 - h^{1/2}\| < \delta_1$$

In particular, if $b \in B$ such that $\|b\| \leq \sqrt{2}$, then $h := b^*b$ has the property that $sp(h) \subset [0,2]$ so

$$\|1 - b^*b\| < \delta_2 \Rightarrow \|1 - |b|\| < \delta_1$$

Now define

$$\delta := \min\left\{\frac{\delta_2}{2}, \frac{\epsilon}{2}, \sqrt{2} - 1\right\}$$

Then if $\|u - b\| < \delta$, then $\|b\| \leq \|u - b\| + 1 \leq \sqrt{2}$, and

$$\begin{aligned}\|1 - b^*b\| &= \|u^*u - b^*b\| \\ &\leq \|u^*u - u^*b\| + \|u^*b - b^*b\| \\ &\leq \|u - b\| + \|u^* - b^*\| = 2\|u - b\| < \delta_2\end{aligned}$$

Hence, $\|1 - |b|\| < \delta_1$, so $\|1 - |b|^{-1}\| < \epsilon/3$. Hence,

$$\begin{aligned}\|u - \omega(b)\| &= \|u - b|b|^{-1}\| \leq \|u - b\| + \|b - b|b|^{-1}\| \\ &\leq \frac{\epsilon}{2} + \|b\|\|1 - |b|^{-1}\| \\ &\leq \frac{\epsilon}{2} + \|b\|\frac{\epsilon}{3} \leq \frac{\epsilon}{2} + \frac{\sqrt{2}\epsilon}{3} \leq \epsilon\end{aligned}$$

Finally, note that $\|u - b\| \leq 1 = \frac{1}{\|u^{-1}\|}$, so b is invertible, whence $\omega(b)$ is a unitary in B.

3. Chapter 3

3.1.

3.2.

3.3.

3.4. Let X be compact Hausdorff

(i) As in Example 3.3.5, for each $x \in X$, there is a map

$$\nu_x : K_0(C(X)) \to \mathbb{Z} \text{ given by } [p]_0 \mapsto Tr(p(x))$$

For $p \in \mathcal{P}_n(C(X))$ fixed, the map $x \mapsto Tr(p(x))$ is continuous from X to $\mathbb{Z}$. Thus, we get a map

$$\dim : K_0(C(X)) \to C(X, \mathbb{Z})$$

This map is surjective: If $X = \bigsqcup_{i=1}^n C_i$, where each C_i is a connected component of X, then any $f \in C(X, \mathbb{Z})$ can be expressed uniquely in the form

$$f = \sum_{i=1}^n f_i \chi_{C_i}$$

where $f_i \in \mathbb{Z}$ the common value taken by f on C_i. Hence, it suffices to show that $\chi_{C_i} \in \mathrm{Im}(\dim)$. But if $p_i \in \mathcal{P}(C(X))$ is the projection χ_{C_i}, then

$$\dim(p_i) = \chi_{C_i}$$

(ii) If there is such a $v_x \in M_{m,n}(C(X))$, then it follows that $p(x) \sim_0 q(x)$, so that $Tr(p(x)) = Tr(q(x))$ for all $x \in X$. Conversely, if $Tr(p(x)) = Tr(q(x))$, then $\exists \widetilde{v_x} \in M_{m,n}(\mathbb{C})$ such that $\widetilde{v_x}^* \widetilde{v_x} = p(x)$ and $\widetilde{v_x}\widetilde{v_x}^* = q(x)$. However, the evaluation map

$$M_{m,n}(C(X)) \to M_{m,n}(\mathbb{C})$$

is surjective, so $\exists v_x \in M_{m,n}(C(X))$ such that $v_x(x) = \widetilde{v_x}$, which solves the problem.

(iii) Suppose $\dim([p]_0) = \dim([q]_0)$, then for each $x \in X, \exists v_x \in M_{m,n}(C(X))$ such that $v_x^*(x)v_x(x) = p(x)$ and $v_x(x)v_x^*(x) = q(x)$. By continuity, $\exists$ a neighbourhood U_x of x such that

$$\|v_x^*(y)v_x(y) - p(y)\| < \frac{1}{2} \text{ and } \|v_x(y)v_x^*(y) - q(y)\| < \frac{1}{2}$$

for all $y \in U_x$. Choose a refinement of $\{U_x : x \in X\}$ made up of mutually disjoint sets, and choose a finite subcover $\{X_1, X_2, \ldots, X_r\}$ such that, for each $1 \leq i \leq r, \exists v_i \in M_{m,n}(C(X))$ such that

$$\|v_i^*(y)v_i(y) - p(y)\| < \frac{1}{2} \text{ and } \|v_i(y)v_i^*(y) - q(y)\| < \frac{1}{2}$$

It follows that $\|p(y) - q(y)\| < 1$ for all $y \in X$, so that $\|p - q\| < 1$. Hence, $p \sim_h q$ as required.

3.5.

3.6.

4. Chapter 4

5. Chapter 5

6. Chapter 6

6.1.

6.2.

6.3.

6.4.

6.5.

6.6. Let A be a C*-algebra, ande define $\rho : \mathcal{P}_\infty(A) \to \mathcal{P}(A \otimes \mathcal{K})$ as follows: If $p \in \mathcal{P}_n(A)$, then $\rho(p) := \kappa_n(p)$. Then ρ induces a bijection

$$\widehat{\rho} : \mathcal{D}(A)/\sim_0 \to \mathcal{P}(A \otimes \mathcal{K})/\sim$$

Proof. (i) Let $p \in \mathcal{P}_n(A)$, then $\kappa_n(p) \in \mathcal{P}(A \otimes \mathcal{K})$ since κ_n is a $*$-homomorphism. Furthermore, if $p \in \mathcal{P}_m(A), q \in \mathcal{P}_n(A)$ such that $p \sim_0 q$, then choose $\ell \geq \max\{n, m\}$ and $p' = \varphi_{\ell,m}(p), q' = \varphi_{\ell,n}(q) \in \mathcal{P}_\ell(A)$ such that $p \sim_0 p'$ and $q \sim_0 q'$ so that

$$p' \sim q' \text{ in } M_n(A)$$

Then

$$\kappa_m(p) = \kappa_\ell \circ \varphi_{\ell,m}(p) = \kappa_\ell(p') \sim \kappa_\ell(q') = \kappa_\ell \circ \varphi_{\ell,n}(q) = \kappa_n(q)$$

Hence, ρ induces a map $\widehat{\rho}$ as required.

To show that $\widehat{\rho}$ is a bijection, it would suffice to prove two things:

(ii) If $p, q \in \mathcal{P}_\infty(A)$ such that $\rho(p) \sim \rho(q)$, then $p \sim_0 q$

Proof. The proof proceeds along the lines of the injectivity part of Theorem 3.5: If $\kappa_n(p) \sim \kappa_m(q)$, then $\exists v \in A \otimes \mathcal{K}$ such that

$$v^*v = \kappa_n(p) \text{ and } vv^* = \kappa_m(q)$$

Choose $\ell \in \mathbb{N}$ and $x \in M_\ell(A)$ such that $\kappa_\ell(x)$ is close enough to v so that

$$\|\kappa_\ell(x^*x) - \kappa_n(p)\| < 1/2 \text{ and } \|\kappa_\ell(xx^*) - \kappa_m(q)\| < 1/2$$

Once again, $\exists k \geq \max\{\ell, n, m\}$ and $y \in M_k(A)$ such that

$$\|y^*y - \varphi_{k,n}(p)\| < 1/2 \text{ and } \|yy^* - \varphi_{k,m}(q)\| < 1/2$$

Hence,

$$\varphi_{k,n}(p) \sim \varphi_{k,m}(q)$$

But $\varphi_{k,n}(p) \sim_0 p$ and $\varphi_{k,m}(q) \sim_0 q$ in $\mathcal{P}_\infty(A)$. □

(iii) If $p \in \mathcal{P}(A \otimes \mathcal{K})$ then $\exists q \in \mathcal{P}_\infty(A)$ such that $\rho(q) \sim p$: The proof proceeds along the same lines as the surjectivity part of Theorem 3.5: Since

$$A \otimes \mathcal{K} \cong \lim(M_n(A), \varphi_n)$$

$\exists k \in \mathbb{N}$ and $b \in M_k(A)$ such that

$$\|\kappa_k(b) - p\| < 1/5$$

Let $a = (b + b^*)/2$, then a is self-adjoint and

$$\|\kappa_k(a) - p\| < 1/5$$

As in that proof, this implies that

$$\|\kappa_k(a^2 - a)\| < 1/4$$

Since

$$\|\kappa_k(d)\| = \lim_{m\to\infty} \|\varphi_{m,k}(d)\|$$

$\exists m \geq k$ and $c := \varphi_{m,k}(a)$ such that

$$\|c^2 - c\| < 1/4$$

By Lemma 3.3, $\exists$ a projection $q \in M_m(A)$ such that

$$\|c - q\| < 1/2$$

Hence,

$$\|\kappa_m(q) - p\| < 1$$

so $p \sim_h \kappa_m(q)$

□

For any C*-algebra A,

$$K_0(A) = \{[p] - [q] : p, q \in \mathcal{P}(A \otimes \mathcal{K})\}$$

Proof. Let A be any C*-algebra (not necessarily stable), and consider $\mathcal{P}(A\otimes\mathcal{K})/\sim$, which we make into an Abelian semigroup as follows: Given $p, q \in \mathcal{P}(A\otimes\mathcal{K})$, choose $n, m \in \mathbb{N}$ and $p' \in \mathcal{P}_n(A), q' \in \mathcal{P}_m(A)$ such that

$$p \sim \rho(p') \text{ and } q \sim \rho(q')$$

Let $\ell = n + m$ and set

$$p'' := \begin{pmatrix} p & 0 \\ 0 & 0_m \end{pmatrix} \text{ and } q'' := \begin{pmatrix} 0_n & 0 \\ 0 & q \end{pmatrix}$$

Then $p'' \perp q''$ and

$$p \sim \rho(p'') \text{ and } q \sim \rho(q'')$$

Define

$$[p] + [q] := [p'' + q'']$$

Then (Check!) that this is a well-defined operation, and that $\mathcal{P}(A \otimes \mathcal{K})/\sim$ is an Abelian semi-group under this operation. Furthermore, the map

$$\widehat{\rho} : \mathcal{D}(A)/\sim_0 \to \mathcal{P}(A \otimes \mathcal{K})/\sim$$

is an isomorphism of Abelian semi-groups. Hence,

$$K_0(A) \cong G(\mathcal{P}(A \otimes \mathcal{K})/\sim) = \{[p] - [q] : p, q \in \mathcal{P}(A \otimes \mathcal{K})\}$$

□

7. Chapter 7

8. Chapter 8

8.1.

8.2.

8.3.

8.4.

8.5.

8.6.

8.7.

8.8.

8.9. Let A be a unital C*-algebra.

(i) Let $u \in \mathcal{U}(A)$ and s an isometry, ie. $s^*s = 1$. Let $w := sus^* + (1 - ss^*)$, then

$$\begin{aligned} ww^* &= sus^*su^*s^* + sus^*(1 - ss^*) + (1 - ss^*)su^*s^* + (1 - ss^*)(1 - ss^*) \\ &= suu^*s^* + sus^* - sus^*ss^* + su^*s^* - ss^*su^*s^* + 1 - ss^* - ss^* + ss^*ss^* \\ &= ss^* + sus^* - sus^* + su^*s^* - su^*s^* + 1 + ss^* \\ &= ss^* + 1 - ss^* = 1 \end{aligned}$$

Similarly, $w^*w = 1$. Now set

$$v = \begin{pmatrix} s & 1 - ss^* \\ 0 & s^* \end{pmatrix}$$

then

$$vv^* = \begin{pmatrix} s & 1 - ss^* \\ 0 & s^* \end{pmatrix} \begin{pmatrix} s^* & 0 \\ 1 - ss^* & s \end{pmatrix} = \begin{pmatrix} ss^* + 1 - ss^* & 0 \\ s^* - s^*ss^* & s - ss^*s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Similarly, $v^*v = 1$, so v is a unitary. Furthermore,

$$\begin{aligned} v\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} v^* &= \begin{pmatrix} s & 1 - ss^* \\ 0 & s^* \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} v \\ &= \begin{pmatrix} su & 1 - ss^* \\ 0 & s^* \end{pmatrix} \begin{pmatrix} s^* & 0 \\ 1 - ss^* & s \end{pmatrix} \\ &= \begin{pmatrix} sus^* + 1 - ss^* & s - ss^*s \\ s^* - s^*ss^* & s^*s \end{pmatrix} = \begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Hence, $[u]_1 = [w]_1$

(ii) Define $v_i := s_ius_i^* + (1 - s_is_i^*)$, then each v_i is a unitary by part (i). Since $s_i^*s_j = \delta_{i,j}$, we have

$$\begin{aligned} v_1v_2 &= (s_1us_1^* + 1 - s_1s_1^*)(s_2us_2^* + 1 - s_2s_2^*) \\ &= s_1us_1^* + s_2us_2^* + 1 - s_2s_2^* - s_1s_1^* \end{aligned}$$

By induction, it follows that $u = \prod_{i=1}^n v_i$, so the result follows.

(iii) Let t as in the question, then

$$t^*t = \begin{pmatrix} s_1^* & 0 & \dots & 0 \\ s_2^* & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ s_n^* & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} s_1 & s_2 & \dots & s_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = 1_n$$

Hence, t is an isometry. Now if $u \in \mathcal{U}_n(A)$, then by part (i),

$$w := tut^* + (1_n - tt^*) \in \mathcal{U}_n(A)$$

Observe that if $u = (a_{i,j})$, then

$$w = \begin{pmatrix} s_1 & s_2 & \dots & s_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} s_1^* & 0 & \dots & 0 \\ s_2^* & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ s_n^* & 0 & \dots & 0 \end{pmatrix} + 1_n - tt^*$$

$$= \begin{pmatrix} \sum_{i=1}^n s_i a_{i1} & \sum_{i=1}^n s_i a_{i2} & \dots & \sum_{i=1}^n s_i a_{in} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} s_1^* & 0 & \dots & 0 \\ s_2^* & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ s_n^* & 0 & \dots & 0 \end{pmatrix} + 1_n - tt^*$$

$$= \begin{pmatrix} \sum_{j=1}^n \sum_{i=1}^n s_i a_{i,j} s_j^* & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} + \begin{pmatrix} 1 - \sum_{i=1}^n s_i s_i^* & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

So if

$$v := \sum_{j=1}^n \sum_{i=1}^n s_i a_{i,j} s_j^* + 1 - \sum_{i=1}^n s_i s_i^*$$

Then $w = v \oplus 1_{n-1}$, and

$$v^* = \sum_{k=1}^n \sum_{\ell=1}^n s_\ell a_{k,\ell}^* s_k^* + 1 - \sum_{i=1}^n s_i s_i^*$$

so one can check (hopefully) that v is a unitary using the fact that $s_i^* s_j = \delta_{i,j}$.

(iv) If A is properly infinite, then for each $n \in \mathbb{N}$, there exist (by [Rørdam, Larsen, and Laustsen, Exercise 4.6]) isometries $\{s_1, s_2, \dots, s_n\}$ such that $s_j s_j^* \perp s_i s_i^*$ when $i \neq j$. The result now follows from the previous steps.

8.10.

8.11. Let A be a unital C*-algebra, p a projection in A, and let $u_0 \in \mathcal{U}((1-p)A(1-p))$. ie.

$$u_0 u_0^* = 1 - p = u_0^* u_0$$

Set $u = u_0 + p$

(i) Then $u_0 \in (1-p)A(1-p)$ implies that $u_0 = (1-p)u_0(1-p)$, so $u_0 p = p u_0 = p u_0^* = u_0^* p = 0$. Hence,

$$uu^* = u_0 u_0^* + p = 1 = u^* u$$

(ii) If $u \sim_h 1$, then $[u]_1 = 0$. Conversely, if $[u]_1 = 0$, then there is a natural number $n \in \mathbb{N}$ such that $u \oplus 1_n \sim_h 1_{n+1}$. Let $t \mapsto w_t$ be a path of unitaries in $\mathcal{U}_{n+1}(A)$ such that $w_0 = 1_{n+1}$ and $w_1 = u \oplus 1_n$. Since p is property infinite and full, by

[Rørdam, Larsen, and Laustsen, Exercise 4.9(i)], $\exists v_0 \in M_{1,n+1}(A)$ such that $v_0^*v_0 = p \oplus 1_n$ and $v_0v_0^* \leq p$. Set

$$v = (1-p\ 0\ \ldots\ 0) + v_0 \in M_{1,n+1}(A)$$

and set $z_t := vw_tv^* + (1 - vv^*)$. Then

$$v^*v = ((1-p\,0\ \ldots\ 0)^* + v_0^*)((1-p\,0\ \ldots\ 0)+v_0) = (1-p)\oplus 0_n + v_0^*v_0 = (1-p)\oplus 0_n + p\oplus 1_n = 1_{n+1}$$

and

$$vv^* = ((1-p\ 0\ \ldots\ 0)+v_0)((1-p\ 0\ \ldots\ 0)^* + v_0^*) = (1-p) + v_0v_0^*$$

Thus, $v^*(1-vv^*) = 0 = (1-vv^*)v$, so

$$z_t^*z_t = (vw_t^*v^* + (1-vv^*)(vw_tv^* + (1-vv^*)) = vw_t^*w_tv^* + (1-vv^*) = 1$$

and

$$z_tz_t^* = (vw_tv^* + (1-vv^*))(vw_t^*v^* + (1-vv^*) = vw_tw_t^*v^* + 1 - vv^* = 1$$

Hence, each z_t is a unitary. Furthermore,

$$z_0 = vv^* + (1-vv^*) = 1$$

and (this needs to be checked)

$$z_1 = v(u\oplus 1_n)v^* + (1-vv^*) = u$$

Hence, $u \sim_h 1$ iff $[u]_1 = 0$.

8.12.

8.13. If A is a purely infinite, simple, unital C*-algebra, then we want to show that

$$\omega : \mathcal{U}(A)/\mathcal{U}_0(A) \to K_1(A)$$

is an isomorphism.

(i) ω is surjective: This follows directly from [Rørdam, Larsen, and Laustsen, Exercise 8.9(iv)]

(ii) ω is injective: Suppose $u \in \mathcal{U}(A)$ is such that $[u]_1 = 0$, then by [Rørdam, Larsen, and Laustsen, Exercise 8.12], $\exists$ a non-zero projection $p \in A$ such that $u \sim_h p + u_0$ for some $u_0 \in U((1-p)A(1-p))$. Since $[u]_1 = 0$, we conclude from [Rørdam, Larsen, and Laustsen, Exercise 8.11] that $u \sim_h 1$. Hence, $u \in \mathcal{U}_0(A)$. Thus, ω is injective.

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