K-theory for C*-Algebras

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Contents

1 Review of Last Week

Let A be a unital C*-algebra.

Definition 1.1. Define $\mathcal{P}_n(A)$ to be the set of projections in $M_n(A)$, and write $\mathcal{P}_\infty(A) := \bigcup_{n=1}^\infty \mathcal{P}_n(A)$ (this is an abuse of notation). For $p, q \in \mathcal{P}_\infty(A)$, write $p \sim_0 q$ if $\exists v \in M_{m,n}(A)$ such that $p = v^*v$ and $q = vv^*$. Write

$$\mathbb{D}(A) := \mathcal{P}_\infty(A)/ \sim_0$$

On $\mathbb{D}(A)$, define an operation

$$p \oplus q := \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

Then $\mathbb{D}(A)$ is an Abelian semi-group.

Definition 1.2. Let $(S, +)$ be an Abelian semi-group, then $\exists$ a pair $(G(S), \gamma)$, where

1.1. $G(S)$ is an Abelian group

1.2. $\gamma : S \to G(S)$ is a semi-group homomorphism.

1.3. (Universal Property) If H is an Abelian group and $\eta : S \to H$ is a homomorphism of Abelian semi-groups, then $\exists$ a unique group homomorphism $\widehat{\eta} : G(S) \to H$ such that

$$\widehat{\eta} \circ \gamma = \eta$$

This last property implies that the pair $(G(S), \gamma)$ is unique, and is called the Grothendieck completion of S.

Definition 1.3. Let A be a unital C*-algebra, then

$$K_0(A) := G(\mathbb{D}(A))$$

and $\gamma : \mathbb{D}(A) \to K_0(A)$ is denoted $[p]_{\mathbb{D}} \mapsto [p]_0$.

Theorem 1.4 (Standard picture of K_0 - unital case). *If A is a unital C*-algebra, then*

$$K_0(A) = \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_\infty(A)\}$$

Theorem 1.5 (Universal Property of K_0). *Let G be an Abelian group and $\nu : \mathcal{P}_\infty(A) \to G$ be a function such that*

1.1. $\nu(p \oplus q) = \nu(p) + \nu(q)$

1.2. $\nu(0) = 0$

1.3. If $p \sim_h q$, then $\nu(p) = \nu(q)$

Then there is a unique group homomorphism $\alpha : K_0(A) \to G$ such that

$$\alpha([p]_0) = \nu(p)$$

The proof follows from the universal property of the Grothendieck construction.

If $\varphi : A \to B$ is a unital $*$-homomorphism between unital C*-algebras, then φ induces a $*$-homomorphism $\varphi_n : M_n(A) \to M_n(B)$ given by $(a_{i,j}) \mapsto (\varphi(a_{i,j}))$. Hence, we get a map $\mathcal{P}_\infty(A) \to \mathcal{P}_\infty(B)$ satisfying the above properties, so we get a map

$$K_0(\varphi) : K_0(A) \to K_0(B)$$

Definition 1.6. Let $\varphi, \psi : A \to B$ be two $*$-homomorphisms. We say that $\varphi \sim_h \psi$ if there is a path $t \mapsto \varphi_t$ such that

1.1. Each $\varphi_t : A \to B$ is a $*$-homomorphism

1.2. $\varphi_0 = \varphi$ and $\varphi_1 = \psi$

1.3. For each $a \in A$, the map $t \mapsto \varphi_t(a)$ is a continuous function $[0, 1] \to B$.

Theorem 1.7 (Homotopy Invariance). *If $\varphi, \psi : A \to B$ are two homotopic $*$-homorphisms, then $K_0(\varphi) = K_0(\psi)$.*

2 The Cuntz Algebra

Definition 2.1. Let $n \geq 2$ and $H = \ell^2(\mathbb{N})$. Decompose $\mathbb{N} = T_1 \sqcup T_2 \sqcup T_2 \ldots \sqcup T_n$ where

$$T_i = \{i, i+n, i+2n, \ldots\}$$

Let $P_i : H \to H$ be the natural projection onto $\ell^2(T_i) \subset H$. Then, P_i is an infinite rank projection, so $P_i \sim I_H$. Furthermore,

$$P_1 + P_2 + \ldots P_n = I_H$$

Choose $s_1, s_2, \ldots, s_n \in \mathcal{B}(H)$ such that

$$s_i^* s_i = 1 \text{ and } s_i s_i^* = P_i$$

Then

$$\sum_{i=1}^n s_i s_i^* = 1$$

(Note that these s_i are isometries). Define

$$\mathcal{O}_n := C^*(s_1, s_2, \ldots, s_n)$$

This is called the Cuntz algebra.

Example 2.2.

$$s_1((x_n)) := (x_1, 0, x_2, 0, x_3, 0, \ldots) \text{ and } s_2((x_n)) := (0, x_1, 0, x_2, 0, x_3, \ldots)$$

Then S_i are both isometries, and (check!)

$$s_1^* s_1 = P_{\text{span}\{e_{2n+1}\}} \text{ and } s_2^* s_2 = P_{\text{span}\{e_{2n}\}}$$

So $\mathcal{O}_2 := C^*(s_1, s_2)$.

Note: An element s in a unital C*-algebra is called an *isometry* if $s^*s = 1$.

Theorem 2.3. *2.1. $\mathcal{O}_n$ is a simple C*-algebra (no non-trivial closed two-sided ideals)*

2.2. (Universal Property of $\mathcal{O}_n$) Given a unital C-algebra A and elements $t_1, t_2, \ldots, t_n \in A$ such that*

$$t_j^* t_j = 1 = \sum_{i=1}^n t_i t_i^*$$

$\exists$ a unique $*$-homomorphism $\varphi : \mathcal{O}_n \to A$ such that $\varphi(s_j) = t_j$

Lemma 2.4. *2.1. Let $u \in \mathcal{U}(\mathcal{O}_n)$, then $\exists$ a unique $*$-homomorphism $\varphi_u : \mathcal{O}_n \to \mathcal{O}_n$ such that*

$$\varphi_u(s_j) = us_j$$

Furthermore,

$$u = \sum_{j=1}^n \varphi_u(s_j) s_j^*$$

2.2. *Let $\varphi : \mathcal{O}_n \to \mathcal{O}_n$ be a unital $*$-homomorphism, then $\exists u \in \mathcal{U}(\mathcal{O}_n)$ such that $\varphi = \varphi_u$*

Proof. 2.1. Follows from the universal property with $t_j = us_j$. Furthermore,

$$\sum_{j=1}^{n} \varphi_u(s_j)s_j^* = \sum_{j=1}^{n} us_js_j^* = u$$

2.2. Given φ, consider

$$u := \sum_{j=1}^{n} \varphi(s_j)s_j^*$$

Then

$$uu^* = \sum_{i,j=1}^{n} \varphi(s_i)s_i^*s_j\varphi(s_j)^*$$

But the P_i are orthogonal projections, and $s_i = P_is_i$ so $s_j^*s_i = \delta_{i,j}$. Hence,

$$uu^* = \sum_{i=1}^{n} \varphi(s_i)\varphi(s_i)^* = \varphi(1) = 1$$

Similarly, $u^*u = 1$. Finally,

$$\varphi_u(s_i) = us_i = \sum_{j=1}^{n} \varphi(s_j)s_j^*s_i = \varphi(s_i)s_i^*s_i = \varphi(s_i)$$

By uniqueness of the universal property, $\varphi_u = \varphi$.

□

Lemma 2.5. *Let $\lambda : \mathcal{O}_n \to \mathcal{O}_n$ be given by*

$$\lambda(x) = \sum_{j=1}^{n} s_jxs_j^*$$

Then

2.1. *λ is an endomorphism of $\mathcal{O}_n$*

2.2. *If $u \in \mathcal{U}(\mathcal{O}_n)$ such that $\lambda = \varphi_u$, then $u = u^*$*

Proof. 2.1. $\lambda(1) = 1$ and $\lambda(x^*) = \lambda(x)^*$. By orthogonality of the P_i

$$\lambda(x)\lambda(y) = \sum_{j=1}^{n} s_jxs_j^*s_jys_j^* = \lambda(xy)$$

since $s_j^*s_j = 1$.

2.2. If $u = \sum_{j=1}^n \lambda(s_j)s_j^*$, then $\lambda = \varphi_u$ and

$$u^* = \sum_{j=1}^n s_j\lambda(s_j^*) = \sum_{j=1}^n s_j \left[\sum_{i=1}^n s_i s_j^* s_i\right] = \sum_{j=1}^n s_j s_j s_j^* s_j = \sum_{j=1}^n s_j^2$$

But

$$\lambda(s_i)s_i = \sum_{j=1}^n s_j s_i s_j^* s_i = s_i s_i s_i^* s_i = s_i^2$$

Hence, $u = u^*$.

□

Lemma 2.6. *Let A be a unital C^*-algebra and $s \in A$ an isometry. Define $\mu : A \to A$ by $\mu(a) = sas^*$. Then $K_0(\mu) = id_{K_0(A)}$*

Proof. Note that $\mu_n : M_n(A) \to M_n(A)$ is given by $\mu_n(a) = s_n a s_n^*$ where

$$s_n = \operatorname{diag}(s, s, \dots, s)$$

and s_n is also an isometry. Furthermore, if $p \in \mathcal{P}_n(A)$, then

$$s_n p s_n = (s_n p)(s_n p)^* \sim (s_n p)^*(s_n p) = p$$

Hence, $[\mu_n(p)]_0 = [p]_0$. □

Theorem 2.7. *If $g \in K_0(\mathcal{O}_n)$, then $(n-1)g = 0$. In particular, $K_0(\mathcal{O}_2) = 0$*

Proof. Let $\lambda : \mathcal{O}_n \to \mathcal{O}_n$ as above, then $\lambda = \sum_{i=1}^n \lambda_i$ where

$$\lambda_i(x) = s_i x s_i^*$$

Then $\lambda_i(x)\lambda_j(y) = 0$ for all $x, y \in \mathcal{O}_n$, so

$$K_0(\lambda) = \sum_{i=1}^n K_0(\lambda_i)$$

By the above lemma, it follows that

$$K_0(\lambda)g = ng \quad \forall g \in K_0(\mathcal{O}_n)$$

However, $\lambda = \varphi_u$, where $u = u^*$. In particular, $u \in \mathcal{U}_0(\mathcal{O}_n)$. Let u_t be a path of unitaries from u to 1, then φ_{u_t} is a path of $*$-homomorphism from

$$\lambda = \varphi_u \text{ to } \mathrm{id}_A = \varphi_1$$

Hence, $K_0(\lambda) = \mathrm{id}_{K_0(\mathcal{O}_n)}$. Hence the result. □

In fact, $K_0(\mathcal{O}_n) \cong \mathbb{Z}_{n-1}$, generated by $[1]_0$.

3 The Irrational Rotation Algebra

Definition 3.1. Let $\theta \in \mathbb{R}$ be fixed, and set $\omega := e^{2\pi i\theta}$. Let $H := L^2(\mathbb{T} \times \mathbb{T})$ equipped with a normalized Haar measure. Let $\zeta_0 \in H$ be the unit vector $\zeta_0(z_1, z_2) := 1$. Define $u, v \in \mathcal{B}(H)$ by

$$(u\zeta)(z_1, z_2) := z_1\zeta(z_1, z_2) \text{ and } (v\zeta)(z_1, z_2) := z_2\zeta(\omega z_1, z_2)$$

Then

$$\langle u\zeta, \eta\rangle = \int_{\mathbb{T}^2} z_1\zeta(z_1, z_2)\overline{\eta(z_1, z_2)} = \int_{\mathbb{T}^2} \zeta(z_1, z_2)\overline{\overline{z_1}\eta(z_1, z_2)}$$

Hence,

$$(u^*\eta)(z_1, z_2) = \overline{z_1}\eta(z_1, z_2)$$

Similarly,

$$(v^*\eta)(z_1, z_2) = \overline{z_2}\eta(\omega^{-1}z_1, z_2)$$

Hence, u and v are unitaries. Furthermore,

$$\begin{aligned}
(vu\zeta)(z_1, z_2) &= z_2(u\zeta)(\omega z_1, z_2) = z_2\omega z_1\zeta(\omega z_1, z_2) \\
(uv\zeta)(z_1, z_2) &= z_1(v\zeta)(z_1, z_2) = z_1 z_2\zeta(\omega z_1, z_2) \\
&\Rightarrow vu = \omega uv
\end{aligned}$$

Define

$$A_\theta := C^*(u, v) \subset \mathcal{B}(H)$$

is called the rotation C*-algebra associated to the angle θ.

(End of Day 1)

We will need the following properties:

Theorem 3.2. *3.1. If θ is irrational, then A_θ is simple, and has a unique tracial state. (see below).*

3.2. (Universal property of A_θ): If D is a unital C-algebra and $u', v' \in D$ are two unitaries such that $v'u' = \omega u'v'$, then $\exists$ a unique $*$-homomorphism $\varphi : A_\theta \to D$ such that $\varphi(u) = u'$ and $\varphi(v) = v'$.*

Note: If $\theta \in \mathbb{Z}$, then A_θ is the universal C*-algebra generated by two commuting unitaries. This is $C(\mathbb{T}^2)$. If $\theta \notin \mathbb{Z}$, A_θ is called a non-commutative two torus.

Remark 3.3. If $\theta, \theta' \in \mathbb{R}$ be irrational.

3.1. Suppose $\theta - \theta' \in \mathbb{Z}$, then $e^{2\pi i\theta} = e^{2\pi i\theta'}$, and so

$$A_\theta \cong A_{\theta'}$$

3.2. If $\theta + \theta' \in \mathbb{Z}$, then $e^{2\pi i\theta} = (e^{2\pi i\theta'})^{-1}$. Hence, there is a surjective $*$-homomorphism $\varphi : A_\theta \to A_{\theta'}$ such that

$$\varphi(u) = v' \text{ and } \varphi(v) = u'$$

Since A_θ is simple, it follows that this map is an isomorphism.

We will now (partially) show that if $A_\theta \cong A_{\theta'}$, then one of the above two conditions must hold.

Define B_θ to be those elements in A_θ of the form

$$\sum_{n,m\in\mathbb{Z}} \alpha_{n,m} u^n v^m$$

where only finitely many coefficients $\alpha_{n,m}$ are non-zero. One thinks of these as Laurent polynomials in u and v. Note that B_θ is a $*$-subalgebra of A_θ, and its closure is thus a C*-algebra containing u and v. Thus, B_θ is dense in A_θ and is called the smooth $*$-subalgebra of A_θ.

Remark 3.4. 3.1. A map $\tau : A \to \mathbb{C}$ is called a *trace* if τ is bounded, linear and $\tau(ab) = \tau(ba)$. Such a map induces a trace $M_n(A) \to \mathbb{C}$ by $(a_{i,j}) \mapsto \sum \tau(a_{i,i})$ [Check!].

3.2. This restricts to a map $\tau : \mathcal{P}_\infty)(A) \to \mathbb{C}$ such that $\tau(p \oplus q) = \tau(p) + \tau(q), \tau(0) = 0$, and if $p \sim_h q$, then $p \sim q$, so $\tau(p) = \tau(q)$. So we get a map

$$K_0(\tau) : K_0(A) \to \mathbb{C}$$

3.3. If τ is a positive trace (ie. $\tau(x^*x) \geq 0$ for all $x \in A$), then $\tau(p) \in \mathbb{R}_+$ for all $p \in \mathcal{P}_\infty(A)$, so we get a map

$$K_0(\tau) : K_0(A) \to \mathbb{R}$$

3.4. If τ is a tracial state (ie. τ is positive and $\tau(1_A) = 1$), then $K_0(\tau)([1]_0) = 1$

We will now construct a trace on A_θ.

Definition 3.5. Define $\tau : A_\theta \to \mathbb{C}$ by

$$\tau(a) := \langle a\zeta_0, \zeta_0\rangle$$

Then τ is a positive linear functional on A_θ of norm 1. Furthermore,

$$\tau\left(\sum_{n,m\in\mathbb{Z}} \alpha_{n,m} u^n v^m\right) = \alpha_{0,0}$$

for elements in B_θ. Hence, it follows that if $x \in B_\theta$ of the above form, then

$$\begin{aligned}\tau(x^*x) &= \tau\left[\left(\sum_{n,m\in\mathbb{Z}} \overline{\alpha_{n,m}} v^{-m}u^{-n}\right)\left(\sum_{n,m\in\mathbb{Z}} \alpha_{n,m}u^n v^m\right)\right] \\ &= \sum_{n,m\in\mathbb{Z}} |\alpha_{n,m}|^2 = \tau(xx^*)\end{aligned}$$

Since B_θ is dense in A_θ, it follows that

$$\tau(x^*x) = \tau(xx^*) \quad \forall x \in A_\theta$$

The exercise from last week implies that τ is a tracial state on A_θ.

We now wish to construct a projection $p \in A_\theta$ such that $\tau(p) = \theta$.

Lemma 3.6. *Let $\varphi : \mathbb{T} \to \mathbb{T}$ is the function $z \mapsto \omega z$. Then, for any $h : \mathbb{T} \to \mathbb{C}$ continuous,*

$$vh(u) = (h \circ \varphi)(u)v, \text{ and } v^*(h \circ \varphi)(u) = h(u)v^*$$

Proof. It suffices to prove the first statement. Note that

$$\omega^k u^k v = vu^k \quad \forall k \in \mathbb{Z}$$

Hence, for any $h : \mathbb{T} \to \mathbb{R}$ Laurent polynomial

$$(h \circ \varphi)(u)v = vh(u)$$

Now approximate any continuous $h : \mathbb{T} \to \mathbb{C}$ by Laurent polynomials. □

If $\theta = 0$, then $C(\mathbb{T}^2) = A_\theta$ has no projections because $\mathbb{T}^2$ is connected. We now assume that $\theta \in (0, 1)$ is irrational, and show that, in this case, A_θ has many projections.

Lemma 3.7. *Let $f, g : \mathbb{T} \to \mathbb{R}$ be continuous functions, and define*

$$p := f(u)v^* + g(u) + vf(u) \in A_\theta$$

Then

3.1. $p = p^*$

3.2. $p = p^2$ *if and only if*

- *(i)* $f \cdot (f \circ \varphi) = 0$
- *(ii)* $f \cdot (g + g \circ \varphi^{-1}) = f$
- *(iii)* $g = g^2 + f^2 + (f \circ \varphi)^2$

3.3. Furthermore,

$$\tau(p) = \int_{\mathbb{T}} g(z)dz$$

Proof. 3.1. Clearly, $p = p^*$ since f and g are real-valued.

3.2. One writes out

$$\begin{aligned}
p^2 &= f(u)v^* f(u)v^* + f(u)v^* g(u) + f(u)v^* v f(u) \\
&\quad + g(u)f(u)v^* + g(u)g(u) + g(u)vf(u) \\
&\quad + vf(u)f(u)v^* + vf(u)g(u) + vf(u)vf(u) \\
&= f \cdot (f \circ \varphi^{-1})(u)v^{-2} + f \cdot (g \circ \varphi^{-1})(u)v^{-1} + f^2(u) \\
&\quad + gf(u)v^{-1} + g^2(u) + g \cdot (f \circ \varphi)(u)v \\
&\quad + (f \circ \varphi)^2(u) + (f \circ \varphi) \cdot (g \circ \varphi)(u)v + (f \circ \varphi) \cdot (f \circ \varphi \circ \varphi)(u)v^2
\end{aligned}$$

Note that

$$p = f(u)v^{-1} + g(u) + (f \circ \varphi)(u)v$$

So comparing coefficients, we get

$$\begin{aligned}
f \cdot (f \circ \varphi^{-1}) &= 0 \\
f \cdot (g \circ \varphi^{-1}) + (g \cdot f) &= f \\
f^2 + g^2 + (f \circ \varphi)^2 &= g \\
g \cdot (f \circ \varphi) + (f \circ \varphi) \cdot (g \circ \varphi) &= (f \circ \varphi) \\
(f \circ \varphi) \cdot (f \circ \varphi \circ \varphi) &= 0
\end{aligned}$$

Since φ is a homeomorphism of $\mathbb{T}$, for any function $h : \mathbb{T} \to \mathbb{R}$, we have

$$h = 0 \Leftrightarrow h \circ \varphi = 0 \Leftrightarrow h \circ \varphi^{-1} = 0$$

So the first and fifth conditions collapse to one, and so do the second and fourth. These are the three conditions mentioned above.

3.3. First we assume that f and g are both Laurent polynomials. Then p is a Laurent polynomial, so we may use the expression for τ on Laurent polynomials. Now approximate f and g by Laurent polynomials, and use the fact that both sides of the equation represent continuous maps.

□

Theorem 3.8. *There exists a projection $p \in A_\theta$ such that $\tau(p) = \theta$*

Proof. Choose $\epsilon > 0$ such that $0 < \epsilon \leq \theta < \theta + \epsilon \leq 1$. Define

$$g(t) := \begin{cases} t/\epsilon & : 0 \leq t \leq \epsilon \\ 1 & : \epsilon \leq t \leq \theta \\ \epsilon^{-1}(\theta + \epsilon - t) & : \theta \leq t \leq \theta + \epsilon \\ 0 & : \theta + \epsilon \leq t \leq 1 \end{cases}$$

and

$$f(t) = \begin{cases} \sqrt{g(t) - g(t)^2} & : \theta \leq t \leq \theta + \epsilon \\ 0 & : \text{ otherwise} \end{cases}$$

Then both f and g define functions on $\mathbb{T}$ because $f(0) = f(1) = 0 = g(0) = g(1)$. The corresponding element p as defined above is a projection, and

$$\tau(p) = \int_{\mathbb{T}} g(z)dz = \frac{1}{2} \cdot \epsilon + (\theta - \epsilon) + \frac{1}{2} \cdot \epsilon = \theta$$

□

Theorem 3.9. *The range of the map*

$$K_0(\tau) : K_0(A_\theta) \to \mathbb{R}$$

contains $(\mathbb{Z} + \mathbb{Z}\theta)$.

Proof. Since $\tau(1) = 1$, the range of $K_0(\tau)$ contains $\mathbb{Z}$. If p_θ is the Rieffel projection from the previous theorem, then $\tau(p_\theta) = \theta$, so the range contains $\mathbb{Z}\theta$. □

Theorem 3.10 (Pimsner-Voiculescu)**.** *If $\theta \in \mathbb{R}$ is irrational, then the map $K_0(\tau)$ induces an isomorphism*

$$K_0(A_\theta) \to \mathbb{Z} + \mathbb{Z}\theta$$

Corollary 3.11. *Let θ and θ' be two irrational numbers. Then $A_\theta \cong A_{\theta'}$ if and only if either $\theta - \theta'$ or $\theta + \theta'$ is an integer.*

Proof. If $\varphi : A_\theta \to A_{\theta'}$ is an isomorphism, and τ' is the trace on $A_{\theta'}$, then by uniqueness of the trace, $\tau' \circ \varphi$ must be the trace on A_θ. Hence, if $p_\theta \in A_\theta$ is the Rieffel projection, then

$$K_0(\tau')([\varphi(p_\theta)]_0) = K_0(\tau)[p_\theta]_0 = \tau(p_\theta) = \theta$$

Hence, $\theta \in \mathbb{Z} + \mathbb{Z}\theta'$, so $\exists a_1, b_1 \in \mathbb{Z}$ such that

$$\theta = a_1 + b_1\theta'$$

Similarly, $\theta' = a_2 + b_2\theta$ for some $a_2, b_2 \in \mathbb{Z}$. Hence,

$$\theta = a_1 + b_1a_2 + b_1b_2\theta$$

Since $\theta \notin \mathbb{Q}$, it follows that $b_1b_2 = 1$, so that $b_1 = b_2 = \pm 1$. Hence the result. □

(End of Day 2)

4 The order structure on $K_0(A)$

Definition 4.1. 4.1. A projection $p \in A$ is said to be *infinite* if $\exists$ a projection q such that $p \sim q$ and $q < p$. If p is not infinite, then it is said to be *finite*.

4.2. A unital C*-algebra A is said to be *finite* if 1_A is finite.

4.3. A is said to be *stably finite* if $M_n(A)$ is finite for all $n \in \mathbb{N}$.

4.4. A non-unital C*-algebra is said to be finite if $\widetilde{A}$ is finite.

Lemma 4.2. *If A is a unital C*-algebra, TFAE:*

4.1. A is finite.

4.2. Every isometry is a unitary.

4.3. All projections in A are finite.

Proof. We prove $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$

(i)⇒ (ii) : If s is an isometry, then $1_A = s^*s \sim ss^* \leq 1$. Since A is finite, $ss^* = 1$ and s is a unitary.

(ii)⇒ (iii) : Suppose every isometry is a unitary, and $p, q \in A$ projections such that

$$p \sim q \text{ and } q \leq p$$

Let $v \in A$ such that $v^*v = p$ and $vv^* = q$, and let

$$s := v + (1 - p)$$

Since $pq = qp = q$, we have $v^*(1-p) = 0 = (1-p)v$. Hence,

$$s^*s = v^*v + (1-p) = 1 \text{ and } vv^* = 1 - (p - q)$$

By hypothesis, s is a unitary, so $p - q = 0$.

(iii)⇒ (i) : If every projection is finite, then 1_A is finite.

□

Definition 4.3. A pair (G, G^+) is called an ordered abelian group if G is an Abelian group, $G^+ \subset G$ such that

4.1. $G^+ + G^+ \subset G^+$

4.2. $G^+ \cap (-G^+) = \{0\}$

4.3. $G^+ - G^+ = G$

We define an order relation on G by $x \leq y$ iff $y - x \in G^+$. This makes $(G, \leq)$ a partially ordered set such that

$$x \leq y \Rightarrow x + z \leq y + z \quad \forall z \in G$$

The converse is also true: If G is a partially ordered group satisfying this condition, we may set $G^+ = \{x \in G : x \geq 0\}$, then it satisfies the above requirements.

Definition 4.4. Define

$$K_0(A)^+ := \{[p]_0 : p \in \mathcal{P}_\infty(A)\}$$

Proposition 4.5. *4.1.* $K_0(A)^+ + K_0(A)^+ \subset K_0(A)+$

4.2. If A is unital, $K_0(A)^+ - K_0(A)^+ = K_0(A)$

4.3. If A is stably finite, then $K_0(A)^+ \cap (-K_0(A)^+) = \{0\}$

Hence, if A is unital and stably finite, then $(K_0(A), K_0(A)^+)$ is an ordered Abelian group.

Proof. 4.1. $[p]_0 + [q]_0 = [p \oplus q]_0$

4.2. This is the standard picture of $K_0(A)$ in the unital case.

4.3. Suppose A is stably finite, and $g \in K_0(A)^+ \cap (-K_0(A)^+)$, then write

$$g = [p]_0 = -[q]_0$$

Hence, $[p \oplus q]_0 = 0$, so $\exists r \in \mathcal{P}_\infty(\widetilde{A})$ such that

$$p \oplus q \oplus r \sim_0 r$$

Choose mutually orthogonal projections p', q', r' such that $p \sim_0 p', q \sim_0 q'$ and $r \sim_0 r'$ and think of them in $M_n(\widetilde{A})$ for some $n \in \mathbb{N}$. Now

$$p' + q' + r' \sim r' \text{ in } M_n(\widetilde{A})$$

But $p' + q' + r' \geq r'$ and $M_n(\widetilde{A})$ is finite, so $p' + q' = 0$. Hence, $p' = q' = 0$, so that

$$g = [p]_0 = [p']_0 = 0$$

□

Definition 4.6. Let (G, G^+) be an ordered abelian group. An element $u \in G^+$ is called an order unit if, for each $x \in G, \exists n \in \mathbb{N}$ such that

$$-nu \leq x \leq nu$$

Note: Not every ordered abelian group has an order unit. For example, $C_c(\mathbb{R})$ with the pointwise order.

Proposition 4.7. *If A is unital, then $[1]_0$ is an order unit of $K_0(A)$*

Proof. If $g \in K_0(A)$, write $g = [p]_0 - [q]_0$ for some $p, q \in \mathcal{P}_n(A)$. Then

$$-n[1]_0 = -[1_n]_0 = -[q]_0 + [1_n - q]_0 \leq -[q]_0 \leq [p]_0 - [q]_0 = g$$

and

$$g \leq [p]_0 \leq [p]_0 + [1_n - p]_0 = [1_n]_0 = n[1]_0$$

□

Example 4.8. If $A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_r}(\mathbb{C})$, then

$$K_0(A) \cong \mathbb{Z}^r$$

In fact, since A is stably finite (since it is finite dimensional) and unital, $(K_0(A), K_0(A)^+, [1_A])$ is an ordered abelian group with order unit, given by

$$\begin{aligned} K_0(A) &= \mathbb{Z}[e_{1,1}^{(1)}] + \mathbb{Z}[e_{1,1}^{(2)}] + \ldots + \mathbb{Z}[e_{1,1}^{(r)}] \cong \mathbb{Z}^r \\ K_0(A)^+ &= \mathbb{Z}^+[e_{1,1}^{(1)}] + \mathbb{Z}^+[e_{1,1}^{(2)}] + \ldots + \mathbb{Z}^+[e_{1,1}^{(r)}] \cong (\mathbb{Z}^+)^r \\ [1_A]_0 &= n_1[e_{1,1}^{(1)}]_0 + n_2[e_{1,1}^{(2)}]_0 + \ldots + n_r[e_{1,1}^{(r)}]_0 \end{aligned}$$

Definition 4.9. Let (G, G^+) and (H, H^+) be ordered Abelian groups. A positive group homomorphism is a map $\alpha : G \to H$ such that $\alpha(G^+) \subset H^+$. It is called an order isomorphism if it is an isomorphism such that $\alpha(G^+) = H^+$. If G and H have distinguished order units u and v respectively, α is said to be order unit preserving if $\alpha(u) = v$

Example 4.10. Let $\varphi : A \to B$ be a $*$-homomorphism, then

$$K_0(\varphi)[p]_0 = [\varphi(p)]_0$$

so $K_0(\varphi)$ is a positive homomorphism. Furthermore, if φ is unital, then $K_0(\varphi)$ preserves the order unit.

Example 4.11. Let τ denote the usual trace on $\mathbb{C}$, then $\tau_n : M_n(\mathbb{C}) \to \mathbb{C}$ is a trace. Furthermore,

$$\tau_n(1_n) = n$$

So τ_n induces an isomorphism

$$(K_0(M_n(\mathbb{C})), K_0(M_n(\mathbb{C}))^+, [1_n]) \to (\mathbb{Z}, \mathbb{Z}^+, n)$$

Thus, $(K_0(A), K_0(A)^+, [1_A]_0)$ is a useful invariant to distinguish C*-algebras.

5 Inductive Limits

Let $\mathcal{C}$ be a category.

Definition 5.1. An inductive sequence in $\mathcal{C}$ is a sequence $\{A_n\}$ of objects in $\mathcal{C}$ together with morphisms $\varphi_n : A_n \to A_{n+1}$, usually written as

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \ldots$$

and denoted (A_n, φ_n). For $m > n$, define

$$\varphi_{m,n} = \varphi_{m-1} \circ \varphi_{m-2} \circ \ldots \circ \varphi_n : A_n \to A_m$$

and write $\varphi_{n,n} = \mathrm{id}_{A_n}, \varphi_{m,n} = 0$ if $m < n$. These are called the connecting maps of the sequence.

Definition 5.2. Given a sequence (A_n, φ_n) in $\mathcal{C}$, and inductive limit is a system $(A, \{\mu_n\})$ where A is an object in $\mathcal{C}$ and $\mu_n : A_n \to A$ are morphisms with the following two properties:

5.1. The following diagram commutes for each $n \in \mathbb{N}$

$$\begin{tikzcd} A_n \arrow[rr, "\varphi_n"] \arrow[dr, "\mu_n"'] & & A_{n+1} \arrow[dl, "\mu_{n+1}"] \\ & A & \end{tikzcd}$$

5.2. If $(B, \{\lambda_n\})$ is another system where B is an object in $\mathcal{C}$ and $\lambda_n : A_n \to B$ are morphisms such that $\lambda_n = \lambda_{n+1} \circ \varphi_n$ for all $n \in \mathbb{N}$, then there exists a unique morphism $\lambda : A \to B$ such that the following diagram commutes

$$\begin{tikzcd} & A_n \arrow[dl, "\mu_n"'] \arrow[dr, "\lambda_n"] & \\ A \arrow[rr, "\lambda"'] & & B \end{tikzcd}$$

Remark 5.3. 5.1. Inductive limits do not always exist. For instance, in the category of finite sets. We will show that they exist in the category of C*-algebras, of abelian groups, and of ordered abelian groups.

5.2. If an inductive limit exists, it is unique by the second property above.

Example 5.4. 5.1. Let D be a C*-algebra and $A_n \subset A_{n+1} \subset D$ be an increasing chain of subalgebras. If $\varphi_n = \iota_n : A_n \hookrightarrow A_{n+1}$, then $(A, \{j_n\})$ is an inductive limit of (A_n, ι_n), where

$$A := \overline{\bigcup_{n=1}^{\infty} A_n}$$

and $\mu_n = j_n : A_n \hookrightarrow A$ is the inclusion map because

(i) $\mu_n = \mu_{n+1} \circ \iota_n$ for all $n \in \mathbb{N}$.

(ii) If $(B, \{\lambda_n\})$ is another system as above, then define $\lambda : A \to B$ by

$$\lambda(a) = \lambda_n(a) \text{ if } a \in A_n$$

This is well-defined, because if $a \in A_n \subset A_{n+1}$, then

$$\lambda_{n+1}(a) = \lambda_{n+1}(\iota_n(a)) = \lambda_n(a)$$

Then it follows that $\lambda \circ \mu_n = \lambda_n$ for all $n \in \mathbb{N}$. Furthermore, this map λ is a $*$-homomorphism, and is unique because $\bigcup A_n$ is dense in A.

5.2. Let $A_n = M_n(\mathbb{C})$ and $\varphi_n : A_n \to A_{n+1}$ is the map

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

Let $\mathcal{K}(H)$ denote the compact operators on $H = \ell^2$, then fix an ONB $\{e_i\}$ of H. Define $p_n \in \mathcal{K}(H)$ to be the canonical rank n projection. If $x, y \in H$, define $x \otimes y \in \mathcal{K}(H)$ by

$$(x \otimes y)(z) = \langle z, x\rangle y$$

Then $p_n = \sum_{i=1}^n e_i \otimes e_i$.

(i) Define $\mu_n : M_n(\mathbb{C}) \to \mathcal{K}(H)$ by

$$\mu_n(a_{i,j}) = \sum_{i,j=1}^n a_{i,j} e_i \otimes e_j$$

Then μ_n is injective, and the range of μ_n is $p_n \mathcal{K}(H) p_n$.

Proof. μ_n is injective because the set $\{e_i \otimes e_j\}$ is linearly independent. As for surjectivity onto $p_n\mathcal{K}(H)p_n$, note that if $u \in p_n\mathcal{K}(H)p_n$, then

$$\begin{aligned} u &= p_n u p_n \\ &= \sum_{i,j=1}^n (e_i \otimes e_i) u (e_j \otimes e_j) \\ &= \sum_{i,j=1}^n \langle u(e_i), e_j\rangle e_i \otimes e_j \\ &= \mu_n(a_{i,j}) \end{aligned}$$

where $a_{i,j} = \langle u(e_i), e_j\rangle$. $\square$

(ii) Check that $\mu_{n+1} \circ \varphi_n = \mu_n$

(iii) Finally, observe that

$$\mathcal{K}(H) = \overline{\bigcup_{n=1}^\infty p_n\mathcal{K}(H)p_n} = \overline{\bigcup_{n=1}^\infty \mu_n(M_n(\mathbb{C}))}$$

(iv) As in the previous example, we see that $(\mathcal{K}(H), \{\mu_n\})$ is an inductive limit of $(M_n(\mathbb{C}), \varphi_n)$.

(End of Day 3)

Proposition 5.5 (Inductive Limits of C*-algebras)**.** *Given an inductive system (A_n, φ_n) of C*-algebras, an inductive limit $(A, \{\mu_n\})$ exists.*

Proof. Consider the quotient map

$$\pi : \prod A_n \to \prod A_n / \sum A_n =: Q$$

and let $\varphi_{m,n} : A_n \to A_m$ as above.

5.1. Define $\nu_n : A_n \to \prod_m A_m$ by

$$\nu_n(a) = (\varphi_{m,n}(a))$$

This is well-defined, because $\|\varphi_{m,n}(a)\| \leq \|a\|$ for all $m \in \mathbb{N}$. Furthermore, ν_n is clearly a $*$-homomorphism.

5.2. Let $\mu_n : A_n \to Q$ by $\mu_n = \pi \circ \nu_n$, then observe that if $a \in A_n$, then

$$c := \nu_n(a) - (\nu_{n+1} \circ \varphi_n)(a)$$

has the form $c_n = a$ and $c_m = 0$ when $m \neq n$. Hence, $c \in \sum A_i$, so that

$$\mu_n(a) - (\mu_{n+1} \circ \varphi_n)(a) = \pi(c) = 0$$

Hence, $\mu_n = \mu_{n+1} \circ \varphi$.

5.3. Thus, $\{\mu_n(A_n)\}$ is an increasing sequence of C*-subalgebras of Q. Define

$$A := \overline{\bigcup_{n=1}^{\infty} \mu_n(A_n)}$$

Then A is a C*-algebra, and $\mu_n : A_n \to A$ is a sequence of $*$-homomorphisms satisfying the first condition of Definition 2.2.

5.4. To prove the second condition, suppose $(B, \{\lambda_n\})$ is another system such that $\lambda_n = \lambda_{n+1} \circ \varphi_n$. Then

$$\lambda_m \circ \varphi_{m,n} = \lambda_n \quad \forall m > n$$

Hence, $\|\lambda_n(a)\| \leq \|\varphi_{m,n}(a)\|$. So

$$\|\lambda_n(a)\| \leq \limsup \|\varphi_{m,n}(a)\| = \|\pi(\nu_n(a))\| = \|\mu_n(a)\|$$

Hence, $\ker(\mu_n) \subset \ker(\lambda_n)$. By the first isomorphism theorem, $\exists$ a unique $*$-homomorphism,

$$\lambda_n' : \mu_n(A_n) \to B \text{ such that } \lambda_n' \circ \mu_n = \lambda_n$$

By uniqueness, $\lambda_{n+1}'|_{\mu_n(A_n)} = \lambda_n'$. Hence, we get a $*$-homomorphism

$$\lambda' : \bigcup_{n=1}^{\infty} \mu_n(A_n) \to B$$

which extends λ_n'. λ is a contraction, so it extends to a $*$-homomorphism

$$\lambda : A \to B$$

such that $\lambda \circ \mu_n = \lambda_n' \circ \mu_n = \lambda_n$. Furthermore, λ is unique with this property because

$$A = \overline{\bigcup_{n=1}^{\infty} \mu_n(A_n)}$$

□

Proposition 5.6. *Let (G_n, α_n) be an inductive system of abelian groups, then an inductive limit (G, β_n) exists. Moreover, one has*

5.1.

$$G = \bigcup_{n=1}^{\infty} \beta_n(G_n)$$

5.2.

$$\ker(\beta_n) = \bigcup_{m=n+1}^{\infty} \ker(\alpha_{m,n})$$

5.3. If (H, γ_n) is another system and $\gamma : G \to H$ the unique group homomorphism as in Definition 2.2, then

- *(i) γ is injective iff $\ker(\gamma_n) = \ker(\beta_n)$ for all $n \in \mathbb{N}$*
- *(ii) γ is surjective iff $H = \bigcup_{n=1}^{\infty} \gamma_n(G_n)$*

Proof. The proof is similar to the one above. □

Example 5.7. 5.1. Consider $G_n = \mathbb{Z}$ and $\alpha_n(1) = n + 1$. ie. We may picture the system as

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \ldots$$

Define $\gamma_n : \mathbb{Z} \to \mathbb{Q}$ by

$$\gamma_n(1) = \frac{1}{n!}$$

Then γ_n is a group homomorphism such that $\gamma_n = \gamma_{n+1} \circ \alpha_n$. Hence, $(\mathbb{Q}, \{\gamma_n\})$ is a system that satisfies (i) in Definition 2.2. Let $(G, \{\beta_n\})$ be an inductive limit of this system, then there is a group homomorphism

$$\gamma : G \to \mathbb{Q} \text{ such that } \gamma \circ \alpha_n = \gamma_n$$

Since

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} \gamma_n(G_n)$$

it follows that γ is surjective. Also, since

$$\ker(\beta_n) = \bigcup_{m=n+1}^{\infty} \ker(\alpha_{m,n})$$

and each α_n is injective, it follows that β_n is injective for all n. We see that each γ_n is also injective. Hence,

$$\ker(\gamma_n) = \ker(\beta_n)$$

for all $n \in \mathbb{N}$. Hence, γ is injective as well.

5.2. Let $G_n = \mathbb{Z}$ and $\alpha_n(1) = 2$ for all $n \in \mathbb{N}$. ie. We may picture the system as

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \ldots$$

Define $\gamma_n : \mathbb{Z} \to \mathbb{Q}$ by

$$\gamma_n(1) = \frac{1}{2^n}$$

Then $\gamma_n = \gamma_{n+1} \circ \alpha_n$. Hence, $(\mathbb{Q}, \{\gamma_n\})$ is a system that satisfies the first condition of Definition 2.2. Hence, if $(G, \{\beta_n\})$ is an inductive limit of the system, then there is a group homomorphism

$$\gamma : G \to \mathbb{Q} \text{ such that } \gamma \circ \alpha_n = \gamma_n$$

As in the previous example, we may check that

$$\ker(\beta_n) = \ker(\gamma_n) = \{0\}$$

so that γ is injective. However, γ is not surjective, but does surject onto

$$H = \bigcup_{n=1}^{\infty} \gamma_n(G_n) \cong \left\{\frac{m}{2^n} : m \in \mathbb{Z}, n \geq 0\right\} \cong \mathbb{Z}\left[\frac{1}{2}\right]$$

This is the inductive limit of the system.

Proposition 5.8 (Inductive Limits of ordered Abelian groups)**.** *Let (G_n, α_n) be an inductive system of ordered abelian groups where $\alpha_n : G_n \to G_{n+1}$ are positive group homomorphisms. Let (G, β_n) be an inductive limit of this system, and define*

$$G^+ = \bigcup_{n=1}^{\infty} \beta_n(G_n^+)$$

Then (G, G^+) is an ordered abelian group, β_n is a positive group homomorphism, and $(G, G^+, \{\beta_n\})$ is an inductive limit in the category of ordered abelian groups.

Proof. Omitted. $\square$

Remark 5.9. Given an inductive sequence

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \dots$$

of C*-algebras, let $(A, \{\mu_n\})$ be the limit of the sequence. (ie. the following diagram commutes

$$\begin{tikzcd} A_n \arrow[rr, "\varphi_n"] \arrow[dr, "\mu_n"'] & & A_{n+1} \arrow[dl, "\mu_{n+1}"] \\ & A & \end{tikzcd}$$

and A is universal with this property). Then we get an inductive sequence of Abelian groups

$$K_0(A_1) \xrightarrow{K_0(\varphi_1)} K_0(A_2) \xrightarrow{K_0(\varphi_2)} K_0(A_3) \xrightarrow{K_0(\varphi_3)} \dots$$

Let $(G, \{\beta_n\})$ be the inductive limit of this sequence. ie. the following diagram commutes

$$\begin{tikzcd} K_0(A_n) \arrow[rr, "K_0(\varphi_n)"] \arrow[dr, "\beta_n"'] & & K_0(A_{n+1}) \arrow[dl, "\beta_{n+1}"] \\ & G_0 & \end{tikzcd}$$

Theorem 5.10 (Continuity of K_0)**.** *Given an inductive system (A_n, φ_n) of C*-algebras with inductive limit A, we have*

$$K_0(A) \cong \lim(K_0(A_n), K_0(\varphi_n))$$

In fact, there is a unique group isomorphism $\gamma : G_0 \to K_0(A)$ such that the following diagram commutes

$$\begin{tikzcd} & K_0(A_n) \arrow[dl, "\beta_n"'] \arrow[dr, "K_0(\mu_n)"] & \\ G_0 \arrow[rr, "\gamma"'] & & K_0(A) \end{tikzcd}$$

In particular,

$$K_0(A) = \bigcup_{n=1}^{\infty} K_0(\mu_n)(K_0(A_n))$$

and

$$\ker(K_0(\mu_n)) = \bigcup_{m=n+1}^{\infty} \ker(K_0(\varphi_{m,n}))$$

Proof. Note that the following diagram commutes

$$\begin{tikzcd} K_0(A_n) \arrow[rr, "K_0(\varphi_n)"] \arrow[dr, "K_0(\mu_n)"'] & & K_0(A_{n+1}) \arrow[dl, "K_0(\mu_{n+1})"] \\ & K_0(A) & \end{tikzcd}$$

Hence, by the universal property of the inductive limit, there is a group homomorphism

$$\gamma : G_0 \to K_0(A)$$

such that $\gamma \circ \beta_n = K_0(\mu_n)$. The proof that γ is bijective is long and technical, so we omit it. □

Definition 5.11. Given a C*-algebra A, consider the inductive sequence $A \to M_2(A) \to M_3(A) \to \ldots$ where the connecting maps are given by the inclusion

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

The inductive limit of this sequence is $A \otimes \mathcal{K}$.

Definition 5.12. Let $e \in \mathcal{K}$ be the fixed projection of rank one

$$e((x_n)) := (x_1, 0, 0, \ldots)$$

and $\kappa : A \to A \otimes \mathcal{K}$ be given by $a \mapsto a \otimes e$. Then κ is an injective $*$-homomorphism, called the canonical inclusion of A into $A \otimes \mathcal{K}$

Lemma 5.13. *Let $p \in \mathcal{K}$ be any rank one projection and $\varphi : A \to A \otimes \mathcal{K}$ be given by $a \mapsto a \otimes p$, then $K_0(\varphi) = K_0(\alpha)$*

Proof. Note that $p \sim e$ and $1 - p \sim 1 - e$, so $\exists u \in \mathcal{U}(\mathcal{B}(H))$ such that $e = upu^*$. By the Borel functional calculus, $\exists h \in \mathcal{B}(H)$ self-adjoint such that $u = e^{ih}$. Hence the path $u_t := e^{ith}$ connects u to the identity. Hence, $e = upu^* \sim_h p$. Furthermore, if $\varphi_t : A \to A \otimes \mathcal{K}$ is given by

$$a \mapsto a \otimes u_t p u_t^*$$

Then φ_t is a path of $*$-homomorphisms such that $\varphi_0 = \varphi$ and $\varphi_1 = \alpha$. Hence, $K_0(\alpha) = K_0(\varphi)$. □

Theorem 5.14 (Stability of K_0)**.** *The map $\kappa : A \to A \otimes \mathcal{K}$ induces an isomorphism $K_0(\kappa) : K_0(A) \to K_0(A \otimes \mathcal{K})$*

Proof. Let $\varphi_n : M_n(A) \to M_{n+1}(A)$ and $\mu_n : M_n(A) \to A \otimes \mathcal{K}$ be the maps as above

5.1. $K_0(\kappa)$ is surjective:

$$K_0(A \otimes \mathcal{K}) = \bigcup_{j=1}^{\infty} K_0(\mu_n)(K_0(M_n(A)))$$

so if $g \in K_0(A \otimes \mathcal{K}), \exists n \in \mathbb{N}$ and $g' \in K_0(M_n(A))$ such that

$$g = K_0(\mu_n)(g')$$

But $\varphi_{n,1} : A \to M_n(A)$ is the map λ_n from the theorem proved last week. Hence, $K_0(\varphi_{n,1}) : K_0(A) \to K_0(M_n(A))$ is an isomorphism, so $\exists h \in K_0(A)$ such that $g' = K_0(\varphi_{n,1})(h)$. Hence,

$$g = K_0(\mu_n \circ \varphi_{n,1})(h) = K_0(\kappa)(h)$$

so $K_0(\kappa)$ is surjective.

5.2. $K_0(\kappa)$ is injective: If $h \in K_0(A)$ is such that $K_0(\kappa)(h) = 0$, then

$$K_0(\mu_n)K_0(\varphi_{n,1})(h) = 0 \quad \forall n \in \mathbb{N}$$

But by the earlier remark,

$$\ker(K_0(\mu_n)) = \bigcup_{m=n+1}^{\infty} \ker(K_0(\varphi_{m,n}))$$

hence,

$$K_0(\varphi_{m,n})(K_0(\varphi_{n,1}(h)) = 0 = K_0(\varphi_{m,1})(h) \text{ in } K_0(M_m(A))$$

But $K_0(\varphi_{m,1})$ is an isomorphism, so $h = 0$ as required.

□

Corollary 5.15. *There is an isomorphism $\alpha : K_0(\mathcal{K}) \to \mathbb{Z}$ such that*

$$\alpha([E]_0) = Tr(E)$$

for every projection $E \in \mathcal{K}$. This isomorphism is denoted by $K_0(Tr)$

Proof. Let $\kappa : \mathbb{C} \to \mathbb{C} \otimes \mathcal{K} \cong \mathcal{K}$ be the map as above, and $\alpha_1 : K_0(\mathbb{C}) \to \mathbb{Z}$ the isomorphism such that

$$\alpha_1([1]_0) = 1$$

Define $\alpha = \alpha_1 \circ K_0(\kappa)^{-1} : K_0(\mathcal{K}) \to \mathbb{Z}$. Then α is an isomorphism. Furthermore, $F := \mathcal{K}(1)$ is a one-dimensional projection in $\mathcal{K}$, and

$$\alpha([F]_0) = \alpha_1([1]_0) = 1$$

If $E \in \mathcal{K}$ is any one-dimensional projection, then $E \sim F$ in $\widetilde{\mathcal{K}(H)}$ as in the case of $\mathcal{B}(H)$. Hence,

$$\alpha([E]_0) = 1$$

If E is any arbitrary n-dimensional projection, then E is a sum of orthogonal rank one projections, so

$$\alpha([E]_0) = n = Tr(E)$$

□

Example 5.16. Consider the short exact sequence

$$0 \to \mathcal{K}(H) \xrightarrow{\iota} \mathcal{B}(H) \to \mathcal{Q}(H) \to 0$$

where $H = \ell^2$. Then $K_0(\mathcal{B}(H)) = 0$, and $K_0(\mathcal{K}(H)) \cong \mathbb{Z}$, so the map

$$K_0(\iota) : K_0(\mathcal{K}(H)) \to K_0(\mathcal{B}(H))$$

is not injective. Therefore, the functor K_0 is not exact.

(End of Day 4)

6 Finite Dimensional C*-Algebras

Definition 6.1. Define $e(n,i,j) \in M_n(\mathbb{C})$ to be the matrix whose $(i,j)^{th}$ entry is 1 and other entries are zero. If

$$A = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \ldots M_{n_r}(\mathbb{C})$$

define

$$e_{i,j}^{(k)} := (0,0,\ldots,e(n_k,i,j),0,0,\ldots,0) \in A$$

These are called the matrix units of A, and they satisfy the following identities

6.1. $e_{i,j}^{(k)} e_{j,\ell}^{(k)} = e_{i,\ell}^{(k)}$

6.2. $e_{i,j}^{(k)} e_{m,n}^{\ell} = 0$ if $k \neq \ell$ or if $j \neq m$

6.3. $(e_{i,j}^{(k)})^* = e_{j,i}^{(k)}$

6.4. $A = \text{span}\{e_{i,j}^{(k)} : 1 \leq k \leq r, 1 \leq i,j \leq n_k\}$

Definition 6.2. Let B be a C*-algebra and $\{f_{i,j}^{(k)}\}$ be a set of elements in B satisfying $(i),(ii)$ and (iii) above. Then this is called a system of matrix units in B of type A.

Note: Given a system of matrix units of type A as above, there is a unique $*$-homomorphism $\varphi : A \to B$ such that $\varphi(e_{i,j}^{(k)}) = f_{i,j}^{(k)}$ for all k,i,j. Furthermore, this map is

6.1. injective if all the $f_{i,j}^{(k)}$ are non-zero.

6.2. surjective if $B = \text{span}\{f_{i,j}^{(k)}\}$

Lemma 6.3. *Suppose that $\{f_{i,i}^{(k)} : 1 \leq k \leq r, 1 \leq i \leq n_k\}$ is a set of mutually orthogonal projections in a C*-algebra B such that*

$$f_{1,1}^{(k)} \sim f_{2,2}^{(k)} \sim \ldots \sim f_{n_k,n_k}^{(k)}$$

for $1 \leq k \leq r$. Then there is a system of matrix units $\{f_{i,j}^{(k)}\}$ in V that extends $\{f_{i,i}^{(k)}\}$.

Proof. Choose partial isometries $f_{1,i}^{(k)}$ such that

$$(f_{1,i}^{(k)})^* f_{1,i}^{(k)} = f_{i,i}^{(k)} \text{ and } f_{1,i}^{(k)}(f_{1,i}^{(k)})^* = f_{1,1}^{(k)}$$

and define

$$f_{i,j}^{(k)} = (f_{1,i}^{(k)})^* f_{1,j}^{(k)}$$

Then this system works [Check!] □

Definition 6.4. A C*-subalgebra $D \subset A$ is called a maximal abelian subalgebra (masa) if it is abelian, and it is not properly contained in any other abelian C*-subalgebra of A.

By Zorn's lemma, every Abelian C*-subalgebra is contained in a masa.

Definition 6.5. Let $X \subset A$. Define

$$X' := \{a \in A : ax = xa \quad \forall x \in X\}$$

Note that X' is a norm-closed subalgebra of A. Furthermore, it is a C*-subalgebra if X is self-adjoint (ie. if $a \in X$, then $a^* \in X$)

Note: $B \subset A$ is Abelian iff $B \subset B'$.

Lemma 6.6. *$D \subset A$ is a masa iff $D = D'$*

Proof. Suppose $D = D'$, then D is Abelian, and if E is Abelian and contains D, then

$$D \subset E \subset E' \subset D' = D$$

so $E = D$. Hence D is a masa.

Conversely, suppose D is a masa, then $D \subset D'$ and D' is a C*-subalgebra. WTS: $D' \subset D$. Since D' and D are C*-algebras, it suffices to show that $(D')_{sa} \subset D$. So fix $a \in D'$ self-adjoint, and set

$$X := D \cup \{a\}$$

Since elements in X commute with each other,

$$X \subset X'$$

Since X is self-adjoint, X' is a C*-subalgebra of A, and so

$$C^*(X) \subset X'$$

So if $y \in C^*(X)$ and $x \in X$, then $xy = yx$. Hence,

$$X \subset C^*(X)'$$

Once again, $C^*(X)'$ is a C*-algebra, so

$$C^*(X) \subset C^*(X)'$$

It follows that $C^*(X)$ is Abelian. Since $D \subset X \subset C^*(X)$, and D is a masa, we conclude that

$$D = C^*(X)$$

In particular, $a \in D$ as required. $\square$

Example 6.7. Let $A = M_n(\mathbb{C})$ and D denote the set of all diagonal matrices. Then D is an Abelian C*-subalgebra of A. Furthermore, if $a \in D'$, then

$$ae_{1,1} = e_{1,1}a$$

So

$$e_{1,1}(a(e_1)) = ae_{1,1}(e_1) = a(e_1)$$

Hence, $a(e_1)$ is an eigen-vector of $e_{1,1}$ with eigen-value 1. So $a(e_1) = \lambda_1 e_1$. Thus continuing, we see that a must be diagonal. Hence, $D' = D$, so D is a masa.

Lemma 6.8. *Let D be a masa in a C^*-algebra A.*

6.1. If D is unital, then A is unital and $1_A = 1_D$

6.2. If p is a projection in D such that $pDp = \mathbb{C}p$, then $pAp = \mathbb{C}p$ (Note: A projection with this property is minimal, in the sense that there is no projection $q \in A$ such that $q < p$ other than $q = 0$)

Proof. 6.1. If $a \in A$, then WTS: $a = a1_D$. Let $z := a - a1_D$, then $zd = 0$ for all $d \in D$. Since D is self-adjoint, this implies $(zd^*)^* = dz^* = 0$ for all $d \in D$. Hence,

$$d(z^*z) = 0 = (z^*z)d \quad \forall d \in D$$

Hence, $(z^*z) \in D' = D$ since D is a masa. Hence,

$$(z^*z)(z^*z) = 0 \Rightarrow \|z\|^4 = 0 \Rightarrow z = 0$$

Hence, $a = a1_D$ for all $a \in A$. Hence,

$$1_D a = (a^* 1_D)^* = (a^*)^* = a \quad \forall a \in A$$

So $1_D = 1_A$

6.2. Let $a \in pAp$, then $a = pa = ap$. So if $d \in D$, we have $pd = dp = pdp = \lambda p$ for some $\lambda \in \mathbb{C}$. Hence,

$$ad = apd = \lambda ap = \lambda a = da$$

Hence, $a \in D' = D$, so $a \in D$. In that case, $a \in pDp$. Hence, $pAp \subset pDp = \mathbb{C}p$.

□

Theorem 6.9. *Any finite dimensional C^*-algebra is isomorphic to*

$$M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \ldots \oplus M_{n_r}(\mathbb{C})$$

for some positive integers $r, n_1, n_2, \ldots, n_r \in \mathbb{N}$

Proof. 6.1. Choose a masa $D \subset A$. By Gelfand, $D \cong C_0(X)$ for some space X. Since D is finite dimensional, it follows that X is finite. In particular, X is compact. Hence, D is unital, and so A is unital and $1_A = 1_D$ by the previous lemma.

6.2. Let $X = \{x_1, x_2, \ldots, x_N\}$ and let $p_i \in D$ denote the corresponding characteristic functions

$$p_i(x_j) = \delta_{i,j}$$

Then $\{p_1, p_2, \ldots, p_N\} \subset D$ are projections such that

$$p_1 + p_2 + \ldots + p_N = 1_D \text{ and } p_j D p_j = \mathbb{C}p_j$$

By the previous lemma, $p_j A p_j = \mathbb{C}p_j$ for all $1 \leq j \leq N$

6.3. Fix $1 \leq i, j \leq N$ such that $p_j A p_i \neq 0$. Choose $v \in p_j A p_i$ such that $\|v\| = 1$, then

$$v^* v \in p_i A p_i$$

is a positive element of norm 1. But $p_i A p_i = \mathbb{C} p_i$. Hence,

$$v^* v = p_i$$

Similarly, $vv^* = p_j$. Hence, we conclude

$$p_j A p_i = \{0\} \text{ or } p_i \sim p_j$$

6.4. Now suppose $p_i \sim p_j$ and $a \in p_j A p_i$, then $a = ap_i = (av^*)v$. As $av^* \in p_j A p_j = \mathbb{C} p_j$, so $av^* = \lambda p_j$ for some $\lambda \in \mathbb{C}$. Furthermore, $p_j v = v$, so

$$a = av^* v = \lambda p_j v = \lambda v$$

Hence, $a \in \mathbb{C} v$, so if $p_i \sim p_j$, then

$$p_j A p_i = \mathbb{C} v$$

6.5. Partition the set $\{p_1, p_2, \ldots, p_N\}$ into Murray von-Neumann equivalence classes. Suppose there are r equivalence equivalence classes, and that the k^{th} class has n_k elements

$$\{f_{1,1}^{(k)}, f_{2,2}^{(k)}, \ldots, f_{n_k,n_k}^{(k)}\}$$

By choice of these projections, we have

$$f_{i,i}^{(k)} A f_{j,j}^{(\ell)} = \{0\} \text{ if } k \neq \ell \text{ and } f_{i,j}^{(k)} \sim f_{j,j}^{(k)}$$

By the earlier lemma, we can extend this collection to a system of matrix units $\{f_{i,j}^{(k)}\}$ in A.

6.6. By Step 4,

$$f_{i,i}^{(k)} A f_{j,j}^{(k)} = \mathbb{C} f_{i,j}^{(k)}$$

and by Step 2,

$$1 = \sum_{i,k} f_{i,i}^{(k)}$$

6.7. Hence if $a \in A$, then

$$\begin{aligned} a &= \left(\sum_{i,k} f_{i,i}^{(k)} \right) a \left(\sum_{i,k} f_{i,i}^{(k)} \right) = \sum_{k=1}^{r} \sum_{i,j=1}^{n_k} f_{i,i}^{(k)} a f_{j,j}^{(k)} \\ &= \sum_{k=1}^{r} \sum_{i,j=1}^{n_k} \lambda_{i,j}^{(k)} f_{i,j}^{(k)} \end{aligned}$$

for some scalars $\lambda_{i,j}^{(k)} \in \mathbb{C}$. Hence,

$$A = \text{span}\{f_{i,j}^{(k)}\}$$

Thus the system of matrix units satisfies all conditions (1) - (4). Hence, by the remark following Definition 1.2,

$$A \cong M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \ldots \oplus M_{n_r}(\mathbb{C})$$

$\square$

(End of Day 5)

7 Classification of AF-Algebras

Definition 7.1. An approximately finite dimensional (AF) algebra is an inductive limit of finite dimensional C*-algebras.

Example 7.2. 7.1. Every finite dimensional C*-algebra is AF

7.2. $\mathcal{K}(\ell^2)$ is AF.

7.3. Fix a sequence $\{n_k\}$ of integers such that $n_k \mid n_{k+1}$. Define $\varphi_k : M_{n_k}(\mathbb{C}) \to M_{n_{k+1}}(\mathbb{C})$ to be the unital map

$$a \mapsto \text{diag}(\underbrace{a, a, \ldots, a}_{d_k \text{ times}})$$

where $d_k = n_{k+1}/n_k$. The inductive limit is a unital AF-algebra, called a Uniformly Hyperfinite Algebra (UHF) algebra of type $\mathfrak{N} := \{n_k\}$

7.4. If $n_k = 2^k$ for all $k \in \mathbb{N}$, then the corresponding UHF algebra of type 2^∞ is called the CAR algebra (Canonical Anticommutation relations)

Lemma 7.3. *Every AF-algebra is stably finite. Hence, $(K_0(A), K_0(A)^+)$ is an ordered abelian group.*

Proof. If A is an AF-algebra, then so is $\widetilde{A}$ and $M_k(A)$. Hence it suffices to show that A is finite when A is unital and AF. We show that every isometry $s \in A$ is a unitary. Suppose $s \in A$ is an isometry, then fix $\epsilon = 1/4$. Since A is an AF-algebra, $\exists$ a finite dimensional C*-subalgebra $B \subset A$ and $x \in B$ such that

$$\|s - x\| < \epsilon$$

It follows that

$$|1 - \|x\|| = |\|s\| - \|x\|| \leq \|s - x\| < \epsilon \Rightarrow \|x\| \leq 1 + \epsilon$$

$$\begin{aligned}\|1_A - x^*x\| &= \|s^*s - x^*x\| \\ &\leq \|s^*s - s^*x\| + \|s^*x - x^*x\| \\ &\leq \|s^*\|\|s - x\| + \|s^* - x^*\|\|x\| \\ &\leq \|s - x\| + \|s - x\|(1+\epsilon) \\ &\leq \epsilon + \epsilon(1+\epsilon) = \epsilon^2 + 2\epsilon \leq \epsilon(3+2\epsilon) < 1\end{aligned}$$

Hence, x^*x is invertible. Replacing B by $B + \mathbb{C}1_A$ (which is also finite dimensional), and using spectral permanence, we can conclude that x^*x is invertible in B. Furthermore, if $z = (x^*x)^{-1}$, then

$$z = \sum_{k=0}^{\infty}(1 - x^*x)^k \Rightarrow \|z\| \leq \sum_{k=0}^{\infty}\|1 - x^*x\|^k = \frac{1}{1 - \|1 - x^*x\|} \leq \frac{1}{1 - \epsilon^2 - 2\epsilon}$$

Hence, if $y = zx^*$, then $yx = 1_A$ and

$$\|y\| < \frac{1+\epsilon}{1 - \epsilon^2 - 2\epsilon}$$

Now x is left-invertible in B. Since B is finite dimensional, it follows that x is right invertible in B (and hence A), and the left and right-inverses coincide. Thus, $xy = 1_A$, so

$$\|sy - 1_A\| = \|sy - xy\| \leq \|s - x\|\|y\| < \frac{\epsilon(1+\epsilon)}{1 - \epsilon^2 - 2\epsilon} < 1$$

because $\epsilon(3 + 2\epsilon) < 1$. Hence, sy is invertible, so s is right invertible as required. □

If A is a unital AF-algebras, we consider the triple

$$\mathcal{E}(A) := (K_0(A), K_0(A)^+, [1_A]_0)$$

If there is a unital $*$-isomorphism $\varphi : A \to B$, then we get an isomorphism of invariants

$$K_0(\varphi) : \mathcal{E}(A) \to \mathcal{E}(B)$$

Theorem 7.4 (Elliott)**.** *Let A and B be two unital AF-algebras. Given an isomorphism $\alpha : \mathcal{E}(A) \to \mathcal{E}(B)$, there is a $*$-isomorphism $\varphi : A \to B$ such that $\alpha = K_0(\varphi)$.*

Proof. The outline of the proof is as follows:

7.1. Write both A and B as inductive limits of finite dimensional C*-algebras

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \ldots \to A$$

$$B_1 \xrightarrow{\psi_1} B_2 \xrightarrow{\psi_2} B_3 \xrightarrow{\psi_3} \ldots \to B$$

This gives an inductive sequence of K_0-groups.

7.2. Given an isomorphism $\alpha : \mathcal{E}(A) \to \mathcal{E}(B)$, we construct an intertwining at the level of K_0 groups.

$$\begin{tikzcd}
& K_0(A_{n_1}) \arrow[rr] \arrow[dr, "\beta_1"] & & K_0(A_{n_2}) \arrow[rr] \arrow[dr, "\beta_2"] & & \dots \arrow[r] & K_0(A) \arrow[d, shift left, "\alpha"] \\
K_0(B_{m_1}) \arrow[ur, "\alpha_1"] \arrow[rr] & & K_0(B_{m_2}) \arrow[ur, "\alpha_2"] \arrow[rr] & & K_0(B_{m_3}) \arrow[r] & \dots \arrow[r] & K_0(B) \arrow[u, shift left, "\alpha^{-1}"]
\end{tikzcd} \tag{.1}$$

This requires a lifting property of the groups $K_0(A_j)$ and $K_0(B_j)$ (which are free Abelian groups) as follows: Given an inductive limit

$$\begin{tikzcd}
K_0(A_k) \arrow[rr, "K_0(\mu_k)"] \arrow[dr, "\alpha"'] & & K_0(A) \\
& K_0(B_j) \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

Once can lift the map γ to a map $\beta : K_0(B_j) \to K_0(A_\ell)$ for some $\ell \geq k$ such that TFDC:

$$\begin{tikzcd}
K_0(A_k) \arrow[r, "K_0(\varphi_{\ell,k})"] \arrow[dr, "\alpha"'] & K_0(A_\ell) \arrow[r, "K_0(\mu_\ell)"] & K_0(A) \\
& K_0(B_j) \arrow[u, "\beta"] \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

We will apply this inductively to construct an intertwining of K_0 groups as above (Equation .1)

7.3. Given an intertwining of K_0 groups as above, we would like to construct $*$-homomorphisms $f_i : B_{m_i} \to A_{n_i}$ and $g_i : A_{n_i} \to B_{m_{i+1}}$ such that

$$K_0(f_i) = \alpha_i \text{ and } K_0(g_i) = \beta_i$$

For this, we need an Existence/Uniqueness theorems:

(i) Given finite dimensional C*-algebras A and B, and a morphism $\eta : K_0(A) \to K_0(B)$, we need to find a $*$-homomorphism $f : A \to B$ such that $K_0(f) = \eta$.

(ii) Furthermore, we would like the f_i and g_i to interact as in Equation .2. Hence, we need a Uniqueness theorem as well: Given finite dimensional C*-algebras A and B and two morphisms $f, g : A \to B$. Suppose $K_0(f) = K_0(g)$, then how are f and g related to each other?

7.4. Finally, we construct an intertwining: two subsequences (A_{n_j}) and (B_{m_j}) and maps between them as below

$$\begin{tikzcd}
& A_{n_1} \arrow[rr] \arrow[dr, "g_1"] & & A_{n_2} \arrow[rr] \arrow[dr, "g_2"] & & A_{n_3} \arrow[r] & \dots \arrow[r] & A \\
B_{m_1} \arrow[ur, "f_1"] \arrow[rr] & & B_{m_2} \arrow[ur, "f_2"] \arrow[rr] & & B_{m_3} \arrow[ur, "f_3"] \arrow[r] & \dots \arrow[rr] & & B
\end{tikzcd} \tag{.2}$$

If such an intertwining exists, then there is an isomorphism $\varphi : A \to B$ (by yesterday's tutorial problem). This isomorphism will have the property that $K_0(\varphi) = \alpha$ as well.

$\square$

Example 7.5. Consider the inductive sequence of C*-algebras

$$\mathbb{C} \to M_2(\mathbb{C}) \to M_4(\mathbb{C}) \to \ldots \to M_{2^n}(\mathbb{C}) \xrightarrow{\varphi_n} M_{2^{n+1}}(\mathbb{C}) \to \ldots$$

where $\varphi_n : M_{2^n}(\mathbb{C}) \to M_{2^{n+1}}(\mathbb{C})$ is given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

Let $(A, \{\mu_n\})$ denote the inductive limit of this system. For each $n \in \mathbb{N}$, define a trace $\tau_n : M_{2^n}(\mathbb{C}) \to \mathbb{C}$ by

$$(a_{i,j}) \mapsto \frac{1}{2^n} \sum_{i=1}^{2^n} a_{i,i}$$

Note that $\tau_{n+1} \circ \varphi_n = \tau_n$. By the universal property of the inductive limit, there is a map $\tau : A \to \mathbb{C}$ such that

$$\tau \circ \mu_n = \tau_n \quad \forall n \in \mathbb{N}$$

Since each τ_n is linear, so is τ. Since each τ is bounded (norm-decreasing), it follows that τ is bounded (Why?). Furthermore, for any $a \in \mu_n(A_n), b \in \mu_m(A_m)$, we write $a = \mu_n(a'), b = \mu_m(b')$. If $m > n$, then $\mu_n = \mu_m \circ \mu_{m-1} \circ \ldots \mu_n$, so we may assume $m = n$, then

$$\tau(ab) = \tau_n(a'b') = \tau_n'(b'a') = \tau(ba)$$

Hence, τ is a trace on A. Similarly, one can check that τ is a positive tracial state. We get a map

$$K_0(\tau) : K_0(A) \to \mathbb{R}$$

Note that

$$K_0(A) = \bigcup_{n=1}^{\infty} K_0(\mu_n)(K_0(A_n))$$

Now,

$$K_0(\tau)(K_0(\mu_n))(K_0(A_n)) = K_0(\tau_n)(K_0(A_n)) = \left\{\frac{a}{2^n} : a \in \mathbb{Z}\right\}$$

Hence, the range of $K_0(\tau)$ is

$$\mathbb{Z}\left[\frac{1}{2}\right] = \left\{\frac{a}{2^n} : a \in \mathbb{Z}, n \in \mathbb{N}\right\}$$

Finally, if $g \in K_0(A)$ is such that $K_0(\tau)(g) = 0$, then $\exists n \in \mathbb{N}$ such that $g \in K_0(\mu_n)(K_0(A_n))$. So write

$$g = K_0(\mu_n)(g')$$

for some $g' \in K_0(A_n)$. Then
$$K_0(\tau_n)(g') = 0$$
But $K_0(\tau_n) : K_0(A_n) \to 2^{-n}\mathbb{Z}$ is an isomorphism. Hence, $g' = 0$, so $g = 0$. Hence,

$$K_0(\tau) : K_0(A) \to \mathbb{Z}\left[\frac{1}{2}\right]$$

is an isomorphism. Furthermore, it is clear that $K_0(\tau)$ maps the positive elements of $K_0(A)$ to the set
$$\left\{\frac{a}{2^n} : a \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}\right\}$$
So the ordered triple
$$(K_0(A), K_0(A)^+, [1]_0)$$
is completely determined.

Remark 7.6. Given a UHF algebra A of type $\mathfrak{N} := \{n_k\}$, A has a trace $\tau : A \to \mathbb{C}$. Furthermore,
$$K_0(\tau) : K_0(A) \cong \bigcup_{k=1}^{\infty} n_k^{-1}\mathbb{Z}$$
Furthermore, we can completely determine the triple $\mathcal{E}(A)$ using $K_0(\tau)$.

8 The Higher K-groups

Definition 8.1. Let A be a C*-algebra. The suspension of A is defined as

$$SA := \{f \in C([0,1], A) : f(0) = f(1) = 0\}$$

For $n > 1$, we define inductively,

$$S^n(A) := S(S^{n-1}A)$$

Note that $S^n(A)$ is a C*-algebra by the point-wise operations; and it is non-unital.

Definition 8.2. For $n \geq 1$, define

$$K_n(A) := K_0(S^n(A))$$

Remark 8.3. 8.1. Given a $*$-homomorphism $\varphi : A \to B$, we get a $*$-homomorphism $S\varphi : SA \to SB$ given by
$$(S\varphi)(f)(t) := \varphi(f(t))$$
Hence, we get a map $K_0(S\varphi) : K_1(A) \to K_1(B)$. We denote this map by $K_1(\varphi)$.

8.2. More generally, we see that K_n is a covariant functor.

8.3. If $\varphi, \psi : A \to B$ are two $*$-homomorphisms such that $\varphi \sim_h \psi$, then $S\varphi \sim_h S\psi$. Therefore, K_1 (and more generally, each K_n) is a homotopy invariant functor as well.

8.4. Given a short exact sequence

$$0 \to J \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$$

of C*-algebras, the induced sequence

$$0 \to SJ \xrightarrow{S\varphi} SA \xrightarrow{S\psi} SB \to 0$$

is also exact. Hence, the sequence

$$K_1(J) \to K_1(A) \to K_1(B)$$

is exact at $K_1(A)$. Hence, K_1 (and hence K_n) is half-exact.

8.5. Similarly, each K_n is a split-exact functor.

8.6. Similarly, all the other properties (continuity, stability, etc.) all carry over from K_0 to K_n.

Definition 8.4. Given a short exact sequence

$$0 \to J \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$$

of C*-algebras, define the mapping cone to be

$$C(A, B) := \{(a, f) : a \in A, f \in C([0, 1], B) \text{ such that } f(0) = 0, f(1) = \psi(a)\}$$

Define $j : J \to C(A, B)$ by $a \mapsto (a, 0)$.

Theorem 8.5. *The map $K_0(j) : K_0(J) \to K_0(C(A, B))$ is an isomorphism.*

Proof. 8.1. Let CB denote the cone of B, ie. the C*-algebra

$$CB := \{f \in C([0, 1], B) : f(0) = 0\}$$

and define $\pi : C(A, B) \to CB$ by $(a, f) \mapsto f$. Then the sequence

$$0 \to J \xrightarrow{j} C(A, B) \xrightarrow{\pi} CB \to 0$$

is exact. We thus get a half-exact sequence

$$K_0(J) \xrightarrow{K_0(j)} K_0(C(A, B)) \xrightarrow{K_0(\pi)} K_0(CB)$$

But CB is contractible, to $K_0(\pi)$ is the zero map. Hence, $K_0(j)$ is surjective.

8.2. For injectivity, define

$$Q := \{f \in C([0,1], A) : f(0) \in J\}$$

We now have maps $\delta : J \to Q$ given by $a \mapsto \overline{a}$, the constant function; and define $\gamma : Q \to J$ given by evaluation at 0. We now have a split exact sequence

$$0 \to \ker(\gamma) \to Q \xrightarrow{\gamma} J \to 0$$

We thus obtain a split exact sequence

$$0 \to K_0(\ker(\gamma)) \to K_0(Q) \xrightarrow{K_0(\gamma)} K_0(J) \to 0$$

Now observe that

$$\ker(\gamma) = \{f \in C([0,1], A) : f(0) = 0\} = CA$$

This is once again contractible, so $K_0(\delta) : K_0(J) \to K_0(Q)$ is an isomorphism.

8.3. Now, we have a map $\eta : Q \to C(A, B)$ given by

$$f \mapsto (f(1), \psi \circ f)$$

This is a surjective $*$-homomorphism, and

$$\ker(\eta) = CJ$$

Hence, $\ker(\eta)$ is contractible, so η is induces an injective map

$$K_0(\eta) : K_0(Q) \to K_0(C(A, B))$$

Now observe that the composition

$$K_0(\eta) \circ K_0(\delta) = K_0(j)$$

which is thus injective.

$\square$

Definition 8.6. Consider a short exact sequence

$$0 \to J \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$$

of C*-algebras, and the short exact sequence

$$0 \to SB \xrightarrow{\alpha} C(A, B) \xrightarrow{\beta} A \to 0$$

where $\alpha(f) := (0, f)$ and $\beta(a, f) := a$ (Observe that this is exact). Therefore, we get a map

$$K_0(\alpha) : K_0(SB) \to K_0(C(A, B))$$

Composing with the map $K_0(j)^{-1}$, we get a map

$$\partial : K_1(B) \to K_0(J)$$

This is called the *boundary map* or *index map.*

Theorem 8.7. *Given a short exact sequence*

$$0 \to J \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$$

the sequence

$$K_1(A) \xrightarrow{K_1(\psi)} K_1(B) \xrightarrow{\partial} K_0(J) \xrightarrow{K_0(\varphi)} K_0(A)$$

is exact.

Theorem 8.8. *Given a short exact sequence of C^*-algebras*

$$0 \to J \to A \to B \to 0$$

there is a natural long exact sequence of K-groups given by

$$\ldots \to K_n(J) \to K_n(A) \to K_n(B) \xrightarrow{\partial} K_{n-1}(J) \to K_{n-1}(A) \to K_{n-1}(B) \to \ldots$$

which ends in $K_0(B)$.

9 Exercises for 9/7/19

9.1. Let X and Y be compact Hausdorff spaces and $\alpha, \beta : X \to Y$ be two continuous functions. We say $\alpha \sim_h \beta$ if there is a continuous function

$$k : [0,1] \times X \to Y$$

such that $k(0,x) = \alpha(x)$ and $k(1,x) = \beta(x)$ for all $x \in X$. Define $A := C(Y), B := C(X)$, and

$$\varphi : A \to B \text{ given by } \varphi(f)(x) := f(\alpha(x))$$

and $\psi : A \to B$ by $\psi(f)(x) := g(\beta(x))$. Use k to construct a homotopy from φ to ψ. Check all the conditions.

9.2. Let $\varphi, \psi : A \to B$ be two $*$-homomorphisms such that $\varphi(x)\psi(y) = 0$ for all $x, y \in A$ (If this happens, we say that φ is *orthogonal* to ψ). Show that $\varphi + \psi : A \to B$ is a $*$-homomorphism, and

$$K_0(\varphi + \psi) = K_0(\varphi) + K_0(\psi)$$

9.3. Let p and q be two projections in a C*-algebra A. Write $p \leq q$ if $(q - p)$ is a positive element in A, and write $p \perp q$ if $pq = 0$.

A non-zero projection p in a C*-algebra A is said to be *properly infinite* if there exist mutually orthogonal projections $e, f \in A$ such that $e \leq p, f \leq p$ and $p \sim e \sim f$. A unital C*-algebra is said to be *properly infinite* if 1_A is a properly infinite projection.

Show that the Cuntz algebra $\mathcal{O}_n$ is properly infinite, and show that $\mathcal{B}(H)$ is properly infinite if and only if H is infinite dimensional.

9.4. Let A be a properly infinite unital C*-algebra.

(i) Show that A contains isometries s_1, s_2 such that $s_1 s_1^* \perp s_2 s_2^*$.

(ii) Show that A contains a sequence of isometries $\{t_j\}_{j=1}^\infty$ such that $t_j t_j^* \perp t_i t_i^*$ when $i \neq j$. [Hint: Look at $s_1, s_2 s_1, s_2^2 s_1, \ldots$]

(iii) For each $n \in \mathbb{N}$, let $v_n \in M_{1,n}(A)$ be the row matrix with entries $t_1, t_2, \ldots, t_n$, where $\{t_i\}$ is as in (ii). Show that $v_n^* v_n = 1$, the unit in $M_n(A)$.

(iv) Let $p \in \mathcal{P}_n(A)$ be given, and let v_n be as in (iii). Show that $v_n p v_n^*$ is a projection in A, and that $p \sim_0 v_n p v_n^*$.

(v) Let p, q be projections in A. Put

$$r := t_1 p t_1^* + t_2(1 - q)t_2^* + t_3(1 - t_1 t_1^* - t_2 t_2^*)t_3^*$$

Show that r is a projection in A and that $[r]_0 = [p]_0 - [q]_0$.

(vi) Show that

$$K_0(A) = \{[p]_0 : p \in \mathcal{P}(A)\}$$

9.5. A trace τ on a C*-algebra A is said to be faithful if $\tau(a) > 0$ for all non-zero, positive elements $a \in A$.

Let $\tau : A \to \mathbb{C}$ be a positive trace on A, and let $\tau_n : M_n(A) \to \mathbb{C}$ be given by

$$\tau_n((a_{i,j})) := \sum_{i=1}^{n} \tau(a_{i,i})$$

(i) Let $x = (a_{i,j}) \in M_n(A)$. Show that

$$\tau_n(x^*x) = \sum_{i,j=1}^{n} \tau(a_{i,j}^* a_{i,j})$$

(ii) Show that τ_n is positive.

(iii) If τ is faithful, show that τ_n is faithful.

(iv) If A is a unital C*-algebra which admits a faithful positive trace, then show that A is stably finite. [Hint: For any projection $p \in A, p \leq 1_A$.]

(v) Conclude that the rotation algebra A_θ is stably finite.

9.6. Let $\{A_i\}_{i\in\mathbb{N}}$ be a sequence of C*-algebras. Define $\prod_{i\in\mathbb{N}} A_i$ to be the set of all sequences $(a_i)_{i=1}^{\infty}$ where $a_i \in A_i$ and

$$\|a\| := \sup_{i\in\mathbb{N}} \|a_i\| < \infty$$

Define

$$\mathcal{I} := \{a \in \prod A_i : a_i = 0 \text{ for all but finitely many } i \in \mathbb{N}\}$$

and define

$$\sum_{i\in\mathbb{N}} A_i := \overline{\mathcal{I}}$$

Show that

(i) $\prod A_i$ is a C*-algebra

(ii) $\sum A_i$ is a closed two-sided ideal of $\prod A_i$

9.7. Let

$$\pi : \prod A_i \to \prod A_i / \sum A_i$$

be the quotient map. For $a \in \prod A_i$, show that

(i) $\|\pi(a)\| = \limsup \|a_n\|$

(ii) Conclude that $a \in \sum A_i$ if and only if $\limsup \|a_n\| = 0$.

10 Exercises for 12/7/19

10.1. Let

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \ldots$$

be an inductive sequence of C*-algebras with inductive limit $(A, \{\mu_n\})$.

(i) Suppose that $1 \leq n_1 < n_2 < n_3 \ldots$, and put $\psi_j := \varphi_{n_{j+1},n_j}$. Show that $(A, \{\mu_{n_j}\})$ is the inductive limit of the sequence

$$A_{n_1} \xrightarrow{\psi_1} A_{n_2} \xrightarrow{\psi_2} A_{n_3} \ldots$$

(ii) Put $B_n := A/\ker(\mu_n)$, and let $\pi_n : A_n \to B_n$ be the quotient map. Justify that there are injective $*$-homomorphisms $\psi_n : B_n \to B_{n+1}$ and a $*$-homomorphism $\pi : A \to \lim B_n$ making the diagram

$$\begin{tikzcd} A_1 \arrow[r, "\varphi_1"] \arrow[d, "\pi_1"'] & A_2 \arrow[r, "\varphi_2"] \arrow[d, "\pi_2"'] & A_3 \arrow[r] \arrow[d, "\pi_3"'] & \ldots \arrow[r] & A \arrow[d, "\pi"] \\ B_1 \arrow[r, "\psi_1"] & B_2 \arrow[r, "\psi_2"] & B_3 \arrow[r] & \ldots \arrow[r] & B \end{tikzcd}$$

commutative. Show that π is a $*$-isomorphism.

(iii) Suppose that each $\varphi_n : A_n \to A_{n+1}$ is injective. Show that each $\mu_n : A_n \to A$ is also injective.

(iv) Suppose that A is unital. Show that there exists a natural number $n_0 \in \mathbb{N}$ such that, for all integers $n \geq n_0$, A_n is unital and the maps $\varphi_n : A_n \to A_{n+1}$ and $\mu_n : A_n \to A$ are unit preserving.

10.2. Given an inductive sequence of Abelian groups

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} G_3 \ldots$$

follow the proof given for C*-algebras, and construct an inductive limit for this sequence.

10.3. Let G_1 and G_2 be the inductive limits of the following two sequences of Abelian groups

$$\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \ldots \text{ and } \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \ldots$$

where the homomorphism $n : \mathbb{Z} \to \mathbb{Z}$ is defined by $1 \mapsto n$. Show that $G_1 \cong \mathbb{Q}$ and determine G_2.

10.4. Let

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \ldots \text{ and } B_1 \xrightarrow{\psi_1} B_2 \xrightarrow{\psi_2} B_3 \ldots$$

be two inductive systems of C*-algebras. Suppose there are $*$-homomorphisms $\alpha_n : A_n \to B_n$ and $\beta_n : B_n \to A_{n+1}$ such that the following diagram commutes

$$\begin{tikzcd} A_1 \arrow[rr, "\varphi_1"] \arrow[dr, "\alpha_1"'] & & A_2 \arrow[rr, "\varphi_2"] \arrow[dr, "\alpha_2"'] & & A_3 \arrow[rr] \arrow[dr, "\alpha_3"'] & & \ldots \arrow[r] & \lim A_n \arrow[d, dotted, shift left, "\alpha"] \\ & B_1 \arrow[rr, "\psi_1"'] \arrow[ur, "\beta_1"] & & B_2 \arrow[rr, "\psi_2"'] \arrow[ur, "\beta_2"] & & B_3 \arrow[r] & \ldots \arrow[r] & \lim B_n \arrow[u, dotted, shift left, "\beta"] \end{tikzcd}$$

Show that there are $*$-isomorphisms α and β as shown in the diagram, making the entire diagram commutative. In particular, A and B are isomorphic.

10.5. Consider the inductive sequence of C*-algebras

$$\mathbb{C} \to M_2(\mathbb{C}) \to M_4(\mathbb{C}) \to \ldots \to M_{2^n}(\mathbb{C}) \xrightarrow{\varphi_n} M_{2^{n+1}}(\mathbb{C}) \to \ldots$$

where $\varphi_n : M_{2^n}(\mathbb{C}) \to M_{2^{n+1}}(\mathbb{C})$ is given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

Let $(A, \{\mu_n\})$ denote the inductive limit of this system. For each $n \in \mathbb{N}$, define a trace $\tau_n : M_{2^n}(\mathbb{C}) \to \mathbb{C}$ by

$$(a_{i,j}) \mapsto \frac{1}{2^n} \sum_{i=1}^{2^n} a_{i,i}$$

(i) Show that there is a positive tracial state $\tau : A \to \mathbb{C}$ such that

$$\tau \circ \mu_n = \tau_n \quad \forall n \in \mathbb{N}$$

(ii) Show that the range of the map $K_0(\tau) : K_0(A) \to \mathbb{R}$ is

$$\mathbb{Z}\left[\frac{1}{2}\right] = \left\{\frac{a}{2^n} : a \in \mathbb{Z}, n \in \mathbb{N}\right\}$$

(iii) Show that one cannot find pairwise orthogonal projections $\{p_1, p_2, p_3\} \in A$ such that $p_1 \sim p_2 \sim p_3$ and $p_1 + p_2 + p_3 = 1$.

Note: The algebra A in this problem is denoted by M_{2^∞}, the UHF algebra of type 2^∞.

Bibliography

[Rordam] Rordam, Larsen, Laustsen, *An Introduction to the K-theory of C*-algebras*