Rokhlin Dimension and Equivariant Bundles

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Contents

Preliminaries

Assumptions

Unless stated otherwise,

- All C*-algebras will be unital and separable (denoted $A, B, C, \ldots$)
- All topological spaces will be compact and Hausdorff (denoted $X, Y, Z, \ldots$)
- All groups will be finite (denoted $G, H, \ldots$)

Motivation

A group action of G on A is a group homomorphism

$$\alpha : G \to \mathrm{Aut}(A)$$

Given such an action, one constructs a crossed product C*-algebra

$$A \rtimes_\alpha G$$

Question: Permanence

Suppose A satisfies a property (P), then can we impose conditions on α so that $A \rtimes_\alpha G$ also satisfies property (P)?

Motivation

Examples of (P) include

1. Finite nuclear dimension/decomposition rank
2. Finite stable rank/real rank
3. Being separable, nuclear and satisfying the UCT.
4. Being simple
5. Stability ($A \otimes \mathcal{K} \cong A$)
6. $\mathcal{Z}$-stability ($A \otimes \mathcal{Z} \cong A$)

The motivation comes from the commutative case.

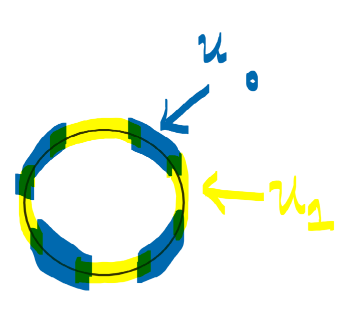
Group Actions on Spaces

Covering Dimension of a Space

Definition

Let $n \in \mathbb{N}$. A finite open cover $\mathcal{U}$ of X is said to be <u>*n*-decomposable</u> if there is a decomposition $\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \ldots \sqcup \mathcal{U}_n$ such that each $\mathcal{U}_i$ consists of mutually disjoint sets.

The following cover of S^1 is 1-decomposable.



Covering Dimension of a Space

One thinks of an n-decomposable cover as a way of covering the space with $(n+1)$ colours, where each colour corresponds to a single $\mathcal{U}_i$.

Definition

The Lebesgue covering dimension of X is the least integer n such that every finite open cover $\mathcal{U}$ of X has a finite refinement $\mathcal{V}$ which is n-decomposable. We denote this number by

$$\dim(X)$$

All spaces in this talk will be assumed to have finite dimension.

Group Actions on Spaces

A group action of G on X is a group homomorphism

$$G \to \mathrm{Homeo}(X)$$

Given such an action, we get an induced action of $\alpha : G \to \mathrm{Aut}(C(X))$ by

$$\alpha_g(f)(x) := f(g^{-1} \cdot x)$$

Furthermore, every action of G on $C(X)$ arises this way.

Definition

An action $G \curvearrowright X$ is said to be free if, for any $x \in X$ and $g \in G$,

$$(g \cdot x = x) \Rightarrow (g = e)$$

Definition

Let $G \curvearrowright X$, $\mathcal{U}$ be a finite open cover of X, and $n \in \mathbb{N}$. We say that $\mathcal{U}$ is *n-decomposable with respect to G* if we can write $\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \ldots \sqcup \mathcal{U}_n$ where, for each $0 \leq i \leq n$, each $\mathcal{U}_i$ consists of $|G|$ mutually disjoint sets

$$\mathcal{U}_i = \{V_i^g : g \in G\}$$

such that

$$g \cdot V_i^h = V_i^{gh}$$

In other words, such a cover of X corresponds to a colouring of X, where each colour respects the action of G.

Free Group Actions on Spaces

Theorem (Gardella, 2017)

An action $G \curvearrowright X$ is free if and only if there exists $n \in \mathbb{N}$ and an open cover $\mathcal{U}$ of X that is n-decomposable with respect to G.

Indeed, we may take $n := \dim(X/G)$.

Rokhlin Dimension

Order Zero Maps

Two elements $a, b \in A$ are said to be orthogonal (in symbols, $a \perp b$) if

$$ab = a^* b = ab^* = ba = 0$$

Definition

A contractive, completely positive (c.c.p.) map $\varphi : A \to B$ is said to have order zero if, for any $a, b \in A$,

$$a \perp b \Rightarrow \varphi(a) \perp \varphi(b)$$

Approximately Decomposable Actions

Definition

Let $n \in \mathbb{N}$. An action $\alpha : G \to \mathrm{Aut}(A)$ is said to be approximately n-decomposable if for every $F \subset A$ finite, $M \subset C(G)$ finite and every $\epsilon > 0$, there are $(n+1)$ c.c.p. order zero linear maps

$$\varphi_0, \varphi_1, \ldots, \varphi_n : C(G) \to A$$

satisfying the following conditions:

Approximately Decomposable Actions

1. Each φ_i is 'approximately equivariant'.

$$\|\alpha_g(\varphi_i(f)) - \varphi_i(\lambda_g(f))\| < \epsilon \quad \forall g \in G, f \in M$$

   where $\lambda_g(f)(s) := f(g^{-1}s)$.

2. Each φ_i is 'approximately central'
   2.1
$$\|\varphi_i(f)a - a\varphi_i(f)\| < \epsilon \quad \forall a \in F \text{ and } f \in M$$
   2.2
$$\|\varphi_i(f_1)\varphi_j(f_2) - \varphi_j(f_2)\varphi_i(f_1)\| < \epsilon \quad \forall f_1, f_2 \in M \text{ and } 0 \leq i,j \leq n$$

3. The $\{\varphi_i\}$ are an 'approximate partition of unity'.

$$\|\sum_{i=0}^{n} \varphi_i(1_{C(G)}) - 1_A\| < \epsilon$$

Rokhlin Dimension

Definition (Hirshberg, Winter, and Zacharias, 2015)

The Rokhlin dimension (with commuting towers) of an action $\alpha : G \to \mathrm{Aut}(A)$ is the least value of $n \in \mathbb{N}$ such that α is approximately n-decomposable. We denote the integer by

$$\dim_{Rok}^{c}(\alpha)$$

Comments on the Definition

1. If $\dim_{Rok}^{c}(\alpha) = 0$, then α has the Rokhlin property .
2. If condition (2.2) is dropped, we get Rokhlin dimension (without commuting towers). This number is denoted by

$$\dim_{Rok}(\alpha).$$

3. Analogous definitions exist for compact groups, and for residually finite discrete groups.

Example 1 (Izumi, 2004)

Let $A := \bigotimes_{n=1}^{\infty} M_2(\mathbb{C})$ be the UHF algebra of type 2^∞ and $\alpha \in \mathrm{Aut}(A)$ be the order two automorphism given by

$$\alpha = \bigotimes_{n=1}^{\infty} \mathrm{Ad} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then α induces an action of $\mathbb{Z}_2$ on A such that

$$\dim_{Rok}^{c}(\alpha) = 0.$$

Example 2 (Hirshberg and Phillips, 2015)

Let $\theta \in \mathbb{R}$ be irrational, and $A = A_\theta$ be the corresponding irrational rotation algebra generated by unitaries $\{u, v\}$ such that

$$uv = e^{2\pi i\theta} vu$$

Let $\alpha \in \mathrm{Aut}(A)$ be the order two automorphism satisfying

$$\alpha(v) = v \text{ and } \alpha(u) = -u$$

Then, α induces an action of $\mathbb{Z}_2$ with the property that

$$\dim^c_{Rok}(\alpha) = 1.$$

Example 3 (Gardella, 2017)

However, if $G \curvearrowright_{\widetilde{\alpha}} X$ is a group action and $\alpha : G \to \mathrm{Aut}(C(X))$ is the induced action, then the following are equivalent:

- $\dim_{Rok}^{c}(\alpha) \leq n$.
- α is n-decomposable in the sense of Definition 1.
- $\widetilde{\alpha}$ is free and $\dim(X/G) \leq n$.

Consequences of Finite Rokhlin Dimension

Theorem (Gardella, Hirshberg, and Santiago, 2021)

Let (P) denote one of the following properties:

1. Finite nuclear dimension/decomposition rank
2. Finite stable rank/real rank
3. Being separable, nuclear and satisfying the UCT.
4. Stability ($A \otimes \mathcal{K} \cong A$)
5. $\mathcal{Z}$-stability ($A \otimes \mathcal{Z} \cong A$)
6. ... etc.

If A satisfies property (P) and $\dim_{Rok}^{c}(\alpha) < \infty$, then $A \rtimes_\alpha G$ also satisfies property (P).

Equivariant Bundles

Vector Bundles

In what follows,

- X will denote a compact metric space with finite covering dimension.
- $p : E \to X$ will be a locally trivial, complex vector bundle, endowed with a fixed hermitian metric.

Vector Bundles

We write

$$\Gamma(E) := \{\xi : X \to E \text{ continuous, such that } p \circ \xi = \mathrm{id}_X\}$$

for the continuous sections of (E, p, X).

Given $\xi \in \Gamma(E)$ and $f \in C(X)$, we may write

$$(f \cdot \xi)(x) := f(x)\xi(x) = (\xi \cdot f)(x)$$

This gives a central action of $C(X)$ on $\Gamma(E)$, so $\Gamma(E)$ is a $C(X)$-module. Moreover, the hermitian metric on E gives $\Gamma(E)$ the structure of a Hilbert $C(X)$-bimodule .

Group Actions on Bundles

An action of a group G on a vector bundle (E, p, X) is a pair

$$\widetilde{\alpha} : G \to \text{Homeo}(X) \text{ and } \widetilde{\gamma} : G \to \text{Homeo}(E)$$

such that

- $p : E \to X$ is G-equivariant.
- For each $s \in G$, the map $E_x \to E_{\widetilde{\alpha}_s(x)}$ is a linear map of vector spaces that preserves the inner product.

The Cuntz-Pimsner Algebra of a Vector Bundle

Using the Hilbert $C(X)$-bimodule $\Gamma(E)$, one can associate a C*-algebra,

$$\mathcal{O}_E$$

called the Cuntz-Pimsner algebra associated to the vector bundle (E, p, X).

Theorem

Given a group action $(\widetilde{\alpha}, \widetilde{\gamma})$ of G on (E, p, X), there is an induced action

$$\beta : G \to \mathrm{Aut}(\mathcal{O}_E)$$

satisfying certain natural properties.

Main Result

Theorem (Vaidyanathan, 2022)

Let $(\widetilde{\alpha}, \widetilde{\gamma})$ be an action of G on (E, p, X) and let

$$\beta : G \to \mathrm{Aut}(\mathcal{O}_E)$$

be the induced action on the corresponding Cuntz-Pimsner algebra.

- If $\widetilde{\alpha}$ is free, then

$$\dim_{Rok}^{c}(\beta) \leq \dim(X/G).$$

- If $\widetilde{\alpha}$ is trivial and the action on each fiber is faithful, then

$$\dim_{Rok}(\beta) \leq 2\dim(X) + 1.$$

Consequences

Suppose that $\dim(X) < \infty$ and the action $\widetilde{\alpha}$ is free, then

- $\mathcal{O}_E \rtimes_\beta G$ has finite nuclear dimension.
- $\mathcal{O}_E \rtimes_\beta G$ has finite stable rank, real rank, etc.
- $\mathcal{O}_E \rtimes_\beta G$ absorbs $\mathcal{Z}$ tensorially.
- $\mathcal{O}_E \rtimes_\beta G$ satisfies the UCT.

Some comments on the proof

The main technical tool in the following theorem

Theorem (Vaidyanathan, 2022)

Let A be a nuclear $C(X)$-algebra and $\alpha : G \to \mathrm{Aut}(A)$ be an action where G acts by $C(X)$-linear automorphisms. Then

$$\dim_{Rok}(\alpha) + 1 \leq (\dim(X) + 1)(\sup_{x \in X} \dim(A(x)) + 1)$$

These theorems works for compact, second countable groups and non-unital C*-algebras as well. The assumption that X is compact, metrizable and finite dimensional is needed in the proof.

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Thank you!