Korovkin's Theorem

Prahlad Vaidyanathan

Department of Mathematics
IISER Bhopal

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Notation

Define

$$C[0,1] := \{f : [0,1] \to \mathbb{C} : f \text{ is continuous}\}.$$

For $f, g \in C[0,1]$, write

$$d(f,g) := \sup_{x\in[0,1]} |f(x) - g(x)|.$$

This defines a metric on $C[0,1]$.

We write $f_n \to f$ in $C[0,1]$ if

$$\lim_{n\to\infty} d(f_n, f) = 0.$$

Positive Operators

An element $f \in C[0,1]$ is *positive* if $f(x) \geq 0$ for all $x \in [0,1]$. If this happens, we write

$$f \geq 0.$$

A linear operator $T : C[0,1] \to C[0,1]$ is *positive* if

$$f \geq 0 \Rightarrow T(f) \geq 0.$$

Examples of Positive Operators

- Define $T : C[0,1] \to C[0,1]$ by

$$T(f)(x) := \int_0^x f(t)dt.$$

- Define $T : C[0,1] \to C[0,1]$ by

$$T(f)(x) := f(0)x + f(1)(1-x)$$

Korovkin's Theorem

Theorem (Korovkin, 1953)

Let $T_n : C[0,1] \to C[0,1]$ be a sequence of positive operators such that

$$T_n(g) \to g \text{ for } g \in \{\mathbf{1}, e_1, e_2\}$$

where $e_i(t) := t^i$. Then

$$T_n(f) \to f$$

for all $f \in C[0,1]$.

For a proof, see [Limaye, 1996].

Example: Weierstrass' Approximation Theorem

Define $B_n : C[0,1] \to C[0,1]$ by

$$B_n(f)(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

Then B_n is linear, and positive. Also,

$$\begin{aligned} B_n(\mathbf{1})(x) &= (x + (1-x))^n = \mathbf{1}(x) \\ B_n(e_1)(x) &= e_1(x) \\ B_n(e_2)(x) &= \frac{n-1}{n} e_2(x) + \frac{1}{n} e_1(x) \end{aligned}$$

Hence, $B_n(g) \to g$ for $g \in \{\mathbf{1}, e_1, e_2\}$, so by Korovkin's theorem,

$$B_n(f) \to f \quad \forall f \in C[0,1].$$

Generalizing Korovkin's Theorem

Let X be a compact Hausdorff space and

$$C(X) := \{f : X \to \mathbb{C} : f \text{ is continuous}\}$$

equipped with the supremum metric

$$d(f, g) := \sup_{x \in X} |f(x) - g(x)|$$

A function $f \in C(X)$ is *positive* if

$$f(x) \geq 0 \quad \forall x \in X.$$

A linear operator $T : C(X) \to C(X)$ is *positive* if

$$f \geq 0 \Rightarrow T(f) \geq 0.$$

Generalizing Korovkin's Theorem

Definition: Korovkin Set

A subset $G \subset C(X)$ is called a *Korovkin set* if, for any sequence $T_n : C(X) \to C(X)$ of positive operators, if

$$T_n(g) \to g \text{ for all } g \in G$$

then

$$T_n(f) \to f \text{ for all } f \in C(X).$$

Hence, if $X = [0, 1]$, then

$$G = \{\mathbf{1}, e_1, e_2\}$$

is a Korovkin set.

Generalizing Korovkin's Theorem

Question A

Given $G \subset C(X)$, how do we determine if G is a Korovkin set?

Given $G \subset C(X)$, the *function system* generated by G is the set

$$A_G := \overline{\text{span}}(G \cup G^*)$$

where $G^* := \{\overline{f} : f \in G\}$.

The answer to Question A may be framed in terms of A_G.

The Choquet Boundary

Representing Measures

Let $A \subset C(X)$ be a vector subspace and a fix a point $x_0 \in X$.

Definition: Representing measures

Let μ be a (Borel, regular) probability measure on X. We say that μ *represents* x_0 *with respect to* A if

$$f(x_0) = \int_X f d\mu$$

for all $f \in A$.

We write $\operatorname{Rep}(x_0, A)$ for the set of all such measures.

Examples of Representing Measures

For any $A \subset C(X)$ and any $x_0 \in X$,

$$\delta_{x_0} \in \mathrm{Rep}(x_0, A)$$

because, by definition

$$\int_X f\,\delta_{x_0} = f(x_0) \quad \forall f \in C(X).$$

Hence,

$$\delta_{x_0} \in \mathrm{Rep}(x_0, A).$$

Question

Can $\mathrm{Rep}(x_0, A)$ contain any other measures?

Examples of Representing Measures

Let $\mathbb{D} \subset \mathbb{C}$ denote the open unit disc, $X := \overline{\mathbb{D}}$ and

$$A := \{f \in C(X) : f \text{ is analytic on } \mathbb{D}\}.$$

If $f \in A$ and $z := re^{it} \in \mathbb{D}$, then $0 \leq r < 1$, and so the Poisson integral formula holds

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})P_r(\theta, t)d\theta$$

This gives a measure $\mu \in \text{Rep}(z, A)$. Note that

$$\mu \neq \delta_z$$

because μ is supported on S^1, the unit circle.

The Choquet Boundary

Let $A \subset C(X)$ be a vector subspace.

Definition

A point $x_0 \in X$ is called a *boundary point* of A if

$$\operatorname{Rep}(x_0, A) = \{\delta_{x_0}\}.$$

The set of all boundary points of A is called the *Choquet Boundary* of A and is denoted by

$$\partial_X(A).$$

Examples of Choquet Boundary

If A is dense in $C(X)$ and $x_0 \in X$, suppose $\mu \in \text{Rep}(x_0, A)$. Then

$$\int_X f d\mu = f(x_0) = \int_X f \delta_{x_0}$$

for all $f \in A$. By continuity, this equation holds for all $f \in C(X)$. Therefore,

$$\mu = \delta_{x_0}.$$

Therefore,

$$\partial_X(A) = X.$$

Examples of Choquet Boundary

Let $\mathbb{D} \subset \mathbb{C}$ denote the open unit disc, $X := \overline{\mathbb{D}}$ and

$$A := \{f \in C(X) : f \text{ is analytic on } \mathbb{D}\}.$$

If $z \in \mathbb{D}$, then by the Poisson integral formula,

$$\text{Rep}(z, A) \neq \{\delta_z\}.$$

Therefore, $z \notin \partial_X(A)$, so

$$\partial_X(A) \subset S^1.$$

Indeed, one can show that $\partial_X(A) = S^1$.

Saskin's Theorem

If $G \subset C(X)$ is a set, write

$$A_G := \overline{\text{span}}(G \cup G^*)$$

where $G^* := \{\overline{f} : f \in G\}$.

Theorem (Saskin, 1967)

A subset $G \subset C(X)$ is a Korovkin set if and only if

$$\partial_X(A_G) = X.$$

For a modern proof, see [Davidson and Kennedy, 2016].

Example: Korovkin's Theorem

Let $X = [0, 1]$ and $G = \{\mathbf{1}, e_1, e_2\}$ where $e_i(t) := t^i$. We wish to prove that

$$\text{Rep}(x, A_G) = \{\delta_x\}$$

for each $x \in [0, 1]$. So fix $x \in [0, 1]$ and suppose $\mu \in \text{Rep}(x, A)$ is such that

$$\mu \neq \delta_x.$$

Then, there exists an open set U not containing x such that

$$\mu(U) > 0.$$

If $f(t) := (t - x)^2$, then $f \in A_G$ and

$$0 = f(x) = \int_X f d\mu \geq \int_U f d\mu > 0.$$

This contradiction proves that $\text{Rep}(x, A) = \{\delta_x\}$.

Example: Fejer's Theorem

Let $X = S^1$ and $G = \{\mathbf{1}, \mathrm{Re}, \mathrm{Im}\}$, then G is a Korovkin set. For a proof, see [Bauer, 1981].

Theorem (Fejer, 1904)

Let $f \in C[-\pi, \pi]$ such that $f(\pi) = f(-\pi)$. Define

$$\widehat{f}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$

$$s_n(f)(x) := \sum_{k=-n}^{n} \widehat{f}(k) e^{ikx}$$

$$\sigma_m(f)(x) := \frac{1}{m} \sum_{n=0}^{m-1} s_n(f)(x).$$

Then,

$$\lim_{m \to \infty} \sigma_m(f) = f$$

Further Avenues of Research

Question 1

Question

If X is a compact Hausdorff space, then

- When does $C(X)$ have a finite Korovkin set?
- For a given X, what is the minimum cardinality of G possible?

Hint: It is related to the Lebesgue covering dimension of X.

Operators on a Hilbert Space

If H is a Hilbert space, then write

$$\mathcal{B}(H) := \{T : H \to H : T \text{ is continuous}\}$$

- $\mathcal{B}(H)$ is a Banach space with the norm is defined by

$$\|T\| := \sup\{\|T(x)\| : x \in H, \|x\| = 1\}.$$

- If $T \in \mathcal{B}(H)$, the *adjoint* of T is defined by

$$\langle T^*(x), y\rangle = \langle x, T(y)\rangle.$$

- If $S, T \in \mathcal{B}(H)$, define $ST := S \circ T$.
- An operator $T \in \mathcal{B}(H)$ is *positive* if

$$\langle T(x), x\rangle \geq 0 \quad \forall x \in H.$$

C*-algebras

Definition

A *C*-algebra* is a norm-closed subalgebra A of $\mathcal{B}(H)$ such that

$$T \in A \Rightarrow T^* \in A.$$

Therefore, we may speak of positive elements and positive operators in the context of C*-algebras.

Question 2

Question

- Can we formulate a definition for 'Korovkin set' in the context of C*-algebras?
- Can we prove a version of Saskin's theorem here?

This question leads to a deep unresolved conjecture called *Arverson's Hyperrigidity Conjecture*. For the details, see [Arveson, 2011] and [Davidson and Kennedy, 2016].

Summary

- Korovkin's theorem is a useful approximation theorem for spaces of continuous functions.
- It may be interpreted using measure theory in the form of Saskin's theorem.
- It may be generalized to C*-algebras, where analogous questions remain open even today.
- There are Korovkin-type approximation theorems for other function spaces ($L^p[0,1], C_b(\mathbb{R}^d)$, etc.). For more on this, see [Altomare, 2010].

References

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Thank you!