SIMPLICITY OF CROSSED PRODUCT C*-ALGEBRAS

Abstract. Given a C*-dynamical system $(A, \mathbb{Z}, \alpha)$, we try to determine when the crossed product $A \rtimes \mathbb{Z}$ is a simple C*-algebra. In doing so, we arrive at the Connes Spectrum of such an action.

Let $G = \mathbb{Z}$ and A be a unital C*-algebra, and let $\alpha : G \to \text{Aut}(A)$ be an action of G on A. We wish to understand when the crossed product C*-algebra $A \rtimes G$ is simple.

Remark 1.

(1) We think of $A \subset A \rtimes G$. So, if $I \lhd A$ is a proper G-invariant ideal, then we have a short exact sequence

$$0 \to I \rtimes G \to A \rtimes G \to A/I \rtimes G \to 0.$$

So, $J := I \rtimes G$ is a proper ideal in $A \rtimes G$. Hence, if $A \rtimes G$ is simple, then A is G-simple (no G-invariant ideals).

(2) The converse is not true. If A is *any* unital C*-algebra and $u \in U(A)$ is a unitary, define $\alpha \in \text{Aut}(A)$ by $\alpha(a) := uau^*$. Then, we claim that

$$A \rtimes G \cong A \otimes C(\mathbb{T}).$$

Indeed, define $\iota : G \to \text{Aut}(A)$ be the trivial action, and let $\varphi : C_c(G, A) \to C_c(G, A)$ by

$$\varphi(f)(t) := f(t)u^t.$$

Then,

$$\begin{aligned}\varphi(f *_\alpha g)(t) &= (f *_\alpha g)(t)u^t \\ &= \sum_{x \in G} f(x)g(t-x)u^x u^{t-x} \\ &= \varphi(f) *_\iota \varphi(g)(t)\end{aligned}$$

So we get a $*$-homomorphism $\varphi : A \rtimes_\alpha G \to A \rtimes_\iota G$. This has an inverse given on $C_c(G, A)$ by $\psi(f)(t) := f(t)u^{-t}$. Hence, φ is an isomorphism, so

$$A \rtimes_\alpha G \cong A \rtimes_\iota G \cong A \otimes C^*(G) \cong A \otimes C(\mathbb{T}).$$

Thus, $A \rtimes_\alpha G$ is not simple (even if A was simple).

Question: If A is G-simple, when can we conclude that $A \rtimes G$ is simple?

Definition 2.

(1) Think of $\alpha = \alpha(1) \in \mathcal{B}(A)$ as a bounded operator, and let $\sigma(\alpha)$ denote its spectrum in $\mathcal{B}(A)$. Note that $\|\alpha\| \leq 1$, so $\sigma(\alpha) \subset \mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$. By the same argument, $\sigma(\alpha^{-1}) \subset \mathbb{D}$, so since $\sigma(\alpha^{-1}) = \sigma(\alpha)^{-1}$,

$$\sigma(\alpha) \subset \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}.$$

(2) Define $S : \ell^1(G) \to \mathcal{B}(A)$ by

$$S(f) := \sum_{t \in G} f(t)\alpha_t.$$

Note that $\|S(f)\| \leq \|f\|_1$, so S defines a bounded operator $S : \ell^1(G) \to \mathcal{B}(A)$. Moreover,

$$\begin{aligned} S(f)S(g) &= \left(\sum_{t\in G} f(t)\alpha_t\right)\left(\sum_{s\in G} g(s)\alpha_s\right) \\ &= \sum_{s,t\in G} f(t)g(s)\alpha_{t+s} \\ &= \sum_{u\in G}\sum_{x\in G} f(x)g(u-x)\alpha_u \\ &= S(f*g). \end{aligned}$$

Hence, S is a homomorphism of Banach algebras.

Lemma 3. *If S is injective, then $\sigma(\alpha) = \mathbb{T}$.*

Proof.

(1) Suppose S is injective and $z \in \mathbb{T}$. Define $\tau : \ell^1(G) \to \mathbb{C}$ by

$$\tau(f) := \widehat{f}(z) = \sum_{t\in G} f(t)z^t.$$

Then, τ is a multiplicative linear functional on $\ell^1(G)$. Since S is injective, we get an induced homomorphism $\overline{\tau} : S(\ell^1(G)) \to \mathbb{C}$ such that

$$\overline{\tau} \circ S = \tau.$$

Moreover, since τ is non-zero, $\overline{\tau}$ is also non-zero.

(2) Let $C := \overline{S(\ell^1(G))}$, then C is a unital commutative Banach subalgebra of $\mathcal{B}(A)$ (with unit $S(\delta_0)$). Moreover,

$$\alpha = S(\delta_1) \in C.$$

Hence, the spectrum $\sigma_C(\alpha)$ is given by

$$\sigma_C(\alpha) = \{\eta(\alpha) : \eta \in \Omega(C)\}.$$

In particular,

$$\overline{\tau}(\alpha) = \overline{\tau}(S(\delta_1)) = \tau(\delta_1) = z \in \sigma_C(\alpha).$$

(3) As before, $\sigma_C(\alpha) \subset \mathbb{T}$. In particular, by the Spectral permanence theorem,

$$\sigma_C(\alpha) = \partial\sigma_C(\alpha) \subset \sigma_{\mathcal{B}(A)}(\alpha).$$

Hence, $z \in \sigma(\alpha)$, so $\sigma(\alpha) = \mathbb{T}$.

□

Lemma 4. *There is a G-invariant state on A. (i.e. a state $\tau : A \to \mathbb{C}$ such that $\tau(\alpha_t(a)) = \tau(a)$ for all $a \in A$ and $t \in G$).*

Proof. Let ψ be any state on A. For each $a \in A$, define $f_a \in \ell^\infty(G)$ by

$$f_a(t) := \psi(\alpha_t(a)).$$

Then,

- If $a, b \in A$, then $f_{a+b} = f_a + f_b$.
- f_{1_A} is the constant function 1.
- If $s \in G$, then

$$f_{\alpha_s(a)}(t) = \varphi(\alpha_{ts}(a)) = f_a(ts) = \sigma_s(f_a)(t)$$

where $\sigma_s : \ell^\infty(G) \to \ell^\infty(G)$ is the map $\sigma_s(g)(t) := g(ts)$.

Let $m \in \ell^\infty(G)^*$ be a G-invariant state (which exists because G is amenable), and define $\tau : A \to \mathbb{C}$ by

$$\tau(a) := m(f_a).$$

Then, τ is linear, $\tau(1_A) = 1$ and for any $s \in G$,

$$\tau(\alpha_s(a)) = m(f_{\alpha_s(a)}) = m(\sigma_s(f_a)) = m(f_a) = \tau(a).$$

Hence, τ is a G-invariant state. □

Remark 5. Let τ be a G-invariant state on A. Let $N := \{a \in A : \tau(a^*a) = 0\}$. Then, N is a G-invariant left-ideal. Let

$$K := A/N$$

Then, K carries an action of A given by $\pi(a)(b+N) := ab+N$ and an action of G given by unitaries

$$u_t(b+N) := \alpha_t(b) + N.$$

Let H denote the completion of K, and we get a triple (π, u, H). Also,

$$\begin{aligned} u_t\pi(a)u_t^*(b+N) &= u_t\pi(a)(\alpha_{t^{-1}}(b) + N) \\ &= u_t(a\alpha_{t^{-1}}(b) + N) \\ &= \alpha_t(a\alpha_{t^{-1}}(b)) + N \\ &= \alpha_t(a)b + N \\ &= \pi(\alpha_t(a))(b+N). \end{aligned}$$

Hence,

$$u_t\pi(a)u_t^* = \pi(\alpha_t(a)).$$

Hence, (π, u, H) is a covariant representation of (A, G, α). The induced $*$-homomorphism $\pi \times u : A \rtimes G \to \mathcal{B}(H)$ is given on $\ell^1(G, A)$ by

$$(\pi \times u)(f) = \sum_{t\in G} f(t)u_t$$

Theorem 6. *If $A \rtimes G$ is simple, then $\sigma(\alpha) = \mathbb{T}$.*

Proof. If $\sigma(\alpha) \neq \mathbb{T}$, then S is not injective. So choose a non-zero $f \in \ell^1(G)$ such that $S(f) = 0$. Since A is unital, we think of $f \in \ell^1(G, A)$. Then, for any $b + N \in K$,

$$(\pi \times u)(f)(b+N) = \sum_{t\in G} f(t)u_t(b+N) = \sum_{t\in G} f(t)\alpha_t(b) + N = S(f)(b) + N = 0.$$

Hence, $f \in \ker(\pi \times u)$, so $J := \ker(\pi \times u) \neq A \rtimes G$. Moreover,

$$(\pi \times u)(\delta_0) = 1_A$$

so $J \neq \{0\}$ either. Hence, J is a proper ideal in $A \rtimes G$. □

Lemma 7 (Connes, 1973)**.** *Let $M \subset \mathcal{B}(H)$ be a von Neumann algebra and $u \in U(M)$ be a unitary and $\alpha : M \to M$ be the automorphism $\alpha(a) := uau^*$. Then, $\sigma(u) = \{\lambda\omega^{-1} : \lambda, \omega \in \sigma(u)\}$.*

Proof. If $\lambda, \omega \in \sigma(u)$, choose a projection $q \in \mathcal{B}(H)$ (by spectral theory) such that $\|uq - \lambda q\| < \epsilon$. Similarly, choose a projection $p \in \mathcal{B}(H)$ such that $\|up - \omega p\| < \epsilon$. Choose a partial isometry $v \in \mathcal{B}(H)$ such that $vv^* \leq q$ and $v^*v \leq p$. Then,

$$\begin{aligned}\|\alpha(v) - \lambda\omega^{-1}v\| &= \|uv - \lambda\omega^{-1}vu\| \\ &= \|uqv - \lambda qv\| + \|\lambda qv - \lambda\omega^{-1}vu\| \\ &= \|(uq - \lambda q)v\| + \|qv - \omega^{-1}vu\| \\ &\leq \epsilon + \|\omega vp - vpu\| \\ &= \epsilon + \|\omega vp - vup\| \\ &= \epsilon + \|v(\omega p - up)\| \\ &< 2\epsilon\end{aligned}$$

Hence,

$$D := \{\lambda\omega^{-1} : \lambda, \omega \in \sigma(u)\} \subset \sigma(\alpha).$$

Conversely, consider two maps $L : M \to M$ by $L(x) := ux$ and $R : M \to M$ by $R(x) = xu^*$, then L, R are bounded linear maps which commute with each other. Therefore,

$$\sigma(\alpha) = \sigma(LR) \subset \{\lambda\zeta : \lambda \in \sigma(L), \zeta \in \sigma(R)\}.$$

Now note that $\sigma(L) = \sigma(u)$ and $\sigma(R) = \sigma(u^*) = \sigma(u)^{-1}$. □

Example 8. Let $H := \ell^2, A = \mathcal{B}(H)$ and $u \in A$ be a unitary with $\sigma(u) = \mathbb{T}$. Define $\alpha : A \to A$ by $\alpha(a) := uau^*$. Then, $A \rtimes G$ is not simple. However, by the previous lemma,

$$\sigma(\alpha) = \mathbb{T}.$$

so the converse of the previous theorem does not hold.

Definition 9. The <u>Connes spectrum</u> of α is the set

$$\Gamma(\alpha) = \bigcap \sigma(\alpha|_B)$$

where the intersection is taken over all G-invariant hereditary subalgebras B of A.

Example 10. In the above example, if $p \in A$ is a spectral projection such that $\|up - \lambda p\| < \epsilon$, then for any $x = pap \in pAp$ with $\|a\| \leq 1$, we have

$$\|\alpha(x) - x\| = \|upap - papu\| = \|upap - paup\| < \|upap - \lambda pap\| + \|\lambda pap - paup\| < 2\epsilon.$$

Hence, restricting α to the G-invariant hereditary subalgebra pAp,

$$\sigma(\alpha|_{pAp}) \subset \{z \in \mathbb{T} : |z - 1| < 2\epsilon\}.$$

This is true for any $\epsilon > 0$, so

$$\Gamma(\alpha) = \{1\}.$$

Theorem 11 (Olesen, Pedersen (1978))**.** *$A \rtimes G$ is simple if and only if A is G-simple and $\Gamma(\alpha) = \mathbb{T}$.*