AMENABLE GROUPS

ABSTRACT. A Banach limit is a specific kind of linear functional on $\ell^\infty(\mathbb{Z})$ that respects the natural action of $\mathbb{Z}$. A generalization of this leads to the notion of an amenable group. We will discuss these groups and give examples. We will also discuss a representation-theoretic question (due to Dixmier) which has been open since 1950.

Define

$$\ell^\infty(\mathbb{Z}) := \{f : \mathbb{Z} \to \mathbb{R} : \sup_{n\in\mathbb{Z}} |f(n)| < \infty\} =: \mathbf{E}$$
$$c := \{f : \mathbb{Z} \to \mathbb{R} : \lim_{n\to\pm\infty} f(n) \text{ both exist}\} =: \mathbf{F}$$

Then, $\mathbf{E}$ is a normed linear space and $\mathbf{F}$ is a subspace of $\mathbf{E}$. Define $\varphi : \mathbf{F} \to \mathbb{R}$ by

$$\varphi(f) := \lim_{n\to\infty} f(n).$$

Then φ is a bounded linear functional on $\mathbf{F}$. By the Hahn-Banach theorem, there is a bounded linear functional $\psi : \mathbf{E} \to \mathbb{R}$ such that

$$\psi|_{\mathbf{F}} = \varphi \text{ and } \|\psi\| = \|\varphi\| = 1.$$

We wish to construct a specific kind of extension of φ. Define the right shift operator by $\sigma : \mathbf{E} \to \mathbf{E}$ by

$$\sigma(f)(n) := f(n-1)$$

Note that S is well-defined, $\sigma(\mathbf{F}) \subset \mathbf{F}$. Moreover, if $f \in \mathbf{F}$, then $\varphi(f) = \varphi(\sigma(f))$.

Theorem 1. *There exists a bounded linear functional $\psi : \mathbf{E} \to \mathbb{R}$ satisfying the following properties:*

(1) $\psi|_{\mathbf{F}} = \varphi$.
(2) $\|\psi\| = 1$.
(3) If $f \in \mathbf{F}$ is such that $f(n) \geq 0$ for all $n \in \mathbb{N}$, then $\psi(f) \geq 0$.
(4) $\psi(f) = \psi(\sigma(f))$ for all $f \in \mathbf{E}$.

(See [CONWAY, 1990] for details)

Proof. Let $\mathbf{F}' := \{f - \sigma(f) : f \in \ell^\infty\}$, and $\mathbf{1}$ be the constant function one. Then, one shows that

$$\|\mathbf{1} + \mathbf{F}'\| = \inf\{\|\mathbf{1} - f\| : f \in \mathbf{F}'\} = 1.$$

Then, one uses a corollary of the Hahn-Banach theorem to produce $\psi : \mathbf{E} \to \mathbb{R}$ such that

(1) $\psi|_{\mathbf{F}'} = 0$
(2) $\psi(\mathbf{1}) = 1$.
(3) $\|\psi\| = 1$.

To see that ψ satisfies (1), choose $f \in \mathbf{F}, \alpha := \varphi(f)$ and $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that $|f(n) - \alpha| < \epsilon$ for all $n \geq N$. If $g := \sigma^N(f)$, then $|g(n) - \alpha| < \epsilon$ for all $n \in \mathbb{N}$. Thus,

$$|\psi(f) - \alpha| = |\psi(g) - \alpha| = |\psi(g - \alpha\mathbf{1})| \leq \|\psi\|\|g - \alpha\mathbf{1}\| \leq \epsilon.$$

Hence, $\psi(f) = \alpha$. □

Let G be a group. If $t \in G$, define $\sigma_t : \ell^\infty(G) \to \ell^\infty(G)$ by

$$\sigma_t(f)(s) := f(t^{-1}s).$$

Definition 2. A group G is said to be amenable if there is a linear functional $\psi : \ell^\infty(G) \to \mathbb{R}$ satisfying the following conditions:

(1) ψ is positive: $\psi(f) \geq 0$ if $f \in \ell^\infty(G)$ is such that $f(s) \geq 0$ for all $s \in G$.
(2) ψ is left translation invariant: If $f \in \ell^\infty(G)$ and $t \in G$, then $\psi(\sigma_t(f)) = \psi(f)$.
(3) ψ is a *mean*: $\psi(\mathbf{1}) = 1$.

Note: The conditions automatically imply that ψ is bounded and that $\|\psi\| = 1$.

Proposition 3. *Let G be amenable and ψ as above. Define $\mu : \mathcal{P}(G) \to \mathbb{R}$ by*

$$\mu(A) := \psi(\chi_A).$$

Then,

(1) μ is positive*: $\mu(A) \geq 0$ for all $A \subset G$.*
(2) μ is finitely additive*: If $A, B \subset G$ are disjoint, then $\mu(A \sqcup B) = \mu(A) + \mu(B)$.*
(3) μ is normalized*: $\mu(G) = 1$.*
(4) μ is translation invariant*: If $A \subset G$ and $t \in G$, then $\mu(tA) = \mu(A)$.*

Proof.

(1) Follows because $\chi_A(s) \geq 0$ for all $s \in G$.
(2) If $A \cap B = \emptyset$, then $\chi_{A \sqcup B} = \chi_A + \chi_B$. So by linearity, $\psi(\chi_{A\sqcup B}) = \psi(\chi_A) + \psi(\chi_B)$.
(3) Follows because $\chi_G = \mathbf{1}$.
(4) Follows because $\chi_{tA} = \sigma_t(\chi_A)$

□

Remark 4. μ is not necessarily countably additive. If $G = \mathbb{Z}$ and μ were countably additive, then by translation invariance,

$$1 = \mu(\mathbb{Z}) = \sum_{n=-\infty}^{\infty} \mu(\{n\}) = \sum_{n=-\infty}^{\infty} \mu(\{0\}).$$

This is clearly impossible.

Example 5.

(1) $\mathbb{Z}$ is amenable.
(2) Any finite group is amenable. Take $\mu(A) := \frac{|A|}{|G|}$.
(3) Let F_2 denote the free group on 2 letters $\{a, b\}$. Then, F_2 is not amenable.

Proof. Let A_0 be the set of all words starting with an odd power of a followed by e or a power of b. Let A_1 be the set of all words starting with an even power of a. Then, $F_2 = A_0 \sqcup A_1$ and $A_1 = aA_0$. By translation invariance,

$$1 = \mu(F_2) = \mu(A_0) + \mu(A_1) = 2\mu(A_0) \Rightarrow \mu(A_0) = \frac{1}{2}.$$

Now for $j \in \{0, 1, 2\}$, define B_j to be the set of words starting with a power of b congruent to $j \pmod 3$. Then, as before,

$$\mu(B_0) = \frac{1}{3}.$$

However, $A_0 \subset B_0$, so it follows that $\frac{1}{2} \leq \frac{1}{3}$. This is absurd. □

(4) Amenability is preserved under taking subgroups, quotients, direct products, direct limits, etc.
(5) In particular, any abelian group is amenable.
(6) Amenability is preserved under taking group extensions $0 \to H \to G \to N \to 0$.
(7) Every solvable group is amenable.

(8) Any group that contains F_2 is non-amenable. In particular, $SO_3(\mathbb{R})$ (with discrete topology) is non-amenable. This is the reason the Banach-Tarski paradox works, which says that there is no $SO_3(\mathbb{R})$-invariant function $\mu : 2^{\mathbb{R}} \to \mathbb{R}$ as above.
(9) Olshanksi (1980) proved that there is a non-amenable group that does not contain F_2 (Tarski Monster groups), settling a conjecture of von Neumann (first stated by M.M. Day (1957)). Now, many more counterexamples are known.

1. Dixmier's Problem

Definition 6. Let G be a (discrete) group and H a Hilbert space.

(1) A <u>representation</u> of G on H is a group homomorphism $\pi : G \to GL(H)$, where $GL(H)$ denotes the group of invertible operators on H.
(2) A representation $\pi : G \to GL(H)$ is <u>unitary</u> if $\pi(s)$ is a unitary for all $s \in G$. (i.e. $\langle \pi(s)x, \pi(s)y \rangle = \langle x, y \rangle$ for all $x, y \in H$).
(3) Two representations $\pi_i : G \to GL(H_i), i = 1, 2$ are <u>equivalent (similar)</u> if there is an invertible $T : H_1 \to H_2$ such that $\pi_1(s) = T^{-1}\pi_2(s)T$ for all $g \in G$.
(4) A representation $\pi : G \to GL(H)$ is <u>uniformly bounded</u> if $\sup_{s\in G} \|\pi(s)\| < \infty$.

Note: If a representation is equivalent to a unitary representation, it must be uniformly bounded because

$$\|\pi(s)\| = \|T\rho(s)T^{-1}\| \leq \|T\|\|T^{-1}\|$$

for all $s \in G$.

Definition 7. A group G is said to be <u>unitarizable</u> if every uniformly bounded representation of G on a Hilbert space is equivalent to a unitary representation.

Question: Is every group unitarizable?
Answer: No. F_2 is not, but this is hard to prove.

Theorem 8 (Day-Dixmier (1950))**.** *Every amenable group is unitarizable.*

Sketch of proof. Let G be amenable, and $\psi : \ell^\infty(G) \to \mathbb{C}$ be an invariant mean. Let $\pi : G \to GL(H)$ be a representation of G on a Hilbert space H such that $C := \sup_{s\in G} \|\pi(s)\| < \infty$.

(1) For $x, y \in H$, define $Q_{x,y} : G \to \mathbb{C}$ by $s \mapsto \langle \pi(s^{-1})x, \pi(s^{-1})y \rangle$. Then,

$$|Q_{x,y}(s)| \leq \|\pi(s^{-1})\|^2\|x\|\|y\| \leq C^2\|x\|\|y\|.$$

Hence, $Q_{x,y} \in \ell^\infty(G)$. Moreover,
- If $x = y$, then $Q_{x,x}(s) = \|\pi(s^{-1})x\|^2 \geq 0$ for all $s \in G$.
- If $t \in G$, then

$$\begin{aligned}\sigma_t(Q_{x,y})(s) &= Q_{x,y}(t^{-1}s)\\ &= \langle \pi(s^{-1}t)x, \pi(s^{-1}t)y \rangle\\ &= \langle \pi(s^{-1})\pi(t)x, \pi(s^{-1})\pi(t)y \rangle\\ &= Q_{\pi(t)x,\pi(t)y}(s).\end{aligned}$$

(2) Define $u : H \times H \to \mathbb{C}$ by $u(x, y) := \psi(Q_{x,y})$. Then,
- u is linear in the first variable.
- u is conjugate linear in the second variable.
- $|u(x, y)| \leq C^2\|x\|\|y\|$ for all $x, y \in H$.

Hence, u is a *bounded sesqui-linear form* on H. Hence, there exists $S \in \mathcal{B}(H)$ such that

$$\langle Sx, y\rangle = u(x,y).$$

for all $x, y \in H$.

(3) If $x \in H$, then $u(x,x) = \psi(Q_{x,x}) \geq 0$ since $Q_{x,x} \geq 0$. Therefore S is a positive operator.

(4) S is invertible (this takes a little work). Hence, $T := S^{1/2}$ is a positive invertible operator and satisfies

$$\langle Tx, Ty\rangle = \langle Sx, y\rangle = u(Q_{x,y}).$$

(5) If $t \in G$, then

$$\begin{aligned}\langle T\pi(t)T^{-1}x, T\pi(t)T^{-1}y\rangle &= u(\pi(t)T^{-1}x, \pi(t)T^{-1}y) \\ &= \psi(Q_{\pi(t)T^{-1}x,\pi(t)T^{-1}y}) \\ &= \psi(\sigma_t(Q_{T^{-1}x,T^{-1}y})) \\ &= \psi(Q_{T^{-1}x,T^{-1}y}) \\ &= u(T^{-1}x, T^{-1}y) \\ &= \langle TT^{-1}x, TT^{-1}y\rangle \\ &= \langle x, y\rangle\end{aligned}$$

Hence, $T\pi(t)T^{-1}$ is a unitary operator.

□

Question: (Dixmier, 1950) Is the converse of this theorem true?
Answer: Unknown! The following is known though:

(1) F_2 is not unitarizable.

(2) A subgroup of a unitarizable group is unitarizable. Therefore, any group that contains F_2 is not unitarizable. tem In 2007, Pisier proved that a 'strongly' unitarizable group is amenable. Here, strongly unitarizable roughly means that if π is uniformly bounded, then the invertible matrix T can be chosen with $\|T\|, \|T^{-1}\| \leq C_\pi$ where $C_\pi := \sup_{s\in G} \|\pi(s)\|$.

(3) In 2010, Monod, Epstein and Ozawa proved that there is a non-unitarizable group that does not contain F_2.

(4) If G is a linear group, then Dixmier's question has a positive answer (by the Tits alternative: Either G contains a solvable subgroup of finite index, or G contains a non-abelian free group).

(5) If all countable subgroups of G are unitarizable, then so is G.

Hence, a counterexample (if it exists) must be a countable, non-amenable group that does not contain F_2. If the conjecture is true, then we may resolve the following questions for unitarizability:

- If G_1 and G_2 are unitarizable, is $G_1 \times G_2$ unitarizable?
- More generally, if N and G/N are both unitarizable, does it follow that G is unitarizable?
- If (G_n) is an inductive sequence of unitarizable groups, is $\lim G_n$ unitarizable?

References

Conway, John B. (1990). *A course in functional analysis*. Second. Vol. 96. Graduate Texts in Mathematics. Springer-Verlag, New York, pp. xvi+399. ISBN: 0-387-97245-5.