Rokhlin Dimension for Group Actions

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Contents

Preliminaries

Assumptions

Unless stated otherwise,

- All C*-algebras will be unital and separable (denoted $A, B, C, \ldots$)
- All topological spaces will be compact, Hausdorff (denoted $X, Y, Z, \ldots$)
- All groups will be finite (denoted $G, H, \ldots$)

Motivation

A group action of G on A is a group homomorphism

$$\alpha : G \to \mathrm{Aut}(A)$$

Given such an action, one constructs a *crossed product* C*-algebra

$$A \rtimes_\alpha G$$

Question: Permanence

Suppose A satisfies a property (P), then can we impose conditions on α so that $A \rtimes_\alpha G$ also satisfies property (P)?

Motivation

Examples of (P) include

1. Simplicity
2. Nuclearity/Exactness
3. Finite nuclear dimension/stable rank/real rank
4. Stability ($A \otimes \mathcal{K} \cong A$)
5. Classifiability (by K-theoretic invariants)

The motivation comes from the commutative case.

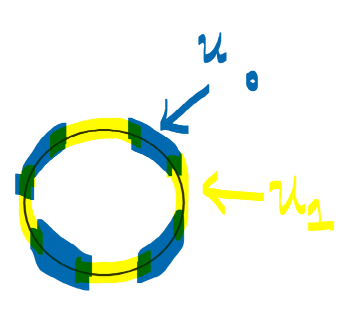
Group Actions on Spaces

Covering Dimension of a Space

Definition

Let $n \in \mathbb{N}$. A finite open cover $\mathcal{U}$ of X is said to be *n-decomposable* if there is a decomposition $\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \ldots \sqcup \mathcal{U}_n$ such that each $\mathcal{U}_i$ consists of mutually disjoint sets.

The following cover of S^1 is 1-decomposable.



Covering Dimension of a Space

One thinks of an n-decomposable cover as a way of covering the space with $(n+1)$ *colours*, where each colour corresponds to a single $\mathcal{U}_i$.

Definition

The *Lebesgue covering dimension* of X is the least integer n such that every finite open cover $\mathcal{U}$ of X has a finite refinement $\mathcal{V}$ which is n-decomposable. We denote this number by

$$\dim(X)$$

All spaces in this talk will be assumed to have finite dimension.

Group Actions on Spaces

A group action of G on X is a group homomorphism

$$\beta : G \to \mathrm{Homeo}(X)$$

Given such an action, we get an induced action of $\alpha : G \to \mathrm{Aut}(C(X))$ by

$$\alpha_g(f)(x) := f(\beta_{g^{-1}}(x))$$

Free Group Actions on Spaces

An action $G \curvearrowright_\beta X$ is said to be *free* if, for any $x \in X$ and $g \in G$,

$$g \cdot x = x \Rightarrow g = e$$

Some examples include:

1. G acts on itself by left-multiplication (where $G = X$ carries the discrete topology). We denote this action by

$$\lambda : G \to \text{Homeo}(G)$$

2. $G = \mathbb{Z}_n$ acts on $X = S^1$ by 'rotation by $2\pi/n$'

$$\overline{k} \cdot z := e^{2\pi i k/n} z$$

Free Group Actions on Spaces

Definition

Let $G \curvearrowright_\beta X$, $\mathcal{U}$ be a finite open cover of X, and $n \in \mathbb{N}$. We say that $\mathcal{U}$ is *n-decomposable with respect to G* if we can write $\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \ldots \sqcup \mathcal{U}_n$ where, for each $0 \leq i \leq n$, each $\mathcal{U}_i$ consists of $|G|$ mutually disjoint sets

$$\mathcal{U}_i = \{V_i^g : g \in G\}$$

such that

$$g \cdot V_i^h = V_i^{gh}$$

In other words, such a cover of X corresponds to a *colouring* of X, where each colour respects the action of G.

Lemma 1

If X has a cover that is n-decomposable with respect to G, then the action is free.

Proof.

If $x \in X$, then there exists $0 \leq i \leq n$ and $h \in G$ such that $x \in V_i^h$. Now if $g \in G$ is such that $g \cdot x = x$, then

$$x = g \cdot x \in g \cdot V_i^h = V_i^{gh}$$

If $g \neq e$, then V_i^g and V_i^{gh} are disjoint. Hence, $g = e$ must hold. □

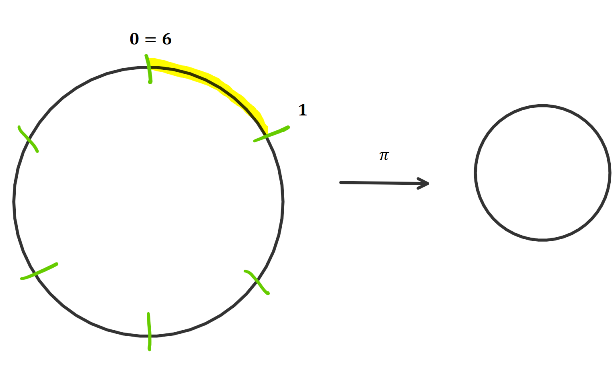
Theorem

Let $G \curvearrowright_\beta X$ be a free action. Then, there exists $n \in \mathbb{N}$ and an open cover $\mathcal{U}$ of X that is n-decomposable with respect to G.

Proof by Example

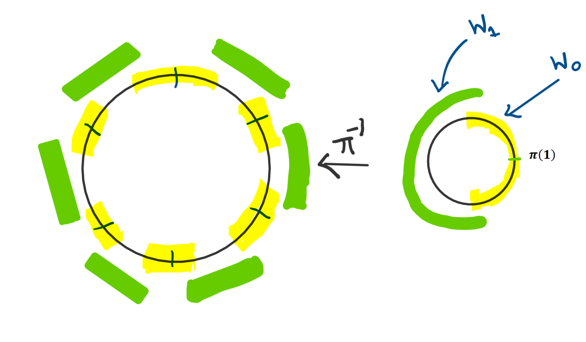
Let $X = S^1$ and $G = \mathbb{Z}_6$ acting by rotation as above.

- Observe that $X/G \cong S^1$



Proof by Example

- Start with an open cover of X/G like $\{W_0, W_1\}$ shown below.



- Lift this cover to get an 1-decomposable cover of X with respect to G.

Decomposable Actions on C*-Algebras

Completely Positive maps

Given a linear map $\varphi : A \to B$ between two C*-algebras, we obtain a linear map

$$\varphi^{(n)} : M_n(A) \to M_n(B)$$

given by

$$(a_{i,j}) \mapsto (\varphi(a_{i,j}))$$

Definition

A linear map $\varphi : A \to B$ is said to be *completely positive* if $\varphi^{(n)}$ is positive for each $n \in \mathbb{N}$.

A *c.c.p.* map is a contractive, completely positive map.

Order Zero Maps

Two elements $a, b \in A$ are said to be *orthogonal* (in symbols, $a \perp b$) if

$$ab = a^* b = ab^* = ba = 0$$

Definition

A c.c.p. map $\varphi : A \to B$ is said to have *order zero* if, for any $a, b \in A$,

$$a \perp b \Rightarrow \varphi(a) \perp \varphi(b)$$

Order Zero Maps

1. Any $*$-homomorphism has order zero.
2. If $\pi : A \to B$ is a $*$-homomorphism and $h \in \pi(A)' \cap B$ is a positive element, then

$$a \mapsto h\pi(a)$$

is an order zero map.

Theorem (Winter and Zacharias, 2009)

Every c.c.p. order zero map has the form of Example 2.

Furthermore, there is a one-to-one correspondence between c.c.p. order zero maps $\varphi : A \to B$ and $*$-homomorphisms

$$\pi_\varphi : C_0[0,1) \otimes A \to B$$

given by $\varphi(a) = \pi_\varphi(\mathrm{id}_{C_0[0,1)} \otimes a)$.

Equivariant Map

Given a group action $G \curvearrowright_\alpha A$, the centre

$$\mathcal{Z}(A) = \{a \in A : ab = ba \quad \forall b \in A\}$$

is G-invariant, so we get an induced action $G \curvearrowright_\alpha \mathcal{Z}(A)$.

Definition

A linear map $\varphi : C(G) \to \mathcal{Z}(A)$ is said to be G-equivariant if

$$\varphi(\lambda_g(f)) = \alpha_g(\varphi(f))$$

for all $f \in C(G)$.

Recall that $\lambda : G \to \mathrm{Aut}(C(G))$ is given by

$$\lambda_g(f)(h) := f(g^{-1}h)$$

Decomposable Actions on C*-algebras

Definition 1

Let $n \in \mathbb{N}$. We say that a group action $G \curvearrowright_\alpha A$ is *n-decomposable* if there exists $(n+1)$ maps

$$\varphi_0, \varphi_1, \ldots, \varphi_n : C(G) \to \mathcal{Z}(A)$$

which are G-equivariant, c.c.p., have order zero, and satisfy

$$\varphi_0(1_{C(G)}) + \varphi_1(1_{C(G)}) + \ldots + \varphi_n(1_{C(G)}) = 1_A$$

We say that α is *decomposable* if it is n-decomposable for some natural number $n \in \mathbb{N}$.

Decomposable ⇒ Free

Lemma 2

Let $G \curvearrowright_\beta X$ be a group action and $\alpha : G \to \mathrm{Aut}(C(X))$ be the induced action. If α is decomposable, then β is free.

Proof.

Let $\varphi_i : C(G) \to C(X)$ be the maps as above. For $g \in G$, define

$$V_i^g := \varphi_i(\delta_g)^{-1}((0, +\infty))$$

and set $\mathcal{U}_i := \{V_i^g : g \in G\}$. Then one can verify that

$$\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \ldots \sqcup \mathcal{U}_n$$

is n-decomposable with respect to G. So β is free by Lemma 1. □

Decomposable $\Leftrightarrow$ Free

Theorem (Gardella, 2014)

Let $G \curvearrowright_\beta X$ be a group action and $\alpha : G \to \mathrm{Aut}(C(X))$ be the induced action. Then, α is decomposable if and only if β is free.

The Structure of the Crossed Product

The Structure of the Crossed Product

Recall that an action $G \curvearrowright_\alpha A$ is n-decomposable if there exist $(n+1)$ G-equivariant, c.c.p., order zero maps

$$\varphi_0, \varphi_1, \ldots, \varphi_n : C(G) \to \mathcal{Z}(A)$$

such that $\sum_{i=0}^{n} \varphi_i(1) = 1_A$.

Theorem 1 (Folklore)

If α is n-decomposable, there is a commuting diagram

$$\begin{array}{ccc} A \rtimes_\alpha G & \xrightarrow{\quad \text{id} \quad} & A \rtimes_\alpha G \\ & {}_{\widetilde{\psi}}\searrow \qquad \nearrow_{\widetilde{\varphi}} & \\ & C(Y) \otimes M_{|G|}(A) & \end{array}$$

where Y is a compact metric space with $\dim(Y) \leq n$.

Conclusion

Theorem 1 allows us to understand properties of $A \rtimes_\alpha G$ by 'factoring through' $C(Y) \otimes M_{|G|}(A)$.

Corollary

Let (P) denote one of the following properties:

1. Finite rank (stable/real rank/nuclear dimension)
2. Stability
3. Nuclear, separable, and satisfying the UCT
4. ... etc.

If A satisfies property (P) and α is decomposable, then $A \rtimes_\alpha G$ also satisfies property (P).

Rokhlin Dimension

Approximately Decomposable Actions

Unfortunately, requiring that an action is decomposable is too restrictive in the noncommutative case. Therefore, one defines an approximate version of it.

Definition

Let $n \in \mathbb{N}$. An action $\alpha : G \to \mathrm{Aut}(A)$ is said to be *approximately n-decomposable* if, for every finite set $F \subset A$ and every $\epsilon > 0$, there are $(n+1)$ c.c.p. order zero linear maps

$$\varphi_0, \varphi_1, \ldots, \varphi_n : C(G) \to A$$

satisfying the following conditions:

Approximately Decomposable Actions

1. Each φ_i is 'approximately equivariant'.

$$\|\alpha_g(\varphi_i(\delta_h)) - \varphi_i(\lambda_g(\delta_h))\| < \epsilon \quad \forall g, h \in G$$

2. Each φ_i is 'approximately central'

$$\|\varphi_i(\delta_h)a - a\varphi_i(\delta_h)\| < \epsilon \quad \forall a \in F, \text{ and } h \in G$$

$$\|\varphi_i(\delta_g)\varphi_j(\delta_h) - \varphi_j(\delta_h)\varphi_i(\delta_g)\| < \epsilon \quad \forall h, g \in G, 0 \leq i, j \leq n$$

3. The $\{\varphi_i\}$ are an 'approximate partition of unity'.

$$\|\sum_{i=0}^{n} \varphi_i(1_{C(G)}) - 1_A\| < \epsilon$$

Rokhlin Dimension

Definition (Hirshberg, Winter, and Zacharias, 2015)

The *Rokhlin dimension (with commuting towers)* of an action $\alpha : G \to \mathrm{Aut}(A)$ is the least value of $n \in \mathbb{N}$ such that α is approximately n-decomposable. We denote the integer by

$$\dim_{Rok}^{c}(\alpha)$$

Note that if $\dim_{Rok}^{c}(\alpha) = 0$, then α has the *Rokhlin property*.

Example 1 (Izumi, 2004)

Let $A := \bigotimes_{n=1}^{\infty} M_2(\mathbb{C})$ be the UHF algebra of type 2^∞ and $\alpha \in \mathrm{Aut}(A)$ be the order two automorphism given by

$$\alpha = \bigotimes_{n=1}^{\infty} \mathrm{Ad} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then α induces an action of $\mathbb{Z}_2$ on A such that $\dim_{Rok}^{c}(\alpha) = 0$.

Example 2 (Hirshberg and Phillips, 2015)

Let $\theta \in \mathbb{R}$ be irrational, and $A = A_\theta$ be the corresponding irrational rotation algebra generated by unitaries $\{u, v\}$ such that

$$uv = e^{2\pi i\theta} vu$$

Let $\alpha \in \mathrm{Aut}(A)$ be the order two automorphism satisfying

$$\alpha(v) = v \text{ and } \alpha(u) = -u$$

Then, α induces an action of $\mathbb{Z}_2$ with the property that $\dim_{Rok}^{c}(\alpha) = 1$.

Note that both actions above are not decomposable in the earlier sense because the underlying algebras are simple, and so have trivial centers.

Example 3 (Gardella, 2014)

However, if $G \curvearrowright_{\beta} X$ is a group action and $\alpha : G \to \mathrm{Aut}(C(X))$ is the induced action, then the following are equivalent:

- $\dim_{Rok}^{c}(\alpha) < +\infty$
- α is decomposable in the sense of Definition 1.
- β is free.

Obstructions (Hirshberg and Phillips, 2015)

- Suppose $G \curvearrowright_\alpha A$ has the property that

$$\dim^c_{Rok}(\alpha) < \infty$$

Then α is point-wise outer: There cannot exist $e \neq h \in G$ and a unitary $u \in \mathcal{U}(A)$ such that

$$\alpha_h(a) = uau^* \quad \forall a \in A$$

- There are also some K-theoretic obstructions. For instance, If $A = \mathcal{Z}$ or $\mathcal{O}_\infty$, and G is a non-trivial finite group, then no action of G on A can have $\dim^c_{Rok}(\alpha) < \infty$.

The Structure of the Crossed Product

The Structure for Decomposable Actions

Recall that we had the following theorem for decomposable actions.

Theorem 1

If α is n-decomposable, there is a commuting diagram

$$\begin{array}{ccc} A \rtimes_\alpha G & \xrightarrow{\quad\mathrm{id}\quad} & A \rtimes_\alpha G \\ & \searrow^{\widetilde{\psi}} \qquad \nearrow_{\widetilde{\varphi}} & \\ & C(Y) \otimes M_{|G|}(A) & \end{array}$$

where Y is a compact metric space with $\dim(Y) \leq n$.

We wish to describe an analogous result for approximately decomposable actions (ie. actions with finite Rokhlin dimension).

A certain 'nice' Crossed Product

Let $G \curvearrowright_\beta X$ be a free group action and $G \curvearrowright_\alpha A$ be a group action. Then, we get a natural diagonal action

$$\gamma : G \to C(X, A)$$

given by $\gamma_g(f)(x) := \alpha_g(f(g^{-1} \cdot x))$.

Lemma (Gardella, Hirshberg, and Santiago, 2021)

The crossed product C*-algebra

$$C(X, A) \rtimes_\gamma G$$

is a continuous $C(X/G)$-algebra with fibers canonically isomorphic to

$$M_{|G|}(A).$$

The Structure of the Crossed Product

Theorem (Gardella, Hirshberg, and Santiago, 2021)

Let $\alpha : G \to \mathrm{Aut}(A)$ be a group action such that

$$\dim^{c}_{Rok}(\alpha) < \infty$$

Then, there is an ‘approximately commuting diagram’

$$\begin{array}{ccc} A \rtimes_\alpha G & \xrightarrow{\quad \mathrm{id} \quad} & A \rtimes_\alpha G \\ & {\scriptstyle \widetilde{\psi}} \searrow \qquad \nearrow {\scriptstyle \widetilde{\varphi}} & \\ & C(X, A) \rtimes_\gamma G & \end{array}$$

where X is a free G-space with $\dim(X) \leq \dim^{c}_{Rok}(\alpha)$.

Conclusion

Many of the properties of C*-algebras listed above are defined in terms of approximations. Hence, we get the following corollary.

Corollary (Gardella, Hirshberg, and Santiago, 2021)

Let (P) denote one of the following properties:

1. Finite rank (stable/real rank/nuclear dimension)
2. Stability
3. Nuclear, separable, and satisfying the UCT
4. ... etc.

If A satisfies property (P) and

$$\dim_{Rok}^{c}(\alpha) < \infty$$

then $A \rtimes_\alpha G$ also satisfies property (P).

Summary

The Goal: Permanence Properties

Given a group action $G \curvearrowright_\alpha A$, we are interested in conditions under which properties pass from A to the crossed product C*-algebra $A \rtimes_\alpha G$.

- The analogue of *free* actions on spaces are *decomposable* actions. Given a decomposable action, one has a structure theorem for the crossed product, which allows us to prove permanence properties.
- This definition is too restrictive in general, so we define an 'approximate' version of it, viz. *finite Rokhlin dimension (with commuting towers)*. If α has this property, then we have an analogous structure theorem for the crossed product. Once again, this allows us to prove permanence properties.

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Thank you!