Stable Ranks for C*-algebras and Rokhlin Actions

(Based on joint work with Ms. Anshu)

Prahlad Vaidyanathan

Department of Mathematics
IISER Bhopal

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Lebesgue Covering Dimension

Lebesgue Covering Dimension

Let X be a compact Hausdorff space and $\mathcal{V}$ be an open cover of X. We say that $\mathcal{V}$ has *order* m if every point in X belongs to atmost m elements of $\mathcal{V}$.

Definition

The (Lebesgue covering) dimension of X is atmost n if every open cover of X has a refinement $\mathcal{V}$, such that

$$\text{order}(\mathcal{V}) \leq n + 1$$

Example: Circle

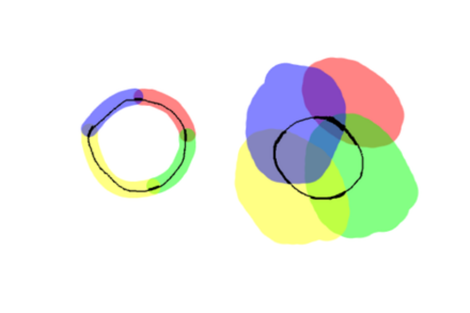

Figure 1: Dimension of a Circle (Source: Wikipedia)

The open cover (on the right) has a refinement (on the left) of order 2. Hence,

$$\dim(S^1) \leq 1$$

Example: Cantor Set

If C denotes the Cantor set, then

C is totally disconnected
(ie. C has a basis consisting of cl-open sets)

Every open cover of C has a refinement consisting of disjoint sets (ie. of order 1). Hence,

$$\dim(C) = 0$$

Alternate Definition

Theorem

Let X be a compact Hausdorff space. Then $\dim(X)$ is the least integer n such that, for any continuous function

$$f : X \to \mathbb{R}^{n+1}$$

and any $\epsilon > 0$, there is a continuous function

$$g : X \to \mathbb{R}^{n+1}$$

such that $\|f - g\|_\infty < \epsilon$ and

$$0 \notin g(X)$$

This says that 0 is an *unstable* value for f.

Alternate Definition

Theorem

Let X be a compact Hausdorff space. Then $\dim(X)$ is the least integer n such that any $(n+1)$-tuple

$$(f_1, f_2, \ldots, f_{n+1}) \in C(X, \mathbb{R})^{n+1}$$

can be approximated arbitrarily closely by a tuple

$$(g_1, g_2, \ldots, g_{n+1}) \in C(X, \mathbb{R})^{n+1}$$

such that

$$\sum_{i=1}^{n+1} g_i^2$$

is a strictly positive function.

Stable Ranks for C*-algebras

Connection to C*-algebras

Definition

A C*-algebra is a Banach algebra A together with an involution $a \mapsto a^*$ satisfying certain conditions.

Example:

- $\mathbb{C}$ is a C*-algebra.
- If X is a compact Hausdorff space, $C(X) = C(X, \mathbb{C})$ is a C*-algebra.
- If H is a Hilbert space, $\mathcal{B}(H)$ is a C*-algebra.

Stable Rank for C*-algebras

Definition (Rieffel (1982))

Let A be a C*-algebra. The *topological stable rank* (tsr) of A is the least integer n such that any tuple

$$(a_1, a_2, \dots, a_n) \in A^n$$

can be approximated arbitrarily closely by a tuple

$$(b_1, b_2, \dots, b_n) \in A^n$$

such that

$$\sum_{i=1}^{n} b_i^* b_i$$

is invertible in A.

If no such integer exists, we write $tsr(A) = +\infty$.

Examples

- Since a complex valued function can be thought of as a pair of real-valued functions, we have

$$tsr(C(X)) = \left\lceil \frac{\dim(X)}{2} \right\rceil + 1$$

  where $\lceil x \rceil$ is the 'integer part' of x.
- $tsr(\mathbb{C}) = 1$
- More generally, if A is a finite dimensional C*-algebra, then

$$tsr(A) = 1$$

Examples

- For any C*-algebra A,

$$tsr(A) = 1 \Leftrightarrow GL(A) \text{ is dense in } A$$

- If $S \in \mathcal{B}(\ell^2)$ denotes the right-shift operator, then S cannot be approximated by invertibles. Hence, if $A := C^*(S)$, then

$$tsr(A) \neq 1$$

  In fact, $tsr(A) = 2$.

- If H is an infinite dimensional Hilbert space, then

$$tsr(\mathcal{B}(H)) = +\infty$$

How are stable ranks useful?

- Knowing the stable rank of an algebra helps answer questions in ‘nonstable’ K-theory. ie. One can use K-theoretic information to extract information about elements (projections or unitaries) in the algebra.
- Algebras with stable rank 1 have nice regularity properties that are useful in classification.

Group Actions on C*-algebras

Group Actions

Standing assumption:

- $G =$ Finite group
- $A =$ Unital, separable C*-algebra.

Define

$$\mathrm{Aut}(A) := \{\text{involution-preserving automorphisms of } A\}$$

Note that $\mathrm{Aut}(A)$ is a group under composition.

Definition

A *group action* of G on A is a group homomorphism

$$\alpha : G \to \mathrm{Aut}(A)$$

Example

- If $G \curvearrowright X$ is an action of G on a compact Hausdorff space X, then it induces an action $\alpha : G \curvearrowright C(X)$ by

$$\alpha_g(f)(x) := f(g^{-1} \cdot x)$$

- Furthermore, every C*-algebra action of G on $C(X)$ arises in this way.
- If $\alpha \in \mathrm{Aut}(A)$ is an automorphism of order N, then it induces a group action

$$\alpha : \mathbb{Z}_N \to \mathrm{Aut}(A)$$

Crossed Product C*-algebras

Given an action $G \curvearrowright X$ on a space X, one can take the quotient

$$X/G$$

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$$X/G$$

If $\alpha : G \curvearrowright A$ is an action of G on a C*-algebra A, then the analogous object to consider is

$$A \rtimes_\alpha G$$

the *crossed product*.

One can think of it like a *semi-direct product* of G with A.

Example

If $\alpha \in \mathrm{Aut}(A)$ has order N, then the crossed product

$$A \rtimes_\alpha \mathbb{Z}_N$$

is the subalgebra of $M_N(A)$ generated by elements of the form

$$\pi(a) := \begin{pmatrix} a & & & \\ & \alpha(a) & & \\ & & \ddots & \\ & & & \alpha^{N-1}(a) \end{pmatrix} \text{ and } U := \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

Once can check that

$$\pi(\alpha(a)) = U\pi(a)U^{-1}$$

The Main Question

Question

If $\alpha : G \to \mathrm{Aut}(A)$ is a group action of a finite group on a C*-algebra, can we estimate

$$tsr(A \rtimes_\alpha G)$$

in terms of $tsr(A)$?

In 2007, Jeong, Osaka, Phillips and Teruya proved that

$$tsr(A \rtimes_\alpha G) \leq tsr(A) + |G| - 1$$

The Main Question

We had two objectives :

- Can we improve this estimate if we impose certain conditions on the action?
- Can we also find estimates for other ‘ranks’? (Topological stable rank is not the only dimension theory for C*-algebras)

Rokhlin Actions

Definition

An action $\alpha : G \curvearrowright A$ is said to have the *Rokhlin property* if, for any finite set $F \subset A$ and any $\epsilon > 0$, there are projections $\{e_g : g \in G\} \subset A$ such that

1. $\sum_{g \in G} e_g = 1_A$ [Partition of Unity]
2. $\alpha_g(e_h) = e_{gh}$ [Permuted by G]
3. $\|e_g a - a e_g\| < \epsilon$ [Approximately commutes with F]

Example: Commutative C*-algebras

If $A = C(X)$, then a projection in A corresponds to a cl-open set in X. Hence, the condition above implies that

X is totally disconnected.
(ie. X has a basis consisting of cl-open sets)

Furthermore, if X is totally disconnected, the Rokhlin property is equivalent to saying that

the action is free.

Example: Noncommutative C*-algebras (Vague)

If A is non-commutative, then A tends to admit a Rokhlin action if

A is low-dimensional.
(ie. A has a lot of projections)

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Not every C*-algebra admits a Rokhlin action of a finite group (for instance, A_θ and $\mathcal{O}_\infty$ do not).

Example: $A = M_{2^\infty}$

Let A be the UHF algebra of type 2^∞

$$A = \bigotimes_{n=1}^{\infty} M_2(\mathbb{C})$$

Let $v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(\mathbb{C})$ and

$$\mathrm{Ad}(v) : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \text{ be given by } a \mapsto vav^*$$

Then

$$\alpha := \bigotimes_{n=1}^{\infty} \mathrm{Ad}(v)$$

is an order 2 automorphism of A, so defines an action

$$\alpha : \mathbb{Z}_2 \curvearrowright M_{2^\infty}$$

This action has the Rokhlin property.

Main Result

Theorem (Anshu, PV (2020))

If $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group on a unital, separable C*-algebra with the Rokhlin property, then

$$tsr(A \rtimes_\alpha G) \leq \left\lceil \frac{tsr(A) - 1}{|G|} \right\rceil + 1$$

Comments on the Theorem

- Analogous inequalities also hold for a number of other 'ranks' for C*-algebras, including
  1. Connected stable rank
  2. General stable rank
  3. Real Rank
- In 2012, Osaka and Phillips proved that, if the action is Rokhlin and $tsr(A) = 1$, then

$$tsr(A \rtimes_\alpha G) = 1$$

  So our theorem is a strengthening of that result.

Thank you!