ROKHLIN DIMENSION FOR GROUP ACTIONS

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ABSTRACT. We discuss a notion of dimension for group actions on C*-algebras, due to Hirshberg, Winter and Zacharias, that allows us to prove permanence properties when passing from the algebra to the crossed product.

CONTENTS

Standing assumption: Unless stated otherwise, all C*-algebras will be unital (denoted by $A, B, C, \ldots$), all topological spaces will be compact and Hausdorff (denoted by $X, Y, Z, \ldots$), and all groups will be finite (denoted by $G, H, \ldots$).

1. MOTIVATION

Definition 1.1. A group action of G on A is a group homomorphism $\alpha : G \to \mathrm{Aut}(A)$

For $g \in G$, we write $\alpha_g := \alpha(g) \in \mathrm{Aut}(A)$. Given a group action $G \curvearrowright_\alpha A$, one constructs a *crossed product* C*-algebra

$$A \rtimes_\alpha G$$

Question: *(Permanence property)* Suppose A satisfies a property (P), then can we impose conditions on α so that that $A \rtimes_\alpha G$ also satisfies (P)?

Some examples of property (P) might be:

(1) Simplicity
(2) Nuclearity/Exactness
(3) Finite nuclear dimension/stable rank/real rank/etc
(4) Stability ($A \otimes \mathcal{K} \cong A$)
(5) Classifiability (by K-theoretic invariants, in the sense of Elliott)

The motivation once again comes from the commutative case.

Definition 1.2. An action of G on X is a group homomorphism $\beta : G \to \mathrm{Homeo}(X)$

Let $G \curvearrowright_\beta X$. For $g \in G$, we write $g \cdot x := \beta(g)(x)$. Given such an action, we get an induced action of G on $C(X)$ by

$$\alpha_g(f)(x) := f(g^{-1} \cdot x)$$

Furthermore, every action of G on $C(X)$ arises in this way.

2. Review of Covering Dimension of a Space

Definition 2.1. An open cover $\mathcal{U}$ of X is said to be *n-decomposable* if there is a decomposition

$$\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \ldots \sqcup \mathcal{U}_n$$

such that each $\mathcal{U}_i$ consists of mutually disjoint sets.

Example 2.2. The following cover of S^1 is 1-decomposable.

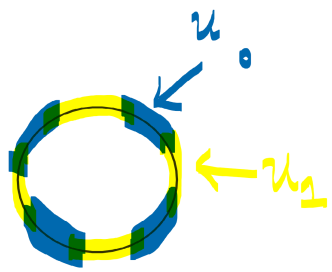

Figure 1. 1-decomposable cover of S^1

One thinks of an n-decomposable cover as a way of covering the space with $(n+1)$ *colours*, where each colour corresponds to a single $\mathcal{U}_i$.

Definition 2.3. The *Lebesgue covering dimension* of X is the least integer n such that every finite open cover $\mathcal{U}$ of X has a finite refinement $\mathcal{V}$ which is n-decomposable. We denote this number by

$$\dim(X)$$

3. Free Group Actions on Spaces

An action $G \curvearrowright_\beta X$ is said to be *free* if, for any $x \in X$ and $g \in G$,

$$g \cdot x = x \Rightarrow g = e$$

Example 3.1.

(1) G acts on itself by left-multiplication (where $G = X$ carries the discrete topology). We denote this action by

$$\lambda : G \to \text{Homeo}(G)$$

(2) $G = \mathbb{Z}_n$ acts on $X = S^1$ by 'rotation by $2\pi/n$'

$$\overline{k} \cdot z := e^{2\pi i k/n} z$$

Definition 3.2. Let $G \curvearrowright_\beta X$ and $\mathcal{U}$ be a cover of X. We say that $\mathcal{U}$ is *n-decomposable with respect to G* if we can write

$$\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \ldots \sqcup \mathcal{U}_n$$

where, for each $0 \leq i \leq n$, each $\mathcal{U}_i$ consists of $|G|$ mutually disjoint sets

$$\mathcal{U}_i = \{V_i^g : g \in G\}$$

such that

$$g \cdot V_i^h = V_i^{gh}$$

In other words, such a cover of X corresponds to a *colouring* of X, where each colour respects the action of G

Lemma 3.3. *If X has a cover that is n-decomposable with respect to G, then the action is free.*

Proof. If $x \in X$, then there exists $0 \leq i \leq n$ and $h \in G$ such that $x \in V_i^h$. Now if $g \in G$ is such that $g \cdot x = x$, then

$$x = g \cdot x \in g \cdot V_i^h = V_i^{gh}$$

If $g \neq e$, then V_i^g and V_i^{gh} are disjoint. Hence, $g = e$ must hold. $\square$

Theorem 3.4. *Let $G \curvearrowright_\beta X$ be a free action. Then, there exists $n \in \mathbb{N}$ such that X has an open cover that is n-decomposable with respect to G.*

We give the idea of the proof by an example.

Example 3.5. Let $X = S^1$ and $G = \mathbb{Z}_6$ acting on X by rotation as above.

(1) Observe that $X/G \cong S^1$

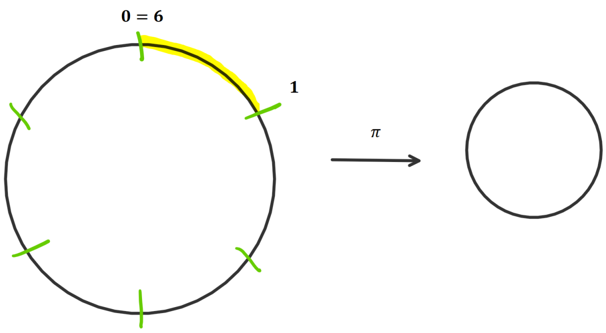

FIGURE 2. Quotient of S^1 by $\mathbb{Z}_6$ action

(2) Start with an open cover of X/G like $\{W_0, W_1\}$ shown below.

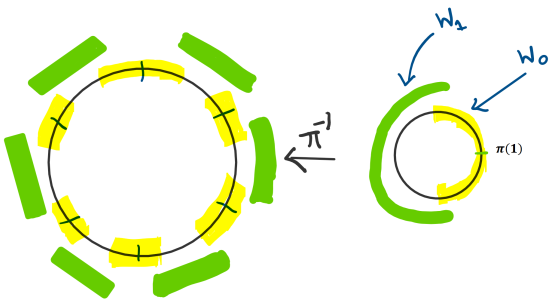

FIGURE 3. 1-decomposable cover of S^1 with respect to $\mathbb{Z}_6$

(3) Lift this cover to get an 1-decomposable cover of X with respect to G.

4. Decomposable Actions on C*-algebras

Let G be a group and $\lambda : G \to \text{Aut}(C(G))$ be the action induced by the left action of G on itself. In other words,

$$\lambda_g(f)(h) := f(g^{-1}h)$$

Let $G \curvearrowright_\alpha A$ and $G \curvearrowright_\beta B$ be two group actions. A linear map $\varphi : A \to B$ is said to be *G-equivariant* if

$$\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$$

for all $a \in A$. Note that if $G \curvearrowright_\alpha A$, then

$$\mathcal{Z}(A) := \{b \in A : ba = ab \quad \forall a \in A\}$$

is G-invariant. So we get an induced action $G \curvearrowright \mathcal{Z}(A)$, which we also denote by α.

Definition 4.1. We say that the group action α is *n-decomposable* if there exist $(n+1)$ linear maps

$$\varphi_0, \varphi_1, \ldots, \varphi_n : C(G) \to \mathcal{Z}(A)$$

such that

(1) Each φ_i is a c.c.p. order zero map.
(2) Each φ_i is G-equivariant.
(3)
$$\sum_{i=0}^{n} \varphi_i(1_{C(G)}) = 1_A$$

Lemma 4.2. *Let $G \curvearrowright_\beta X$ be a group action, and $\alpha : G \to Aut(C(X))$ be the induced action. If α is n-decomposable, then β is free.*

Proof. Let $\varphi_i : C(G) \to C(X)$ be the maps as above. For $g \in G$, define

$$V_i^g := \varphi_i(\delta_g)^{-1}\left((0,1]\right)$$

Then

- $V_i^g \cap V_i^h = \emptyset$ if $g \neq h$, since $\varphi_i(\delta_g) \perp \varphi_i(\delta_h)$.
- $$g \cdot V_i^h = V_i^{gh}$$
  since the φ_i are G-equivariant.
- Furthermore, $\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \ldots \sqcup \mathcal{U}_n$ is a cover for X since
$$\sum_{i=0}^{n} \varphi_i(1_{C(G)}) = 1_{C(X)}$$

Hence, the action is free by 3.3. □

In fact, we have the following theorem.

Theorem 4.3. *Let $G \curvearrowright_\beta X$ be a group action, and $\alpha : G \to Aut(C(X))$ the induced action. Then, α is n-decomposable if and only if β is free.*

Once again, we give an example to illustrate the idea of the proof.

Example 4.4. Let $X = S^1$ and $G = \mathbb{Z}_6$ with the action as in 3.5. We show that this action is 1-decomposable by constructing maps

$$\varphi_0, \varphi_1 : C(\mathbb{Z}_6) \to C(S^1)$$

as above.

Consider the open cover

$$\{W_0, W_1\}$$

of X/G obtained in Step (ii). Choose a partition of unity $\{f_0, f_1\}$ subordinate to this cover. Then, this partition of unity lifts to functions

$$\{f_0^0, f_0^1, \ldots, f_0^5\} \text{ and } \{f_1^0, f_1^1, \ldots, f_1^5\}$$

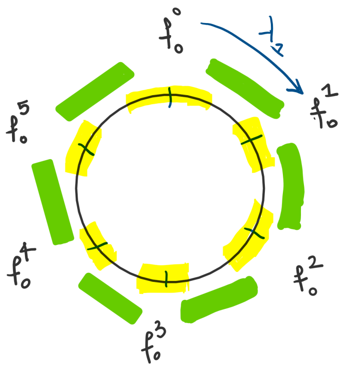

FIGURE 4. Left-action of G permutes the f_0^j

Define $\varphi_i : C(G) \to C(X)$ by

$$\varphi_i(\delta_j) = f_i^j$$

Observe that

- $0 \leq f_i^j \leq 1$. Hence, φ_i is c.c.p.
- $\lambda_k(f_i^j) = f_i^{k+j}$. Hence, φ_i is G-equivariant.
- For $i \in \{0, 1\}$ fixed, we have $f_i^j f_i^k = 0$ if $j \neq k$. Hence, φ_i has order zero.
- $$\sum_{i=0}^{1} \varphi_i(1_{C(G)}) = \sum_{i=0}^{1} \sum_{j=0}^{5} f_i^j = 1_{C(X)}$$

Hence, this action of $\mathbb{Z}_6$ on $C(S^1)$ is 1-decomposable.

Hence, the notion of n-decomposability can be thought of as a generalization of 'freeness' of an action on space.

5. The Structure of the Crossed Product

The goal of this section is to understand what this condition of n-decomposability implies for the structure of the crossed product. Our goal is to factor the identity map

$$\begin{tikzcd} A \rtimes_\alpha G \arrow[rr, "\mathrm{id}"] \arrow[dr, "\widetilde{\psi}"'] & & A \rtimes_\alpha G \\ & B \arrow[ur, "\widetilde{\varphi}"'] & \end{tikzcd}$$

where B is a 'nice' C*-algebra and $\widetilde{\psi}$ and $\widetilde{\varphi}$ are $*$-homomorphisms. We will do this in the following steps:

(1) Since α is n-decomposable, we have $(n+1)$ c.c.p. order zero maps

$$\varphi_0, \varphi_1, \ldots, \varphi_n : C(G) \to \mathcal{Z}(A)$$

as above.

By a structure theorem for order zero maps [Winter and Zacharias, 2009], we get $*$-homomorphisms

$$\rho_i : C_0[0,1) \otimes C(G) \to \mathcal{Z}(A) \text{ such that } \mathrm{id}_{C[0,1)} \otimes a \mapsto \varphi_i(a)$$

Furthermore, these maps ρ_i are G-invariant where the action of G on $C_0[0,1) \otimes C(G)$ is given by

$$g \cdot (f \otimes a) := f \otimes \lambda_g(a)$$

(2) Since the maps ρ_i have commuting ranges, we get a single G-equivariant $*$-homomorphism

$$\rho : C \to \mathcal{Z}(A)$$

where $C = [C_0[0,1) \otimes C(G)]^{\otimes(n+1)}$.

(3) If $\eta_i : C_0[0,1) \otimes C(G) \to C$ are the inclusion maps (for $0 \leq i \leq n$), then for

$$x := \sum_{i=0}^{n} \eta_i(\mathrm{id}_{C[0,1)} \otimes 1_{C(G)}) \in C$$

we have $\rho(x) = 1$. Therefore, if I is the ideal of C generated by elements of the form $\{xf - f : f \in C\}$, then ρ induces a map

$$\overline{\rho} : D \to \mathcal{Z}(A)$$

where $D := C/I$ is a unital and commutative C*-algebra. Furthermore, D is of the form

$$D \cong C(Y) \otimes C(G)$$

for some compact metric space Y with $\dim(Y) \leq n$, and where the action of G on the $C(G)$ component is by λ.

(4) Tensoring with A, we get a G-equivariant map

$$\widehat{\rho} : D \otimes A \to \mathcal{Z}(A) \otimes A$$

(5) Now, the map

$$\theta : \mathcal{Z}(A) \otimes A \to A \text{ given by } x \otimes a \mapsto xa$$

is a G-equivariant $*$-homomorphism. Composing, we get a map

$$\widetilde{\rho} : D \otimes A \to A$$

Finally, the condition $\sum_{i=0}^{n} \varphi_i(1_{C(G)}) = 1_A$ implies that, for all $a \in A$,

$$\widetilde{\rho}(1_D \otimes a) = a$$

(6) So we have a diagram of G-equivariant $*$-homomorphisms

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi"'] & & A \\ & D \otimes A \arrow[ur, "\widetilde{\rho}"'] & \end{tikzcd}$$

where $\psi(a) = 1_D \otimes a$.

(7) Finally, we need one more fact. If $G \curvearrowright_\gamma E$, then the tensor product $C(G) \otimes E$ carries two different actions of G,

$$\lambda \otimes \text{id and } \lambda \otimes \gamma$$

It turns out, both crossed products are the same

$$(C(G) \otimes E) \rtimes_{\lambda \otimes \gamma} G \cong (C(G) \otimes E) \rtimes_{\lambda \otimes \mathrm{id}} G \cong M_{|G|}(E)$$

In other words, tensoring with $(C(G), \lambda)$ kills the dynamical structure!

Since

$$D \otimes A \cong C(Y) \otimes C(G) \otimes A$$

where the action on $C(G)$ is by λ, we conclude that

$$(D \otimes A) \rtimes G \cong C(Y) \otimes M_{|G|}(A)$$

Hence, we arrive at the following theorem

Theorem 5.1. *Let $\alpha : G \to Aut(A)$ be an n-decomposable action. Then, there is a commuting diagram*

$$\begin{tikzcd}
A \rtimes_\alpha G \arrow[rr, "id"] \arrow[dr, "\widetilde{\psi}"'] & & A \rtimes_\alpha G \\
& C(Y) \otimes M_{|G|}(A) \arrow[ur, "\widetilde{\varphi}"'] &
\end{tikzcd}$$

where Y is a compact metric space with $\dim(Y) \leq n$.

This allows us to understand properties of $A \rtimes_\alpha G$ by 'factoring through' $C(Y) \otimes M_{|G|}(A)$.

Corollary 5.2. *Let (P) denote one of the following properties:*

(1) Finite rank (stable/real/decomposition rank/nuclear dimension)
(2) Stability
(3) Classifiability (ie. Simple, separable, and satisfying the UCT)
(4) ... etc.

If A satisfies property (P) and α is n-decomposable, then $A \rtimes_\alpha G$ also satisfies property (P).

6. Approximately Decomposable Actions and Rokhlin Dimension

Unfortunately, requiring that an action is n-decomposable is too restrictive in the non-commutative case. Therefore, one defines an approximate version of it.

Definition 6.1. An action $\alpha : G \to \mathrm{Aut}(A)$ is said to be *approximately n-decomposable* if, for every finite set $F \subset A$ and every $\epsilon > 0$, there are $(n+1)$ linear maps

$$\varphi_0, \varphi_1, \ldots, \varphi_n : C(G) \to A$$

such that

(1) Each φ_i is a c.c.p. order zero map.
(2) Each φ_i is 'approximately equivariant'.

$$\|\alpha_g(\varphi_i(f)) - \varphi_i(\lambda_g(f))\| < \epsilon \quad \forall g \in G, f \in C(G)$$

(3) Each φ_i is 'approximately central'

$$\|\varphi_i(f)a - a\varphi_i(f)\| < \epsilon \quad \forall a \in F, \text{ and } f \in C(G)$$

$$\|\varphi_i(f_1)\varphi_j(f_2) - \varphi_j(f_2)\varphi_i(f_1)\| < \epsilon \quad \forall f_1, f_2 \in C(G), \text{ and } 0 \leq i, j \leq n$$

(4) The $\{\varphi_i\}$ are an 'approximate partition of unity'.

$$\|\sum_{i=0}^{n} \varphi_i(1_{C(G)}) - 1_A\| < \epsilon$$

Definition 6.2. [Hirshberg, Winter, and Zacharias, 2012] The *Rokhlin dimension (with commuting towers)* of an action $\alpha : G \to \mathrm{Aut}(A)$ is the least value of $n \in \mathbb{N}$ such that α is approximately n-decomposable. We denote the integer by

$$\dim_{Rok}^{c}(\alpha)$$

Example 6.3. We give some examples without proof.

(1) [Phillips, 2017, Example 13.6] Let $A := \bigotimes_{n=1}^{\infty} M_2(\mathbb{C})$ be the UHF algebra of type 2^{∞} and $\alpha \in \mathrm{Aut}(A)$ be the order two automorphism given by

$$\alpha = \bigotimes_{n=1}^{\infty} \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then α induces an action of $\mathbb{Z}_2$ on A such that $\dim_{Rok}^{c}(\alpha) = 0$.

(2) [Hirshberg and Phillips, 2015, Example 1.12] Let $\theta \in \mathbb{R}$ be irrational, and $A = A_\theta$ be the corresponding irrational rotation algebra generated by unitaries $\{u, v\}$ such that

$$uv = e^{2\pi i\theta} vu$$

Let $\alpha \in \mathrm{Aut}(A)$ be the order two automorphism satisfying

$$\alpha(v) = v \text{ and } \alpha(u) = -u$$

Then, α induces an action of $\mathbb{Z}_2$ with the property that $\dim_{Rok}^{c}(\alpha) = 1$

> Note that both actions above are not n-decomposable as in 4.1 because the underlying algebras are simple, and so have trivial centers.

(3) [Gardella, 2014, Theorem 4.1] However, if A is commutative and $\alpha : G \to \mathrm{Aut}(A)$ is an action such that

$$n := \dim_{Rok}^{c}(\alpha) < \infty$$

then α is n-decomposable in the sense of 4.1.

(4) [Hirshberg and Phillips, 2015, Lemma 1.20] Let $G \curvearrowright_\alpha A$. If there exists $e \neq h \in G$ and $u \in U(A)$ a unitary such that

$$\alpha_h(a) = uau^* \quad \forall a \in A$$

Then $\dim_{Rok}^{c}(\alpha) = +\infty$.

The following is now an analog of 5.1.

> **Theorem 6.4.** *[Gardella, Hirshberg, and Santiago, 2017] If*
>
> $$\dim_{Rok}^{c}(\alpha) < \infty$$
>
> *Then, there is an approximate/local analog of 5.1.*

Many of the properties of C*-algebras listed above are defined in terms of approximations. Hence, we get the following corollary.

Corollary 6.5. *Let (P) denote one of the following properties:*

(1) Finite rank (stable/real/decomposition rank/nuclear dimension)
(2) Stability

(3) Simple, separable, and satisfying the UCT
(4) ... etc.

If A satisfies property (P) and

$$\dim_{Rok}^{c}(\alpha) < \infty$$

then $A \rtimes_\alpha G$ also satisfies property (P).

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