Rokhlin Dimension and Equivariant Bundles

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Contents

Preliminaries

Assumptions

Unless stated otherwise,

- All C*-algebras will be unital and separable (denoted $A, B, C, \ldots$)
- All topological spaces will be compact, Hausdorff (denoted $X, Y, Z, \ldots$)
- All groups will be finite (denoted $G, H, \ldots$)

Motivation

A group action of G on A is a group homomorphism

$$\alpha : G \to \mathrm{Aut}(A)$$

Given such an action, one constructs a *crossed product* C*-algebra

$$A \rtimes_\alpha G$$

Question: Permanence

Suppose A satisfies a property (P), then can we impose conditions on α so that $A \rtimes_\alpha G$ also satisfies property (P)?

Motivation

Examples of (P) include

1. Simplicity
2. Nuclearity/Exactness
3. Finite nuclear dimension/stable rank/real rank
4. Stability ($A \otimes \mathcal{K} \cong A$)
5. Classifiability (by K-theoretic invariants)

The motivation comes from the commutative case.

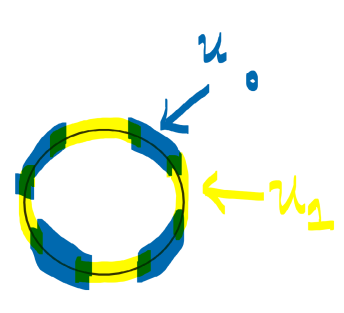
Group Actions on Spaces

Covering Dimension of a Space

Definition

Let $n \in \mathbb{N}$. A finite open cover $\mathcal{U}$ of X is said to be *n-decomposable* if there is a decomposition $\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \ldots \sqcup \mathcal{U}_n$ such that each $\mathcal{U}_i$ consists of mutually disjoint sets.

The following cover of S^1 is 1-decomposable.



Covering Dimension of a Space

One thinks of an n-decomposable cover as a way of covering the space with $(n+1)$ *colours*, where each colour corresponds to a single $\mathcal{U}_i$.

Definition

The *Lebesgue covering dimension* of X is the least integer n such that every finite open cover $\mathcal{U}$ of X has a finite refinement $\mathcal{V}$ which is n-decomposable. We denote this number by

$$\dim(X)$$

All spaces in this talk will be assumed to have finite dimension.

Group Actions on Spaces

A group action of G on X is a group homomorphism

$$\beta : G \to \text{Homeo}(X)$$

Given such an action, we get an induced action of $\alpha : G \to \text{Aut}(C(X))$ by

$$\alpha_g(f)(x) := f(\beta_{g^{-1}}(x))$$

Furthermore, every action of G on $C(X)$ arises this way.

Free Group Actions on Spaces

An action $G \curvearrowright_{\beta} X$ is said to be *free* if, for any $x \in X$ and $g \in G$,

$$g \cdot x = x \Rightarrow g = e$$

Some examples include:

1. G acts on itself by left-multiplication (where $G = X$ carries the discrete topology). We denote this action by

$$\lambda : G \to \mathrm{Homeo}(G)$$

2. $G = \mathbb{Z}_n$ acts on $X = S^1$ by 'rotation by $2\pi/n$'

$$\overline{k} \cdot z := e^{2\pi i k/n} z$$

Free Group Actions on Spaces

Definition

Let $G \curvearrowright_\beta X$, $\mathcal{U}$ be a finite open cover of X, and $n \in \mathbb{N}$. We say that $\mathcal{U}$ is *n-decomposable with respect to G* if we can write $\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \ldots \sqcup \mathcal{U}_n$ where, for each $0 \leq i \leq n$, each $\mathcal{U}_i$ consists of $|G|$ mutually disjoint sets

$$\mathcal{U}_i = \{V_i^g : g \in G\}$$

such that

$$g \cdot V_i^h = V_i^{gh}$$

In other words, such a cover of X corresponds to a *colouring* of X, where each colour respects the action of G.

Free Group Actions on Spaces

Lemma 1

If X has a cover that is n-decomposable with respect to G, then the action is free.

Proof.

If $x \in X$, then there exists $0 \leq i \leq n$ and $h \in G$ such that $x \in V_i^h$. Now if $g \in G$ is such that $g \cdot x = x$, then

$$x = g \cdot x \in g \cdot V_i^h = V_i^{gh}$$

If $g \neq e$, then V_i^g and V_i^{gh} are disjoint. Hence, $g = e$ must hold. □

Free Group Actions on Spaces

Theorem

Let $G \curvearrowright_\beta X$ be a free action. Then, there exists $n \in \mathbb{N}$ and an open cover $\mathcal{U}$ of X that is n-decomposable with respect to G.

Decomposable Actions on C*-Algebras

Completely Positive maps

Given a linear map $\varphi : A \to B$ between two C*-algebras, we obtain a linear map

$$\varphi^{(n)} : M_n(A) \to M_n(B)$$

given by

$$(a_{i,j}) \mapsto (\varphi(a_{i,j}))$$

Definition

A linear map $\varphi : A \to B$ is said to be *completely positive* if $\varphi^{(n)}$ is positive for each $n \in \mathbb{N}$.

A *c.c.p.* map is a contractive, completely positive map.

Order Zero Maps

Two elements $a, b \in A$ are said to be *orthogonal* (in symbols, $a \perp b$) if

$$ab = a^* b = ab^* = ba = 0$$

Definition

A c.c.p. map $\varphi : A \to B$ is said to have *order zero* if, for any $a, b \in A$,

$$a \perp b \Rightarrow \varphi(a) \perp \varphi(b)$$

Order Zero Maps

1. Any $*$-homomorphism has order zero.
2. If $\pi : A \to B$ is a $*$-homomorphism and $h \in \pi(A)' \cap B$ is a positive element, then

$$a \mapsto h\pi(a)$$

is an order zero map.

Theorem (Winter and Zacharias, 2009)

Every c.c.p. order zero map has the form of Example 2.

Furthermore, there is a one-to-one correspondence between c.c.p. order zero maps $\varphi : A \to B$ and $*$-homomorphisms

$$\pi_\varphi : C_0[0,1) \otimes A \to B$$

given by $\varphi(a) = \pi_\varphi(\mathrm{id}_{C_0[0,1)} \otimes a)$.

Equivariant Map

Given a group action $G \curvearrowright_\alpha A$, the centre

$$\mathcal{Z}(A) = \{a \in A : ab = ba \quad \forall b \in A\}$$

is G-invariant, so we get an induced action $G \curvearrowright_\alpha \mathcal{Z}(A)$.

Definition

A linear map $\varphi : C(G) \to \mathcal{Z}(A)$ is said to be G-equivariant if

$$\varphi(\lambda_g(f)) = \alpha_g(\varphi(f))$$

for all $f \in C(G)$.

Recall that $\lambda : G \to \mathrm{Aut}(C(G))$ is given by

$$\lambda_g(f)(h) := f(g^{-1}h)$$

Decomposable Actions on C*-algebras

Definition 1

Let $n \in \mathbb{N}$. We say that a group action $G \curvearrowright_\alpha A$ is *n-decomposable* if there exists $(n+1)$ maps

$$\varphi_0, \varphi_1, \ldots, \varphi_n : C(G) \to \mathcal{Z}(A)$$

which are G-equivariant, c.c.p., have order zero, and satisfy

$$\varphi_0(1_{C(G)}) + \varphi_1(1_{C(G)}) + \ldots + \varphi_n(1_{C(G)}) = 1_A$$

We say that α is *decomposable* if it is n-decomposable for some natural number $n \in \mathbb{N}$.

Decomposable $\Rightarrow$ Free

Lemma 2

Let $G \curvearrowright_\beta X$ be a group action and $\alpha : G \to \mathrm{Aut}(C(X))$ be the induced action. If α is decomposable, then β is free.

Proof.

Let $\varphi_i : C(G) \to C(X)$ be the maps as above. For $g \in G$, define

$$V_i^g := \varphi_i(\delta_g)^{-1}((0, +\infty))$$

and set $\mathcal{U}_i := \{V_i^g : g \in G\}$. Then one can verify that

$$\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \ldots \sqcup \mathcal{U}_n$$

is n-decomposable with respect to G. So β is free by Lemma 1. $\square$

Decomposable $\Leftrightarrow$ Free

Theorem (Gardella, 2014)

Let $G \curvearrowright_\beta X$ be a group action and $\alpha : G \to \mathrm{Aut}(C(X))$ be the induced action. Then, α is decomposable if and only if β is free.

Rokhlin Dimension

Unfortunately, requiring that an action is decomposable is too restrictive in the noncommutative case. Therefore, one defines an approximate version of it.

Definition

Let $n \in \mathbb{N}$. An action $\alpha : G \to \mathrm{Aut}(A)$ is said to be *approximately n-decomposable* if, for every finite set $F \subset A$ and every $\epsilon > 0$, there are $(n+1)$ c.c.p. order zero linear maps

$$\varphi_0, \varphi_1, \ldots, \varphi_n : C(G) \to A$$

satisfying the following conditions:

Approximately Decomposable Actions

1. Each φ_i is 'approximately equivariant'.

$$\|\alpha_g(\varphi_i(\delta_h)) - \varphi_i(\lambda_g(\delta_h))\| < \epsilon \quad \forall g, h \in G$$

2. Each φ_i is 'approximately central'

$$\|\varphi_i(\delta_h)a - a\varphi_i(\delta_h)\| < \epsilon \quad \forall a \in F, \text{ and } h \in G$$

$$\|\varphi_i(\delta_g)\varphi_j(\delta_h) - \varphi_j(\delta_h)\varphi_i(\delta_g)\| < \epsilon \quad \forall h, g \in G, 0 \leq i, j \leq n$$

3. The $\{\varphi_i\}$ are an 'approximate partition of unity'.

$$\|\sum_{i=0}^{n} \varphi_i(1_{C(G)}) - 1_A\| < \epsilon$$

Definition (Hirshberg, Winter, and Zacharias, 2015)

The *Rokhlin dimension (with commuting towers)* of an action $\alpha : G \to \mathrm{Aut}(A)$ is the least value of $n \in \mathbb{N}$ such that α is approximately n-decomposable. We denote the integer by

$$\dim_{Rok}^{c}(\alpha)$$

Note that if $\dim_{Rok}^{c}(\alpha) = 0$, then α has the *Rokhlin property*.

Example 1 (Izumi, 2004)

Let $A := \bigotimes_{n=1}^{\infty} M_2(\mathbb{C})$ be the UHF algebra of type 2^∞ and $\alpha \in \mathrm{Aut}(A)$ be the order two automorphism given by

$$\alpha = \bigotimes_{n=1}^{\infty} \mathrm{Ad} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then α induces an action of $\mathbb{Z}_2$ on A such that $\dim_{Rok}^{c}(\alpha) = 0$.

Example 2 (Hirshberg and Phillips, 2015)

Let $\theta \in \mathbb{R}$ be irrational, and $A = A_\theta$ be the corresponding irrational rotation algebra generated by unitaries $\{u, v\}$ such that

$$uv = e^{2\pi i\theta} vu$$

Let $\alpha \in \mathrm{Aut}(A)$ be the order two automorphism satisfying

$$\alpha(v) = v \text{ and } \alpha(u) = -u$$

Then, α induces an action of $\mathbb{Z}_2$ with the property that $\dim^c_{Rok}(\alpha) = 1$.

Note that both actions above are not decomposable in the earlier sense because the underlying algebras are simple, and so have trivial centers.

Example 3 (Gardella, 2014)

However, if $G \curvearrowright_\beta X$ is a group action and $\alpha : G \to \mathrm{Aut}(C(X))$ is the induced action, then the following are equivalent:

- $\dim^c_{Rok}(\alpha) < +\infty$
- α is decomposable in the sense of Definition 1.
- β is free.

Consequences of Finite Rokhlin Dimension

Theorem (Gardella, Hirshberg, and Santiago, 2021)

Let (P) denote one of the following properties:

1. Finite rank (stable/real rank/nuclear dimension)
2. Stability
3. Nuclear, separable, and satisfying the UCT
4. ... etc.

If A satisfies property (P) and

$$\dim_{Rok}^{c}(\alpha) < \infty$$

then $A \rtimes_\alpha G$ also satisfies property (P).

Equivariant Bundles

Vector Bundles

In what follows,

- X will denote a compact metric space.
- $p : E \to X$ will be a locally trivial, complex vector bundle, endowed with a fixed hermitian metric. The fibers of E, denoted by $\{E_x : x \in X\}$, are finite dimensional Hilbert spaces.

We write

$$\Gamma(E) := \{\xi : X \to E \text{ continuous, such that } p \circ \xi = \mathrm{id}_X\}$$

for the continuous sections of (E, p, X).

Vector Bundles

Given $\xi \in \Gamma(E)$ and $f \in C(X)$, we may write

$$(f \cdot \xi)(x) := f(x)\xi(x) = (\xi \cdot f)(x)$$

This gives a central action of $C(X)$ on $\Gamma(E)$, so $\Gamma(E)$ is a $C(X)$-module.

Theorem (Serre-Swan)

$\Gamma(E)$ is a finitely generated, projective module over $C(X)$. Furthermore, every finitely generated, projective module over $C(X)$ has this form.

Furthermore, the hermitian metric on E gives $\Gamma(E)$ the structure of a *Hilbert $C(X)$-bimodule*.

The Cuntz-Pimsner Algebra of a Vector Bundle

Fact

Using the Hilbert $C(X)$-bimodule $\Gamma(E)$, one can associate a C*-algebra,

$$\mathcal{O}_E$$

called the Cuntz-Pimsner algebra associated to the vector bundle (E, p, X).

The Cuntz-Pimsner Algebra of a Vector Bundle

Examples:

- If $X = \{*\}$ is a point, then $\mathcal{O}_E \cong \mathcal{O}_n$, the usual Cuntz algebra (Here, $n = \dim(E)$ as a vector space).
- More generally, $\mathcal{O}_E$ is a locally trivial unital $C(X)$-algebra, each of whose fibers are of the form $\mathcal{O}_{n(x)}$, where $n : E \to \mathbb{Z}$ is the rank function of E.

Group Actions on Bundles

An action of a group G on a vector bundle (E, p, X) is a pair

$$\widetilde{\alpha} : G \to \text{Homeo}(X), \text{ and } \widetilde{\gamma} : G \to \text{Homeo}(E)$$

such that

- $p : E \to X$ is G-equivariant.
- For each $s \in G$, the map $E_x \to E_{\widetilde{\alpha}_s(x)}$ is a linear map of vector spaces.

Group Actions on the Cuntz-Pimsner Algebra

Theorem

Given a group action $(\widetilde{\alpha}, \widetilde{\gamma})$ of G on (E, p, X), there is an induced action

$$\beta : G \to \mathrm{Aut}(\mathcal{O}_E)$$

satisfying certain natural properties.

Main Result

Theorem (Vaidyanathan, 2020)

Let $(\widetilde{\alpha}, \widetilde{\gamma})$ be an action of G on (E, p, X) and let

$$\beta : G \to \mathrm{Aut}(\mathcal{O}_E)$$

be the induced action on the corresponding Cuntz-Pimsner algebra.

- If $\widetilde{\alpha}$ is free, then

$$\dim^c_{Rok}(\beta) \leq \dim(X/G)$$

- If $\widetilde{\alpha}$ is trivial, then

$$\dim_{Rok}(\beta) \leq 2\dim(X) + 1$$

Consequences

Suppose that $\dim(X) < \infty$ and the action $\widetilde{\alpha}$ is free or trivial, then

- $\mathcal{O}_E \rtimes_\beta G$ has finite nuclear dimension.
- $\mathcal{O}_E \rtimes_\beta G$ is classifiable by K-theoretic invariants.

If $\widetilde{\alpha}$ is free, then

- $\mathcal{O}_E \rtimes_\beta G$ has finite stable rank, real rank, etc.

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Thank you!