NUCLEAR DIMENSION FOR C*-ALGEBRAS

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ABSTRACT. We discuss a notion of dimension for C*-algebras, due to Winter and Zacharias, that generalizes the Lebesgue covering dimension for topological spaces.

CONTENTS

Standing assumption: All C*-algebras will be unital (denoted by $A, B, C, \ldots$), and all topological spaces will be compact and Hausdorff (denoted by $X, Y, Z, \ldots$).

1. Completely Positive Maps

Definition 1.1. An element $a \in A$ is said to be *positive* if there exists $x \in A$ such that $a = x^*x$. Equivalently, a is self-adjoint and $\sigma(a) \subset [0, \infty)$.

We write A_+ for the set of all positive elements of A.

Definition 1.2. If A is a C*-algebra, so is $M_n(A)$. Given any linear map $\varphi : A \to B$, we may define

$$\varphi^{(n)} : M_n(A) \to M_n(B) \text{ given by } (a_{i,j}) \mapsto (\varphi(a_{i,j})$$

Note that $\varphi^{(n)}$ is also a linear map.

Definition 1.3. Let $\varphi : A \to B$ be a linear map (not necessarily a $*$-homomorphism). We say that

(1) φ is *unital* if $\varphi(1_A) = 1_B$.
(2) φ is *positive* if $\varphi(A_+) \subset B_+$.
(3) φ is *completely positive* if $\varphi^{(n)}$ is positive for each $n \in \mathbb{N}$.
(4) φ is *contractive* if $\|\varphi\| \leq 1$.
(5) φ is *u.c.p.* if it is unital and completely positive.
(6) φ is *c.c.p.* (or c.p.c.) if it is contractive and completely positive.

Example 1.4.

(1) Every $*$-homomorphism is positive and contractive. Furthermore, if φ is a $*$-homomorphism, then so is $\varphi^{(n)}$. Hence, every $*$-homomorphism is c.c.p.
(2) If $\varphi : A \to B$ is a $*$-homomorphism, and $x \in B$ is any element. Define $\psi : A \to B$ by

$$\psi(a) := x^*\varphi(a)x$$

Then ψ is not necessarily multiplicative, but

(a) ψ is positive: If $a = y^*y$, then $\psi(a) = (\varphi(y)x)^*(\varphi(y)x)$.
(b) For $n \in \mathbb{N}$, write

$$X = \text{diag}(x, x, \ldots, x) \in M_n(B)$$

Then

$$\psi^{(n)}(T) = X^*\varphi^{(n)}(T)X$$

Hence, ψ is completely positive.

(3) The 'transpose' map $M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is positive, but not completely positive.

Lemma 1.5. *Let $\varphi : A \to B$ be a positive map. If A is commutative, then φ is completely positive.*

Example 1.6. Let $\mathcal{U} = \{U_1, U_2, \ldots, U_k\}$ be a collection of open subsets of a space X (not necessarily a cover). Let $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ be a partition of unity subordinate to $\mathcal{U}$. Define $F(\mathcal{U}) = \mathbb{C}^k$ and define

$$\varphi_{\mathcal{U}} : F(\mathcal{U}) \to C(X)$$

given by

$$(c_1, c_2, \ldots, c_k) \mapsto \sum_{j=1}^{k} c_k\sigma_j$$

This map is clearly positive (a positive element of $\mathbb{C}^k$ is one whose entries are all positive real numbers). By Lemma 1.5, it is completely positive. Furthermore,

$$\|\sigma_j\|_\infty \leq 1$$

for each $1 \leq j \leq k$, so $\varphi_{\mathcal{U}}$ is c.c.p.

2. Nuclear C*-algebras

Definition 2.1. A linear map $\theta : A \to B$ is said to be *nuclear* if there exist c.c.p maps

$$\varphi_n : A \to M_{k(n)}(\mathbb{C}) \text{ and } \psi_n : M_{k(n)}(\mathbb{C}) \to B$$

such that

$$\lim_{n\to\infty} \|\psi_n \circ \varphi_n(a) - \theta(a)\| = 0$$

Lemma 2.2. *For a map $\theta : A \to B$, the following are equivalent:*

(1) θ is nuclear.
(2) There are finite dimensional C-algebras C_n and c.c.p. maps $\varphi_n : A \to C_n$ and $\psi_n : C_n \to B$ such that*

$$\lim_{n\to\infty} \|\psi_n \circ \varphi_n(a) - \theta(a)\| = 0$$

for all $a \in A$.

(3) For any finite set $G \subset A$ and $\epsilon > 0$, there is a finite dimensional C-algebra C and c.c.p. maps*

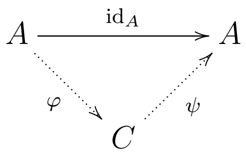
$$\varphi : A \to C \text{ and } \psi : C \to B$$

such that $\|\psi \circ \varphi(a) - \theta(a)\| < \epsilon \quad \forall a \in G$.

Definition 2.3. A C*-algebra A is said to be *nuclear* if $\text{id}_A : A \to A$ is a nuclear map. Equivalently, for every finite set $G \subset A$ and $\epsilon > 0$, there exists a finite dimensional C*-algebra C and c.c.p. maps $\varphi : A \to C$ and $\psi : C \to A$ such that

$$\|\psi \circ \varphi(a) - a\| < \epsilon \quad \forall a \in G$$

We represent this by a diagram that "approximately (G, ϵ)-commutes"



Theorem 2.4. *$C(X)$ is nuclear.*

Proof. Let $G \subset A$ be a finite set, and $\epsilon > 0$. Choose an open cover $\mathcal{U} = \{U_1, U_2, \ldots, U_m\}$ and points $\lambda_i \in U_i$ such that, for any $f \in A$,

$$|f(x) - f(\lambda_i)| < \epsilon \quad \forall x \in U_i$$

Let $C := F(\mathcal{U}) = \mathbb{C}^m$ and define $\varphi : A \to C$ by

$$\varphi(f) := (f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_m))$$

Then φ is a $*$-homomorphism, so it is c.c.p.

Let $\{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ be a partition of unity subordinate to $\mathcal{U}$ and define $\psi : C \to A$ by $\psi = \varphi_{\mathcal{U}}$. In other words,

$$\psi(c_1, c_2, \ldots, c_m) = \sum_{i=1}^{m} c_i \sigma_i$$

Then ψ is c.c.p. by Example 1.6. Finally, if $f \in G$, then

$$\begin{aligned}
\|\psi(\varphi(f)) - f\| &= \sup_{x \in X} |\psi((f(\lambda_i)) - f(x)| \\
&= \sup_{x \in X} \left| \sum_{i=1}^{m} f(\lambda_i)\sigma_i(x) - f(x) \right| \\
&= \sup_{x \in X} \left| \sum_{i=1}^{m} f(\lambda_i)\sigma_i(x) - \sum_{i=1}^{m} \sigma_i(x) f(x) \right| \\
&\leq \sup_{x \in X} \sum_{i=1}^{m} |\sigma_i(x)(f(\lambda_i) - f(x))| \\
&< \epsilon \sup_{x \in X} \sum_{i=1}^{m} |\sigma_i(x)| \\
&= \epsilon
\end{aligned}$$

This completes the proof. □

3. Covering Dimension of Spaces

Consider Example 1.6 more carefully. Given an open cover $\mathcal{U}$ of X, we get a map

$$\varphi_{\mathcal{U}} : F(\mathcal{U}) \to C(X)$$

Now suppose $\mathcal{U} = \mathcal{U}_1 \sqcup \mathcal{U}_2$ for two disjoint subsets of $\mathcal{U}$. Then we may write

$$F(\mathcal{U}) = F(\mathcal{U}_1) \oplus F(\mathcal{U}_2)$$

and

$$\varphi_{\mathcal{U}}|_{F(\mathcal{U}_i)} = \varphi_{\mathcal{U}_i}$$

Definition 3.1. [Kirchberg and Winter, 2004, Definition 1.4] An open cover $\mathcal{U}$ of X is said to be *n-decomposable* if there is a decomposition

$$\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \ldots \sqcup \mathcal{U}_n$$

such that each $\mathcal{U}_i$ consists of mutually disjoint sets.

Example 3.2.

(1) If X is a finite set and $\mathcal{U}$ consists of singleton sets, then $\mathcal{U}$ is 0-decomposable.
(2) If $X = [0, 1]$ and $\mathcal{U} = \{[0, 1/2), (1/4, 3/4), (1/2, 1]\}$, then with

$$\mathcal{U}_0 := \{[0, 1/2), (1/2, 1]\} \text{ and } \mathcal{U}_1 := \{(1/4, 3/4)\}$$

we see that $\mathcal{U}$ is 1-decomposable.
(3) The following cover of S^1 is 1-decomposable.

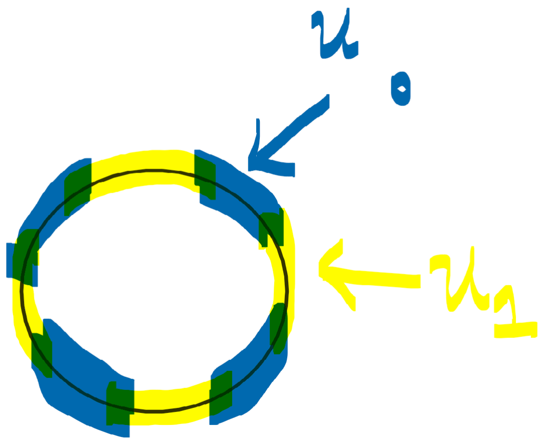


Figure 1. 1-decomposable cover of S^1

Note: One thinks of an n-decomposable cover as a way of covering the space with $(n+1)$ *colours*, where each colour corresponds to a single $\mathcal{U}_i$.

Definition 3.3. The *Lebesgue covering dimension* of X is the least integer n such that every finite open cover $\mathcal{U}$ of X has a finite refinement $\mathcal{V}$ which is n-decomposable. We denote this number by

$$\dim(X)$$

Example 3.4.

(1) If X is finite, then $\dim(X) = 0$
(2) If $X = [0, 1]$ or $X = S^1$, then $\dim(X) = 1$
(3) If $X = [0, 1]^m$, then $\dim(X) = m$
(4) If X is a manifold, then $\dim(X)$ coincides with its manifold dimension.

Remark 3.5. Suppose $n = \dim(X)$, and suppose we are given a finite subset $G \subset C(X)$ and $\epsilon > 0$. In the proof of Theorem 2.4, we may choose a refinement of the original cover to assume that the cover $\mathcal{U}$ is itself n-decomposable. Hence, we write

$$\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \ldots \sqcup \mathcal{U}_n$$

so that we have

$$F(\mathcal{U}) = F(\mathcal{U}_0) \oplus F(\mathcal{U}_1) \oplus \ldots \oplus F(\mathcal{U}_n)$$

Now consider the maps

$$\psi = \varphi_{\mathcal{U}_0} : F(\mathcal{U}_0) \to C(X)$$

given by

$$\psi(c_1, c_2, \ldots, c_k) = \sum_{i=1}^{k} c_i \sigma_i$$

where $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ is a collection of positive functions such that

(1) $0 \leq \sigma_i \leq 1$
(2) If $i \neq j$, then $\sigma_i \sigma_j = 0$.

Hence, for any $a, b \in F(\mathcal{U}_0)$,

$$ab = 0 \Rightarrow \psi(a)\psi(b) = 0$$

This condition captures the fact that $\mathcal{U}_0$ is made of mutually disjoint sets.

4. Nuclear Dimension

Definition 4.1.

(1) For any two elements $a, b \in A$, we say that a and b are *orthogonal* if

$$ab = a^*b = ab^* = ba = 0$$

If this happens, we write $a \perp b$.

(2) A c.p. map $\theta : A \to B$ is said to have *order zero* if, for any $a, b \in A$,

$$a \perp b \Rightarrow \varphi(a) \perp \varphi(b)$$

In other words, φ preserves orthogonality.

Example 4.2.

(1) Any $*$-homomorphism has order zero.
(2) If $\pi : A \to B$ is a $*$-homomorphism, and $h \in \pi(A)'$ is a positive element, then the map

$$\varphi : A \to B \text{ given by } a \mapsto h\pi(a)$$

is an order zero map. (In fact, every order zero map has this form [Winter and Zacharias, 2009])

(3) Let $\mathcal{U}$ be a collection of open sets in X, and consider the map

$$\varphi_{\mathcal{U}} : F(\mathcal{U}) \to C(X)$$

as in Example 1.6. If members of $\mathcal{U}$ are mutually disjoint, then $\varphi_{\mathcal{U}}$ is an order zero map.

Definition 4.3. A c.p. map $\varphi : A \to B$ is said to be n-*decomposable* if A can be expressed as a direct sum

$$A = A_0 \oplus A_1 \oplus \ldots \oplus A_n$$

such that $\varphi|_{A_i}$ has order zero for each $0 \leq i \leq n$.

Definition 4.4. [Winter and Zacharias, 2010] The *nuclear dimension* of a C*-algebra A is defined as the least integer $n \in \mathbb{N}$ such that, for any finite set $G \subset A$, and for any $\epsilon > 0$, there exists a finite dimensional C*-algebra C and c.p. maps

$$\varphi : A \to C \text{ and } \psi : C \to A$$

such that

(1)
$$\|\psi \circ \varphi(a) - a\| < \epsilon \quad \forall a \in G$$
(2) ψ is n-decomposable.
(3) φ is contractive. (ψ need not be contractive).

If such a number exists, we denote it by
$$\dim_{nuc}(A)$$

Theorem 4.5.
$$\dim_{nuc}(C(X)) \leq \dim(X)$$
If X is second countable (or equivalently, metrizable), then equality holds.

Proof. The inequality $\leq$ holds from Remark 3.5. The reverse inequality is quite technical. This result is originally due to [Winter, 2003], and there is a somewhat shorter proof due to [Castillejos, 2018] as well.

□

Example 4.6.
(1) For a C*-algebra A, $\dim_{nuc}(A) = 0$ if and only if A is an AF-algebra.
(2)
$$\dim_{nuc}(A_\theta) = \begin{cases} 1 & : \theta \text{ is irrational} \\ 2 & : \theta \text{ is rational} \end{cases}$$
(3) If $\mathcal{T}$ denotes the Toeplitz algebra, then $\dim_{nuc}(\mathcal{T}) = 1$
(4) For $n \in \mathbb{N} \cup \{\infty\}$,
$$\dim_{nuc}(\mathcal{O}_n) = 1$$
(5) In fact, if A is a simple, separable C*-algebra, then
$$\dim_{nuc}(A) \in \{0, 1, +\infty\}$$

Remark 4.7. Note that, in the definition of nuclear dimension, we do not require that the second map ψ be contractive.
(1) Therefore, it is not obvious (but it is true) that $\dim_{nuc}(A) < \infty$ implies that A is nuclear.
(2) If we require both φ and ψ to be contractive, then we arrive at the definition of *decomposition rank* due to [Kirchberg and Winter, 2004]. The two ranks coincide for commutative C*-algebras. An important difference is that $\dim_{nuc}$ is well-behaved with respect to extensions, while dr is not. Given an extension
$$0 \to J \to E \to A \to 0$$
we have
$$\dim_{nuc}(E) \leq \dim_{nuc}(A) + \dim_{nuc}(J) + 1$$
There is no equivalent inequality for dr. In fact, if $\mathcal{T}$ denotes the Toeplitz algebra, then
$$\dim_{nuc}(\mathcal{T}) = 1 \text{ while } \mathrm{dr}(\mathcal{T}) = +\infty$$

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