Non-Stable K-theory for C*-algebras

(Joint work with Ms. Apurva Seth)

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Jan 07, 2020

Homotopy Groups of Unitary Groups

The unitary group

Let A be a unital C*-algebra. Fix $n \in \mathbb{N}$ and write

$$\mathcal{U}_n(A) := \{u \in M_n(A) : uu^* = u^*u = 1_n\}$$

Then $\mathcal{U}_n(A)$ is a topological group.

Treating 1_n as a base point, we may define the m^{th} homotopy group of $\mathcal{U}_n(A)$.

Definition

$$\pi_m(\mathcal{U}_n(A)) := [S^m, \mathcal{U}_n(A)]_*$$

We wish to calculate/understand these groups better.

Example: The Complex numbers

- $\mathcal{U}_1(\mathbb{C}) = S^1$. So

$$\pi_m(\mathcal{U}_1(\mathbb{C})) = \begin{cases} \mathbb{Z} & : n = 1 \\ 0 & : n \neq 1 \end{cases}$$

- However, for $n > 1$, $\pi_m(\mathcal{U}_n(\mathbb{C}))$ is complicated, and typically has torsion. For instance,

$$\pi_6(\mathcal{U}_2(\mathbb{C})) = \mathbb{Z}_{12}$$

- By Bott periodicity, if $m > 1$ and $n \geq \frac{m+1}{2}$,

$$\pi_m(\mathcal{U}_n(\mathbb{C})) = \begin{cases} 0 & : m \text{ even} \\ \mathbb{Z} & : m \text{ odd} \end{cases}$$

Connection to K-theory

Define

$$\theta_A : \mathcal{U}_n(A) \to \mathcal{U}_{n+1}(A) \text{ given by } a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

Definition

$K_{m+1}(A)$ is the inductive limit of the sequence

$$\ldots \to \pi_m(\mathcal{U}_n(A)) \xrightarrow{\theta_A} \pi_m(\mathcal{U}_{n+1}(A)) \to \ldots$$

Hence, the study these homotopy groups is called *non-stable K-theory*.

Example: The UHF algebra of type 2^∞

$A = M_{2^\infty}$ is an inductive limit

$$\mathbb{C} \to M_2(\mathbb{C}) \to M_4(\mathbb{C}) \to M_8(\mathbb{C}) \to \dots$$

where the connecting maps are

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

The map $\theta_A : \mathcal{U}(A) \to \mathcal{U}_2(A)$ is then a homotopy equivalence because

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \sim_h \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix} \text{ in } \mathcal{U}_2(B)$$

for any C*-algebra B.

Example: The UHF algebra of type 2^∞

More generally,

$$\theta_A : \mathcal{U}_{2^n}(A) \xrightarrow{\simeq} \mathcal{U}_{2^{n+1}}(A)$$

The same is true if A is replaced by $M_n(A)$, so passing to the limit, for each $n \in \mathbb{N}$,

$$\pi_m(\mathcal{U}_n(A)) \cong K_{m+1}(A) \cong \begin{cases} \mathbb{Z}\left[\frac{1}{2}\right] & : m \text{ odd} \\ 0 & : m \text{ even} \end{cases}$$

K-stable C*-algebras

Definition

A C*-algebra A is called *K-stable* if the map

$$\theta_A : \pi_m(\mathcal{U}_n(A)) \to \pi_m(\mathcal{U}_{n+1}(A))$$

is an isomorphism for all $m, n \in \mathbb{N}$.

Note that

- $\mathbb{C}$ is not K-stable. In fact, no finite dimensional C*-algebra is.
- M_{2^∞} is K-stable.

Examples

- [Rieffel, 1987] Non-commutative tori A_θ with θ irrational.
- [Thomsen, 1991] Infinite dimensional, simple AF-algebras (including the UHF algebras)
- [Zhang, 1991] Any purely infinite, simple C*-algebra (including the Cuntz algebras $\mathcal{O}_n$)
- [Jiang, 1997] The Jiang-Su algebra $\mathcal{Z}$

K-stable C*-algebras

If A is a K-stable C*-algebra, then, for all $m, n \in \mathbb{N}$,

$$\pi_m(\mathcal{U}_n(A)) \cong K_{m+1}(A)$$

Contrast this with the observation that

$$K_1(\mathbb{C}) = 0 \text{ but } \pi_6(\mathcal{U}_2(\mathbb{C})) = \mathbb{Z}_{12}$$

Goal of this project

To enlarge the class of K-stable C*-algebra to include some non-simple C*-algebras, by proving some permanence properties.

Some Permanence Properties

Theorem [Thomsen (1991)]

Given a short exact sequence $0 \to J \to A \to B \to 0$ of C*-algebras, if J and B are K-stable, then A is K-stable.

Proposition [Apurva Seth, PV (2019)]

- Given a pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \gamma \\ C & \underset{\delta}{\longrightarrow} & D \end{array}$$

  If B, C, and D are K-stable, and one of δ or γ is surjective, then A is K-stable.

- If A is K-stable and B is commutative, then $A \otimes B$ is K-stable.

$C(X)$-algebras

$C(X)$-algebras

Definition

Let X be a compact Hausdorff space and A a unital C*-algebra. A is called a *$C(X)$-algebra* if there is a unital $*$-homomorphism

$$\Delta : C(X) \to Z(A)$$

where $Z(A)$ denotes the center of A.

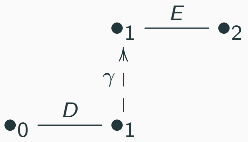
In other words, A carries a central $C(X)$-action.

Examples

- $A := C(X) \otimes D$ for any unital C*-algebra D.
- If $\gamma : D \to E$ is a $*$-homomorphism, then

  $$A := \{(f, g) \in C[0,1] \otimes D \oplus C[1,2] \otimes E : \gamma(f(1)) = g(1)\}$$

  is a $C[0,2]$-algebra. Pictorially,



- If A is a unital, separable C*-algebra with $X := \mathrm{Prim}(A)$ Hausdorff, then A is a $C(X)$-algebra. [Dauns-Hoffmann, Fell]

Continuous $C(X)$-algebras

Let A be a $C(X)$-algebra. For $x \in X$, define

$$I_x := \{f \in C(X) : f(x) = 0\}$$

Then I_x is an ideal of $C(X)$, so $(I_x \cdot A)$ is an ideal of A.

Definition

The *fiber* of A at x is the quotient

$$A_x := \frac{A}{(I_x \cdot A)}$$

Continuous $C(X)$-algebras

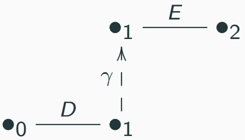
For $a \in A$, define $a(x) \in A_x$ to be the image of a in A_x. Define

$$N_a : X \to \mathbb{R} \text{ given by } x \mapsto \|a(x)\|$$

Definition

A is called a *continuous* $C(X)$-algebra if each N_a is continuous.

In the example above, if A is given by the picture



then A is a continuous $C[0,2]$-algebra if and only if γ is injective.

Main Result

Theorem [Apurva Seth, PV (2019)]

Let X be a compact metric space of finite covering dimension, and let A be a continuous $C(X)$-algebra. If each fiber of A is K-stable, then A is K-stable.

One may think of this as another permanence property for the class of K-stable C*-algebras.

Remarks

- If X is zero-dimensional, then the metrizability condition on X may be dropped.
- If X is not metrizable, we may replace covering dimension with inductive dimension (All notions of dimension coincide for compact metric spaces).
- If A is separable and each fiber is non-zero, then X is automatically metrizable.
- That X has finite dimension is crucial for the proof, as it works by induction on the dimension.
- Everything I have said works for non-unital C*-algebras as well, with appropriate changes to the definitions.

Thank you!