

ROKHLIN DIMENSION AND INDUCTIVE LIMIT ACTIONS ON AF-ALGEBRAS

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ABSTRACT. Given a separable, AF-algebra A and an inductive limit action on A of a finitely generated abelian group with finite Rokhlin dimension with commuting towers, we give a local description of the associated crossed product C^* -algebra. In particular, when A is unital and $\alpha \in \text{Aut}(A)$ is approximately inner and has the Rokhlin property, we conclude that $A \rtimes_{\alpha} \mathbb{Z}$ is an AT-algebra.

Given a group action on a C^* -algebra, one is often interested in the structure of the associated crossed product C^* -algebra. The crossed product is a flexible and powerful construction, and it is often difficult to describe it unless we impose some conditions on the action or algebra. One such condition that has proved to be helpful in such investigations is that of the Rokhlin property and its higher dimensional analogue, Rokhlin dimension. Given an action of a compact group with finite Rokhlin dimension with commuting towers, Gardella, Hirshberg and Santiago [GHS21] have been successful in describing the structure of the associated crossed product. In this paper, we wish to extend their ideas to the context of residually finite groups.

This is a complicated in general, so it felt prudent to begin by analyzing actions of abelian groups on AF-algebras, which may be thought of as ‘zero-dimensional’ C^* -algebras. Here, we were able to partially describe the crossed product when the action is built up from actions on finite dimensional subalgebras (Theorem 2.5). When restricted to the action of a single automorphism, we obtain the following result, which is a partial generalization of results of Kishimoto et al, which were proved for UHF algebras (see [BEK95], [Kis95] and [Bra+93]).

Theorem A. *Let A be a separable, unital AF-algebra and let $\alpha \in \text{Aut}(A)$ be an approximately inner automorphism. If α has the Rokhlin property, then $A \rtimes_{\alpha} \mathbb{Z}$ is an AT-algebra. Moreover, if A is simple, then $A \rtimes_{\alpha} \mathbb{Z}$ is simple and has real rank zero and stable rank one.*

In Section 1, we begin by describing twisted C^* -dynamical systems and twisted crossed products, which play a crucial role for us. We then discuss the profinite completion of a residually finite group which, together with the notion of a topological join, allows us to give a local description of crossed products when the action has finite Rokhlin dimension with commuting towers. In Section 2, we focus our attention on inductive limit actions on AF-algebras. We prove Theorem 2.5, a local approximation theorem for crossed products by such actions and we end with a proof of Theorem A.

1. PRELIMINARIES

We begin with a number of preliminary notions that we will need as we go along. For convenience, we first fix some notation. Henceforth, all groups (denoted G, H, K , etc.) will be countable and discrete, and we will write e for the identity of the group. We write $H \leq G$ if H is a subgroup of G and $H \leq_{fin} G$ if H is a subgroup of G of finite index.

If Z is a topological space, we write $\dim(Z)$ for its Lebesgue covering dimension. If a group G acts on Z by homeomorphisms, we denote this by $G \curvearrowright Z$ or $\beta : G \curvearrowright Z$ if $\beta : G \rightarrow \text{Homeo}(Z)$ denotes the corresponding homomorphism. If F' is a set, then we write $F \subset F'$ if F is a finite

subset of F' .

If A is a C^* -algebra and $a, b \in A$, we write $[a, b] := ab - ba$, and we write $a \approx_\epsilon b$ if $\|a - b\| < \epsilon$. If A is unital, we write 1_A for the unit of A . We write $A_{\leq 1}$ for the closed unit ball in A . Finally, all C^* -algebras discussed in this paper will be assumed to be separable and nuclear unless otherwise stated.

1.1. Twisted C^* -Dynamical System. The main object of our investigation is a C^* -dynamical system (A, G, α) where G is a discrete, countable group, A is a separable, nuclear C^* -algebra and $\alpha : G \rightarrow \text{Aut}(A)$ is a homomorphism. Given such a triple, there is an associated crossed product C^* -algebra $A \rtimes_\alpha G$, which is the object we wish to describe. However, in the course of our investigations, we naturally come across a twisted C^* -dynamical system. The construction of the twisted crossed product mirrors the construction of the usual crossed product, but is quite a bit more flexible. We now briefly revisit these notions from [PR89] in the context of discrete groups.

Definition 1.1. [PR89, Definition 2.1 and 2.3]

- (1) A twisted action of a discrete group G on C^* -algebra A is a pair of maps $\alpha : G \rightarrow \text{Aut}(A)$ and $u : G \times G \rightarrow \mathcal{UM}(A)$ such that for all $r, s, t \in G$, we have
 - (a) $\alpha_e = \text{id}_A, u(e, s) = u(s, e) = 1$,
 - (b) $\alpha_s \circ \alpha_t = \text{Ad } u(s, t) \circ \alpha_{st}$, and
 - (c) $\alpha_r(u(s, t))u(r, st) = u(r, s)u(rs, t)$.

We shall refer to the quadruple (A, G, α, u) as a twisted C^* -dynamical system.

- (2) A covariant representation of a twisted dynamical system (A, G, α, u) is a triple (π, U, H) consisting of a non-degenerate $*$ -representation (π, H) of A and a map $U : G \rightarrow \mathcal{U}(H)$ such that for all $s, t \in G$ and $a \in A$, we have
 - (a) $U_s U_t = \pi(u(s, t))U_{st}$, and
 - (b) $\pi(\alpha_s(a)) = U_s \pi(a) U_s^*$.

We now describe the (reduced) twisted crossed product C^* -algebra associated to a twisted C^* -dynamical system (A, G, α, u) . Consider $C_c(G, A)$ to be the set of all finitely supported functions from G to A , whose elements are written as $f = \sum_{s \in G} a_s s$. We equip $C_c(G, A)$ with an (α, u) -twisted convolution product and $*$ -operation given by

$$(f \times_u g)(s) = \sum_{t \in G} f(t) \alpha_t(g(t^{-1}s)) u(t, t^{-1}s), \text{ and } f^*(s) = u(s, s^{-1})^* \alpha_s(f(s^{-1})^*).$$

Then, $C_c(G, A)$ is a $*$ -algebra and every covariant representation (π, U, H) of (A, G, α, u) induces a $*$ -representation $\pi \times U : C_c(G, A) \rightarrow \mathcal{B}(H)$ given by

$$(\pi \times U)(f) := \sum_{s \in G} \pi(f(s)) U_s.$$

Suppose A is faithfully represented on a Hilbert space H . Define a representation $\pi_\alpha : A \rightarrow \mathcal{B}(H \otimes \ell^2(G))$ by $\pi_\alpha(a)(\zeta \otimes \delta_t) = \alpha_{t^{-1}}(a)\zeta \otimes \delta_t$, and a function $\lambda_u : G \rightarrow \mathcal{B}(H \otimes \ell^2(G))$ by $\lambda_u(g)(\zeta \otimes \delta_t) = u(t^{-1}g^{-1}, g)\zeta \otimes \delta_{gt}$. Then, $(\pi_\alpha, \lambda_u, H \otimes \ell^2(G))$ is a covariant representation of (A, G, α, u) , and so it gives rise to a $*$ -representation $\pi_\alpha \times \lambda_u : C_c(G, A) \rightarrow \mathcal{B}(H \otimes \ell^2(G))$. We define the (reduced) crossed product C^* -algebra $A \rtimes_{\alpha, u} G$ as the norm closure of $C_c(G, A)$ under this map. For convenience, we write a and λ_s^α for elements in $M(A \rtimes_{\alpha, u} G)$ corresponding to elements $a \in A$ and $s \in G$ respectively. Note that $\{a \lambda_s^\alpha : a \in A, s \in G\}$ spans a dense subset of $A \rtimes_{\alpha, u} G$.

From our point of view, the most important reason to discuss twisted C^* -dynamical systems is that they arise naturally when trying to decompose the crossed product associated to an (untwisted)

C*-dynamical system. We now give the version of [PR89, Theorem 4.1] that we need for our purposes.

Theorem 1.2. [PR89, Theorem 4.1] *Let G be an abelian group, $N \leq G$ be a subgroup and let $Q := G/N$. Let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of G on a C*-algebra A , let $\alpha_N : N \rightarrow \text{Aut}(A)$ be the restriction and let $c : Q \rightarrow G$ be a section such that $c(N) = e$. Define $\alpha_Q : Q \rightarrow \text{Aut}(A \rtimes_{\alpha_N} N)$ by $(\alpha_Q)_{sN}(a\lambda_r^{\alpha_N}) := \alpha_{c(sN)}(a)\lambda_r^{\alpha_N}$ and $v_Q : Q \times Q \rightarrow \mathcal{UM}(A \rtimes_{\alpha_N} N)$ by $v_Q(sN, tN) := \lambda_{c(sN)c(tN)c(stN)}^{\alpha_N}$. Then (α_Q, v_Q) is a twisted action of Q on $A \rtimes_{\alpha_N} N$, and*

$$A \rtimes_{\alpha} G \cong (A \rtimes_{\alpha_N} N) \rtimes_{(\alpha_Q, v_Q)} Q.$$

The next lemma is now an immediate consequence.

Lemma 1.3. *Let G be an abelian group, $N \leq G$ be a subgroup and let $Q := G/N$. Let (A, G, α) be a C*-dynamical system such that α_N is trivial. Given a C*-dynamical system (B, G, β) , let $\gamma : G \rightarrow \text{Aut}(A \otimes B)$ be the tensor product action $\gamma_s = \alpha_s \otimes \beta_s$. Then, there is a twisted action (γ'_Q, v'_Q) of Q on $A \otimes (B \rtimes_{\beta_N} N)$ such that*

$$(A \otimes B) \rtimes_{\gamma} G \cong (A \otimes (B \rtimes_{\beta_N} N)) \rtimes_{\gamma'_Q, v'_Q} Q.$$

Proof. Let $R := B \rtimes_{\beta_N} N$ and define $\gamma'_Q : Q \rightarrow \text{Aut}(A \otimes R)$ by $(\gamma'_Q)_{sN}(a \otimes f) := \alpha_s(a) \otimes (\beta_Q)_{sN}(f)$ for $a \in A, f \in C_c(N, B)$, where $\beta_Q : Q \rightarrow \text{Aut}(B \rtimes_{\beta_N} N)$ is defined as in Theorem 1.2. Define $v'_Q : Q \rightarrow \mathcal{UM}(A \otimes R)$ by $v'_Q(sN, tN) := 1 \otimes \lambda_r^{\beta_N}$. Define $\Theta : A \otimes R \rightarrow (A \otimes B) \rtimes_{\gamma_N} N$ to be the isomorphism given by

$$\Theta \left(a \otimes \left(\sum_{t \in N} b_t \lambda_t^{\beta_N} \right) \right) = \sum_{t \in N} (a \otimes b_t) \lambda_t^{\gamma_N}$$

for $a \in A, f = \sum_{t \in N} b_t \lambda_t^{\beta_N} \in C_c(N, B)$. Now, let (γ_Q, v_Q) denote the twisted action of Q on $(A \otimes B) \rtimes_{\gamma_N} N$ as in Theorem 1.2. Then, $\Theta \circ v'_Q(sN, tN) = \lambda_{c(sN)c(tN)c(stN)}^{\beta_N} = v_Q(sN, tN)$. Moreover, a short calculation shows that $\Theta \circ (\gamma'_Q)_{sN} = (\gamma_Q)_{sN} \circ \Theta$ for all $s \in G$. Hence Θ induces an isomorphism $(A \otimes (B \rtimes_{\beta_N} N)) \rtimes_{\gamma'_Q, v'_Q} Q \rightarrow ((A \otimes B) \rtimes_{\gamma_N} N) \rtimes_{\gamma_Q, v_Q} Q$. The result then follows from Theorem 1.2. \square

We now turn our attention to twisted actions on continuous $C(X)$ -algebras. Our goal is to understand the associated crossed product as a continuous field of C*-algebras whose fibers are themselves twisted crossed product C*-algebras.

Definition 1.4. Let X be a compact Hausdorff space. A C*-algebra A is said to be a $C(X)$ -algebra if there is a non-degenerate *-homomorphism $\theta : C(X) \rightarrow Z(M(A))$, where $Z(M(A))$ denotes the center of the multiplier algebra of A .

For a function $f \in C(X)$ and $a \in A$, we will write $fa := \theta(f)(a)$. If $Y \subset X$ is a closed subspace, let $C_0(X, Y)$ denote the ideal of functions that vanish on Y . Then $C_0(X, Y)A$ is a closed ideal in A . We write $A(Y) := A/C_0(X, Y)A$ for the corresponding quotient and $\pi_Y : A \rightarrow A(Y)$ for the quotient map. If $Y = \{x\}$ is a singleton set, then we write I_x for the ideal $C_0(X, \{x\})A$, the algebra $A(x) := A(\{x\})$ is called the fiber of A at x , and we write $\pi_x : A \rightarrow A(x)$ for the corresponding quotient map. If $a \in A$, we simply write $a(x) := \pi_x(a) \in A(x)$. For each $a \in A$, we have a map $\Gamma_a : X \rightarrow \mathbb{R}$ given by $x \mapsto \|a(x)\|$. This map is upper semicontinuous (by [Rie89, Proposition 1.2]). We say that A is a continuous $C(X)$ -algebra if Γ_a is continuous for each $a \in A$.

Definition 1.5. Let X be a compact Hausdorff space and A be a separable continuous $C(X)$ -algebra. A twisted action (α, u) of a group G on A is said to act fiberwise if the ideal I_x is invariant under α for each $x \in X$.

Given such a fiberwise twisted action, it is clear that there is a natural twisted action $(\alpha(x), u(x))$ of G on $A(x)$. Then next theorem due to Packer and Raeburn gives us the expected theorem concerning the corresponding twisted crossed products.

Theorem 1.6. [PR90, Theorem 2.1] *Let X be a compact Hausdorff space. Let A be a separable continuous $C(X)$ -algebra and (α, u) be a twisted action of an amenable group G on A acting fiberwise. Then for each $x \in X$, there is a twisted action $(\alpha(x), u(x))$ of G on $A(x)$, and $A \rtimes_{\alpha, u} G$ is a continuous $C(X)$ -algebra with fibers isomorphic to $A(x) \rtimes_{\alpha(x), u(x)} G$.*

The next result is something we will need during the course of our analysis. While it may be possible to state it in more generality, we only give the version that we need here.

Theorem 1.7. *Let G be a finite group and (γ, w) be a twisted action of G on a C^* -algebra D . If $\sigma : G \rightarrow \text{Aut}(C(G))$ denotes the left translation action, then*

$$(C(G, D) \rtimes_{\sigma \otimes \gamma, 1 \otimes w} G) \otimes \mathcal{K}(\ell^2(G)) \cong D \otimes \mathcal{K}(\ell^2(G)) \otimes \mathcal{K}(\ell^2(G))$$

Proof. Let $A = C(G, D)$, then $\alpha := \sigma \otimes \gamma$ and $u := 1 \otimes w$ makes (A, G, α, u) a separable twisted dynamical system. By [PR89, Theorem 3.7], there is an action $\beta : G \rightarrow \text{Aut}(A \otimes \mathcal{K}(\ell^2(G)))$ such that

$$(A \rtimes_{\alpha, u} G) \otimes \mathcal{K}(\ell^2(G)) \cong (A \otimes \mathcal{K}(\ell^2(G))) \rtimes_{\beta} G.$$

If β were of the form $\beta = \sigma \otimes \delta$ for some action $\delta : G \rightarrow \text{Aut}(D \otimes \mathcal{K}(\ell^2(G)))$, then the result would follow immediately by Green's Theorem [Gre80, Corollary 2.9].

To prove this, we follow the notation of [PR89]. Define $v : G \rightarrow \mathcal{UM}(A \otimes \mathcal{K})$ by $v_s := (1 \otimes \lambda_s)(\text{id} \otimes M(u(s, \cdot)^*))$. Then β is given by $\beta_s = \text{Ad}(v_s) \circ (\alpha_s \otimes \text{id})$. To understand this, we think of A as faithfully represented on a Hilbert space H and think of $A \otimes \mathcal{K}$ as faithfully represented on $H \otimes \ell^2(G)$. Then, if $a \in A, T \in \mathcal{K}(\ell^2(G)), h \in H$ and $\zeta \in \ell^2(G)$, we have

$$\begin{aligned} \beta_s(a \otimes T)(h \otimes \zeta) &= (1 \otimes \lambda_s)(\text{id} \otimes M(u(s, \cdot)^*)[\alpha_s(a) \otimes T](\text{id} \otimes M(u(s, \cdot)))(1 \otimes \lambda_{s^{-1}})(h \otimes \zeta) \\ &= (1 \otimes \lambda_s)(\text{id} \otimes M(u(s, \cdot)^*)[\alpha_s(a) \otimes T](u(s, \cdot)(h) \otimes \lambda_{s^{-1}}(\zeta)) \\ &= u(s, \cdot)^*(\alpha_s(a)(u(s, \cdot)(h))) \otimes \lambda_s \circ T \circ \lambda_{s^{-1}}(\zeta) \end{aligned}$$

Now suppose $a = f \otimes d$ for some $f \in C(G)$ and $d \in D$. Then

$$u(s, \cdot)^*(\alpha_s(a)(u(s, \cdot))) = \sigma_s(f) \otimes [w(s, \cdot)^* \gamma_s(d) w(s, \cdot)].$$

Therefore, $\beta = \sigma \otimes \delta$ for some action $\delta : G \rightarrow \text{Aut}(D \otimes \mathcal{K}(\ell^2(G)))$, and this completes the proof. \square

This leads us to the main result of this section. Recall that a corner of a C^* -algebra B is a hereditary subalgebra of the form pBp for some projection $p \in M(B)$.

Corollary 1.8. *Let G be a finite group acting freely on a compact Hausdorff space X and let D be a separable C^* -algebra carrying a twisted action (α, u) of G . Let $A := C(X, D)$, and let $(\gamma, v) : G \rightarrow \text{Aut}(A)$ be the induced twisted diagonal action given by*

$$\gamma_s(f)(x) := \alpha_s(f(s^{-1} \cdot x)) \text{ and } v_{s,t}(x) := u_{s,t}.$$

Then, $A \rtimes_{\gamma, v} G$ is a continuous $C(X/G)$ -algebra each fiber of which is isomorphic to a corner of $M_{|G|^2}(D)$.

Proof. By the proof of [GHS21, Proposition 4.5], A is a continuous $C(X/G)$ -algebra whose fiber over a point $y = \pi(x) \in X/G$ can be identified with $C(G \cdot x, D) \cong C(G, D)$, because the action on X is free. Note that the action of G is fiberwise. Indeed, fix $y \in Y$, and consider the twisted action $(\gamma(y), v(y)) : G \rightarrow \text{Aut}(A(y))$ on the corresponding fiber. Then for any $f \in C(G \cdot x, D)$,

$$\gamma(y)_s(f)(rx) = \alpha_s(f(s^{-1}rx)) \text{ and } v(y)_{s,t}(f)(rx) := u_{s,t}f(rx).$$

Under the isomorphism $C(G \cdot x, D) \cong C(G, D)$, this twisted action translates to the natural twisted action $(\hat{\gamma}, \hat{v}) : G \rightarrow \text{Aut}(C(G, D))$ given by $\hat{\gamma}_s(f)(r) := \alpha_s(f(s^{-1}r))$ and $\hat{v}_{s,t}(r) := u_{s,t}$. Hence, $A(y) \rtimes_{\gamma(y), v(y)} G \cong C(G, D) \rtimes_{\hat{\gamma}, \hat{v}} G$. By [Theorem 1.7](#), there is an isomorphism

$$(C(G, D) \rtimes_{\hat{\gamma}, \hat{v}} G) \otimes \mathcal{K}(\ell^2(G)) \cong D \otimes \mathcal{K}(\ell^2(G)) \otimes \mathcal{K}(\ell^2(G)) \cong M_{|G|^2}(D).$$

Therefore, the result follows from [Theorem 1.6](#). \square

1.2. The Topological Join. Given two compact Hausdorff spaces X and Y , the topological join of X and Y is defined as

$$X * Y := ([0, 1] \times X \times Y) / \sim$$

where \sim is the equivalence relation defined by $(0, x, y) \sim (0, x', y)$ and $(1, x, y) \sim (1, x, y')$ for all $x, x' \in X$ and $y, y' \in Y$. Elements of $X * Y$ are denoted by the symbol $[t, x, y]$ for the equivalence class containing (t, x, y) .

Given three compact Hausdorff spaces X, Y and Z , we may also define $(X * Y) * Z$ and $X * (Y * Z)$ as above. Since all spaces are compact and Hausdorff, these two spaces are naturally homeomorphic, so the join operation is associative. Thus if X_1, X_2, \dots, X_n are compact Hausdorff spaces, then $X_1 * X_2 * \dots * X_n$ may be defined unambiguously. If $X_i = X$ for all $1 \leq i \leq n$, we denote the space $X_1 * X_2 * \dots * X_n$ by $X^{*(n)}$.

Given two continuous maps $f : X_1 \rightarrow X_2$ and $g : Y_1 \rightarrow Y_2$, their join $f * g : X_1 * Y_1 \rightarrow X_2 * Y_2$ is defined by $(f * g)([t, x_1, x_2]) := [t, f(x_1), g(x_2)]$. It is evident that $f * g$ is well-defined and continuous. If $f = g$, then we denote the map $f * f$ by $f^{*(2)}$. Given a continuous function $f : X \rightarrow X$, this process may be repeated inductively to define $f^{*(n)} : X^{*(n)} \rightarrow X^{*(n)}$.

Now suppose G is a group and $\alpha : G \curvearrowright X$ and $\beta : G \curvearrowright Y$ are two group actions of G on X and Y respectively. Then, there is a natural action $\alpha * \beta : G \curvearrowright (X * Y)$ given by $(\alpha * \beta)_g([t, x, y]) = [t, \alpha_g(x), \beta_g(y)]$. Moreover, this action is free if each individual action is free. In particular, if $\gamma : G \curvearrowright X$ is a free action, then we get an induced action $\gamma^{*(n)} : G \curvearrowright X^{*(n)}$ which is also free.

1.3. The Profinite Completion. An inverse system of compact topological spaces over a directed partially ordered set (\mathcal{I}, \leq) consists of a collection $\{X_i : i \in \mathcal{I}\}$ of compact Hausdorff topological spaces indexed by \mathcal{I} , and a collection of continuous maps $\varphi_{i,j} : X_i \rightarrow X_j$, defined whenever $i \geq j$, such that $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$ whenever $i, j, k \in \mathcal{I}$ with $i \geq j \geq k$. Moreover, we assume that $\varphi_{i,i} = \text{id}_{X_i}$ for all $i \in \mathcal{I}$. The inverse limit of this system is

$$\varprojlim (X_i, \varphi_{i,j}, \mathcal{I}) = \{(x_i)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} X_i : x_j = \varphi_{i,j}(x_i) \text{ whenever } i \geq j\}$$

Note that $\varprojlim (X_i, \varphi_{i,j}, \mathcal{I})$ is a closed subset of $\prod_{i \in \mathcal{I}} X_i$ and is therefore a compact Hausdorff space. Moreover, there are maps $\pi_i : X \rightarrow X_i$ satisfying $\varphi_{i,j} \circ \pi_i = \pi_j$ whenever $i \geq j$, which satisfy the following universal property: Whenever Y is a topological space and $\eta_i : Y \rightarrow X_i$ are continuous maps such that $\varphi_{i,j} \circ \eta_i = \eta_j$ whenever $i \geq j$, there is a unique continuous map $\psi : Y \rightarrow X$ such that $\pi_i \circ \psi = \eta_i$ for all $i \in \mathcal{I}$. When the directed system \mathcal{I} is understood, we write $\varprojlim (X_i, \varphi_{i,j})$ for this inverse limit.

Note that if we begin with an inverse system of compact topological groups and group homomorphisms, then the inverse limit defined above is also a compact topological group, the maps π_i mentioned above are continuous group homomorphisms. Similarly, if each topological space X_i is equipped with an action of a fixed group G , and each connecting map is G -equivariant, then the inverse limit is also equipped with a natural G -action in such a way that the maps $\pi_i : X \rightarrow X_i$

are G -equivariant.

Before we proceed, we need a short lemma that will be useful to us.

Lemma 1.9. *Let $(X, \pi_i) = \varprojlim(X_i, f_{i,j})$ and $(Y, \rho_i) = \varprojlim(Y_i, g_{i,j})$ be two inverse limits over a common directed set. Then $(X_i * Y_i, f_{i,j} * g_{i,j})$ forms an inverse system and $\varprojlim(X_i * Y_i, f_{i,j} * g_{i,j}) \cong (X * Y, \pi_i * \rho_i)$. Moreover, if there is a G -action on X_i and Y_i and the connecting maps are G -equivariant, then this homeomorphism may be chosen to be G -equivariant.*

Proof. Let \mathcal{I} denote the common directed set. It is easy to verify that $(X_i * Y_i, f_{i,j} * g_{i,j})$ forms an inverse system, so let $(Z, p_i) = \varprojlim(X_i * Y_i, f_{i,j} * g_{i,j})$. Note that the maps $\pi_i * \rho_i : X * Y \rightarrow X_i * Y_i$ satisfy $(f_{i,j} * g_{i,j}) \circ (\pi_i * \rho_i) = \pi_j * \rho_j$ whenever $i \geq j$. Hence, by the universal property of the inverse limit, there is a continuous map $\Theta : X * Y \rightarrow Z$ such that $p_i \circ \Theta = \pi_i * \rho_i$. Identifying Z as a subset of $\prod_{i \in \mathcal{I}} (X_i * Y_i)$, we see that

$$\Theta([t, x, y]) = ([t, \pi_i(x), \rho_i(y)])_{i \in \mathcal{I}}.$$

We wish to prove that Θ is a homeomorphism. Since both spaces are compact and Hausdorff, it suffices to show that Θ is bijective. For injectivity, suppose $\Theta([t, x, y]) = \Theta([s, x', y'])$, then we consider three cases:

- (1) If $0 < t < 1$, then $0 < s < 1$ and $s = t$ must follow. Moreover, $\pi_i(x) = \pi_i(x')$ and $\rho_i(y) = \rho_i(y')$ for all $i \in \mathcal{I}$. Therefore, $x = x'$ and $y = y'$ as well.
- (2) If $t = 0$ then $s = 0$ must hold. Since $(0, \pi_i(x), \rho_i(y)) \sim (0, \pi_i(x'), \rho_i(y'))$ for all $i \in \mathcal{I}$, it follows that $\rho_i(y) = \rho_i(y')$ for all $i \in \mathcal{I}$. Therefore, $y = y'$ and $(0, x, y) \sim (0, x', y')$ holds.
- (3) The argument when $t = 1$ is similar.

This proves that Θ is injective. To prove that Θ is surjective, fix $i_0 \in \mathcal{I}$ and consider the subsystem $\mathcal{J} := \{j \in \mathcal{I} : i_0 \leq j\}$. Then, \mathcal{J} is cofinal in \mathcal{I} and therefore $\varprojlim(X_i * Y_i, f_{i,j} * g_{i,j}, \mathcal{I}) \cong \varprojlim(X_i * Y_i, f_{i,j} * g_{i,j}, \mathcal{J})$ by [RZ10, Lemma 1.1.9]. Replacing \mathcal{I} by \mathcal{J} , we may assume that $i_0 \leq i$ for all $i \in \mathcal{I}$. Now fix $z = (z_i)_{i \in \mathcal{I}} \in Z$ and write $z_i = [t_i, x_i, y_i]$ for some $(t_i, x_i, y_i) \in [0, 1] \times X_i \times Y_i$. We know that if $i \geq j$, then $[t_i, f_{i,j}(x_i), g_{i,j}(y_i)] = [t_j, x_j, y_j]$. We once again break it into cases:

- (1) If $0 < t_{i_0} < 1$, then $t_i = t_{i_0}$ for all $i \in \mathcal{I}$ and therefore $f_{i,j}(x_i) = x_j$ and $g_{i,j}(y_i) = y_j$ whenever $i \geq j$. Hence, $x = (x_i)_{i \in \mathcal{I}} \in X$ and $y = (y_i)_{i \in \mathcal{I}} \in Y$ and clearly $\Theta([t_{i_0}, x, y]) = z$.
- (2) If $t_{i_0} = 0$ then $t_i = 0$ for all $i \in \mathcal{I}$. It follows that $g_{i,j}(y_i) = y_j$ whenever $i \geq j$. So if $y = (y_i)_{i \in \mathcal{I}} \in Y$, then for any $x \in X$, we have $\Theta([0, x, y]) = z$.
- (3) The case when $t_{i_0} = 1$ is analogous.

This proves that Θ is surjective and thus a homeomorphism. Note that if the X_i and Y_i all carry a G -action and all connecting maps are G -equivariant, then Θ is also clearly G -equivariant. \square

We now recall the definition of the profinite completion of a discrete, residually finite group.

Definition 1.10. Recall that a discrete group G is said to be residually finite if for each non-identity element $g \in G$, there is a subgroup H of G of finite index such that $g \notin H$. Given such a group G , let \mathcal{I}_G denote the set of all normal subgroups of G of finite index, partially ordered by reverse inclusion. In other words, $H \leq K$ if and only if $K \subset H$. Whenever $H \leq K$, there is a homomorphism $\varphi_{K,H} : G/K \rightarrow G/H$ given by $gK \mapsto gH$, and this makes the collection $\{G/H, \varphi_{K,H}\}_{\mathcal{I}_G}$ an inverse system of groups. The inverse limit of this system is called the profinite completion of G and is denoted by \overline{G} . By definition,

$$\overline{G} = \{(g_H H)_{H \in \mathcal{I}_G} : g_K H = g_H H \text{ for all } H, K \in \mathcal{I}_G \text{ with } K \subset H\}$$

Note that \overline{G} is a group under componentwise multiplication and is a topological space as a subspace of $\prod_{H \in \mathcal{I}_G} G/H$. Indeed, \overline{G} is compact, Hausdorff and totally disconnected. A profinite group

is, by definition, the inverse limit of a surjective inverse system of finite groups, and \overline{G} is therefore a profinite group.

For each $H \in \mathcal{I}_G$, there is a natural action $\beta^H : G \curvearrowright G/H$ given by $\beta_t^H(gH) := tgH$. The maps $\varphi_{K,H}$ respect these actions, so we have an induced left-translation action $\beta : G \curvearrowright \overline{G}$ given by $\beta_t((g_H H)_{H \in \mathcal{I}_G}) := (tg_H H)_{H \in \mathcal{I}_G}$. Moreover, this action is both free and minimal (in the sense that \overline{G} has no non-trivial closed β -invariant subsets). Hence, for each $n \in \mathbb{N}$, we obtain actions $(\beta^H)^*(n) : G \curvearrowright (G/H)^*(n)$ and $\beta^{*(n)} : G \curvearrowright \overline{G}^{*(n)}$. Note that the action $\beta^{*(n)}$ is free but not necessarily minimal. For a fixed $n \in \mathbb{N}$, [Lemma 1.9](#) now ensures that there is a natural G -equivariant homeomorphism

$$\overline{G}^{*(n)} \cong \varprojlim ((G/H)^*(n), \varphi_{K,H}^{*(n)}).$$

Now consider the commutative C^* -algebra $C(\overline{G}^{*(n)})$. From the above argument, it follows that $C(\overline{G}^{*(n)})$ is an inductive limit of the system $(C((G/H)^*(n)), (\varphi_{K,H}^{*(n)})^*)$. For each $H \in \mathcal{I}_H$, let $(\sigma^H)^{(n)} : G \rightarrow \text{Aut}(C((G/H)^*(n)))$ denote the action induced by $(\beta^H)^*(n)$ and let $\sigma^{(n)} : G \rightarrow \text{Aut}(C(\overline{G}^{*(n)}))$ be the action induced by $\beta^{*(n)}$. The maps $(\pi_K^{*(n)})^* : C((G/K)^*(n)) \rightarrow C(\overline{G}^{*(n)})$ and $(\varphi_{K,H}^{*(n)})^* : C((G/H)^*(n)) \rightarrow C((G/K)^*(n))$ are all G -equivariant and therefore we see that

Corollary 1.11. *For each $n \in \mathbb{N}$,*

$$(C(\overline{G}^{*(n)}), \sigma^{(n)}) \cong \varinjlim_{\mathcal{I}_G} (C((G/H)^*(n)), (\sigma^H)^{(n)})$$

where this denotes an inductive limit of G -algebras.

1.4. Rokhlin dimension. We now turn to the definition of Rokhlin dimension for actions of residually finite groups on C^* -algebras. In their original work on the subject [[HWZ15](#)], Hirshberg, Winter and Zacharias defined two notions: Rokhlin dimension with and without commuting towers. The latter is, in principle, a stronger requirement than the former and was introduced as a technical artefact to prove certain results. However, it was unclear at that time if the two definitions are necessarily distinct. For actions of compact groups, these are now known to be different in general. However, for actions of \mathbb{Z} , for instance, it is unknown whether these two notions are distinct. Since we only need the notion of Rokhlin dimension with commuting towers for our main results, that is the one we give here. Indeed, structure theorems such as the one we describe are unknown even for actions of compact groups without this assumption.

Given two C^* -algebras A and B , a linear map $\varphi : A \rightarrow B$ is said to be contractive and completely positive (abbreviated to c.c.p.) if the natural induced map $\varphi^{(n)} : M_n(A) \rightarrow M_n(B)$ is contractive and positive for each $n \in \mathbb{N}$. Two elements $a, b \in A$ are said to be orthogonal (in symbols, $a \perp b$) if $ab = ba = a^*b = b^*a = 0$. A c.c.p. map $\varphi : A \rightarrow B$ is said to have order zero if $\varphi(a) \perp \varphi(b)$ whenever $a \perp b$.

Definition 1.12. [[SWZ19](#), Definition 5.4] Let A be a C^* -algebra, G be a discrete, countable, residually finite group, H be a subgroup of G of finite index, and $\alpha : G \rightarrow \text{Aut}(A)$ be an action of G on A . We say that α has Rokhlin dimension d with commuting towers relative to H if d is the least integer such that for each $F \subset A, M \subset G, S \subset C(G/H)$ and $\epsilon > 0$, there exist $(d+1)$ c.c.p. maps

$$\psi_0, \psi_1, \dots, \psi_d : C(G/H) \rightarrow A$$

satisfying the following properties:

- (1) $[\psi_\ell(f), a] \approx_\epsilon 0$ for all $a \in F, f \in S$ and $0 \leq \ell \leq d$.
- (2) $\psi_\ell(\sigma_g^H(f))a \approx_\epsilon \alpha_g(\psi_\ell(f))a$ for all $a \in F, f \in S, g \in M$ and $0 \leq \ell \leq d$.

- (3) $\psi_\ell(f_1)\psi_\ell(f_2)a \approx_\epsilon 0$ for all $a \in F$ and $f_1, f_2 \in S$ such that $f_1 \perp f_2$ and for all $0 \leq \ell \leq d$.
- (4) $\sum_{\ell=0}^d \psi_\ell(1_{C(G/H)})a \approx_\epsilon a$ for all $a \in F$.
- (5) $[\psi_k(f_1), \psi_\ell(f_2)]a \approx_\epsilon 0$ for all $f_1, f_2 \in S, 0 \leq k, \ell \leq d$ and all $a \in F$.

We denote the Rokhlin dimension of α with commuting towers relative to H by $\dim_{\text{Rok}}^c(\alpha, H)$. If no such integer d exists, then we write $\dim_{\text{Rok}}^c(\alpha, H) = +\infty$. We define the Rokhlin dimension of α with commuting towers as

$$\dim_{\text{Rok}}^c(\alpha) = \sup\{\dim_{\text{Rok}}^c(\alpha, H) : H \leq_{\text{fin}} G\}.$$

When $\dim_{\text{Rok}}^c(\alpha) = 0$, then we say that α has the Rokhlin property.

The version of the definition given above is somewhat unwieldy for our purposes. To remedy this, we need a fact proved in [MV23]. First, we need the following definition.

Definition 1.13. Given a C*-algebra A , let $\ell^\infty(\mathbb{N}, A)$ be the C*-algebra of all bounded sequences in A and $c_0(\mathbb{N}, A)$ be the ideal of sequences that vanish at infinity. If $A_\infty := \ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A)$, then A embeds in A_∞ as the set of all constant sequences so we identify A with its image in A_∞ . For a C*-subalgebra $D \subset A$, we define

$$\begin{aligned} A_\infty \cap D' &= \{x \in A_\infty : xd = dx \text{ for all } d \in D\} \text{ and} \\ \text{Ann}(D, A_\infty) &= \{x \in A_\infty : xd = dx = 0 \text{ for all } d \in D\}. \end{aligned}$$

$\text{Ann}(D, A_\infty)$ is an ideal in $A_\infty \cap D'$, so we write

$$F(D, A) := (A_\infty \cap D') / \text{Ann}(D, A_\infty)$$

and $\kappa_{D,A} : A_\infty \cap D' \rightarrow F(D, A)$ for the corresponding quotient map. When $D = A$, we write $F(A)$ for $F(A, A)$ and κ_A for $\kappa_{A,A}$. Note that $F(D, A)$ is unital if D is σ -unital.

Let G be a discrete group and $\alpha : G \rightarrow \text{Aut}(A)$ be an action of G on a C*-algebra A . If D is an α -invariant subalgebra of A , there is a natural induced action of G on A_∞ and on $F(D, A)$ which we denote by α_∞ and $\tilde{\alpha}_\infty$ respectively. Moreover, there is a G -equivariant *-homomorphism $(F(D, A) \otimes_{\max} D, \tilde{\alpha}_\infty \otimes \alpha|_D) \rightarrow (A_\infty, \alpha_\infty)$ given on elementary tensors by $\kappa_{D,A}(x) \otimes a \rightarrow x \cdot a$. Under this *-homomorphism $1_{F(D,A)} \otimes a$ is mapped to a for all $a \in D$, so we think of it as a way to multiply elements of $F(D, A)$ with elements of D to obtain elements of A_∞ (in a way that respects the action of G).

With all this notation in place, we are now ready to state the theorem that connects the notion of Rokhlin dimension to that of the profinite completion. Before we state the result, recall that if G is a discrete, residually finite group that is also finitely generated, then \mathcal{I}_G is countable. Therefore, \overline{G} is metrizable and $C(\overline{G}^{*(n)})$ is a separable C*-algebra (see [MV23, Lemma 1.3]). This fact is used in the proof of this result.

Theorem 1.14. [MV23, Theorem 2.10] *Let A be a separable C*-algebra, G be a finitely generated, residually finite group and $\alpha : G \rightarrow \text{Aut}(A)$ be an action of G on A . Then $\dim_{\text{Rok}}^c(\alpha) \leq d$ if and only if there is a unital, G -equivariant *-homomorphism*

$$\varphi : (C(\overline{G}^{*(d+1)}), \sigma^{(d+1)}) \rightarrow (F(A), \tilde{\alpha}_\infty).$$

Moreover, $\dim(\overline{G}^{*(d+1)}) \leq d$.

The above theorem allows us to analyze the crossed product C*-algebra along the lines of [GHS21, Section 4]. Suppose $\alpha : G \rightarrow \text{Aut}(A)$ is an action with $d := \dim_{\text{Rok}}^c(\alpha) < \infty$. Set $Y := \overline{G}^{*(d+1)}$ and let $C(Y, A)$ be equipped with the diagonal action $\gamma : G \rightarrow \text{Aut}(C(Y, A))$ given by $\gamma_g(f \otimes a) := \sigma_g^{(d+1)}(f) \otimes \alpha_g(a)$. Then, the inclusion map $\eta : A \rightarrow C(Y, A)$ (as constant functions) is a G -equivariant map. As it turns out, this map is the key to understanding the crossed product.

Definition 1.15. [BS16, Definition 3.3] Let G be a discrete group and let (A, G, α) and (B, G, β) be two separable C^* -dynamical systems. A G -equivariant $*$ -homomorphism $\varphi : A \rightarrow B$ is said to be (G -equivariantly) sequentially split if there is a commutative diagram of the form

$$\begin{array}{ccc} (A, \alpha) & \xrightarrow{\iota} & (F(A), \alpha_\infty) \\ & \searrow \varphi & \nearrow \\ & (B, \beta) & \end{array}$$

where the horizontal map is the natural inclusion.

We now obtain a direct analogue of [GHS21, Proposition 4.11].

Proposition 1.16. *If $d := \dim_{\text{Rok}}^c(\alpha) < \infty$, then the inclusion map $\eta : A \rightarrow C(\overline{G}^{*(d+1)}, A)$ induces a sequentially split $*$ -homomorphism $\overline{\eta} : A \rtimes_\alpha G \rightarrow C(\overline{G}^{*(d+1)}, A) \rtimes_{\sigma^{(d+1)} \otimes \alpha} G$.*

Proof. There is a map $F(A) \otimes A \rightarrow F(A)$ given on elementary tensors by $(a + \text{Ann}(A_\infty, A)) \otimes b \mapsto ab + \text{Ann}(A_\infty, A)$. Also, this is G -equivariant with respect to the diagonal action $\alpha_\infty \otimes \alpha$ on $F(A) \otimes A$. By Theorem 1.14, there is a G -equivariant unital $*$ -homomorphism $\varphi : C(Y) \rightarrow F(A)$, which induces a G -equivariant $*$ -homomorphism $\varphi : C(Y) \otimes A \rightarrow F(A) \otimes A$. Composing with the previous map, we get a G -equivariant $*$ -homomorphism

$$\psi : (C(Y) \otimes A, \gamma) \rightarrow (F(A), \alpha_\infty).$$

If $\eta : A \rightarrow C(Y) \otimes A$ denotes the natural inclusion, then it is clear that $\psi \circ \eta(a) = \iota(a)$ for all $a \in A$. Thus, η is G -equivariantly sequentially split. By [BS16, Proposition 3.11], the induced map $\overline{\eta} : A \rtimes_\alpha G \rightarrow C(Y, A) \rtimes_\gamma G$ is sequentially split. \square

We now wish to have a slight refinement of Proposition 1.16, which necessitates the introduction of a new notion, that of a dominating regular approximation from [SWZ19]. Let G be a finitely generated abelian group. A decreasing sequence $\tau := (H_n)$ of finite index subgroups of G is said to be a regular approximation if for each $g \in G$, there exists $n \in \mathbb{N}$ such that $g \notin H_n$. Moreover, τ is said to be dominating if, for every $H \leq_{\text{fin}} G$, there exists $n \in \mathbb{N}$ such that $H_n \subset H$.

Now, suppose (A, G, α) is a C^* -dynamical system and $\tau = (H_n)$ is a dominating regular approximation. If \mathcal{I}_G denotes the directed set of all finite index subgroups of G , then τ can be thought of as a cofinal subset of \mathcal{I}_G . If $\overline{G}_\tau := \varprojlim (G/H, \varphi_{K,H}, \tau)$, then by [RZ10, Lemma 1.1.9], there is a homeomorphism $\overline{G} \rightarrow \overline{G}_\tau$. Moreover, this homeomorphism is easily seen to be G -equivariant. Therefore, by Lemma 1.9, we conclude that $C(\overline{G}^{*(n)})$ is isomorphic to $C(\overline{G}_\tau^{*(n)})$ and that the isomorphism is G -equivariant (For simplicity, we denote the action on both algebras by $\sigma^{(n)}$). We thus obtain the following consequence of Proposition 1.16.

Proposition 1.17. *Let (A, G, α) be a C^* -dynamical system with G finitely generated and abelian. If $d := \dim_{\text{Rok}}^c(\alpha) < \infty$ and τ is a dominating regular approximation in G , then the inclusion map $\eta : (A, \alpha) \rightarrow (C(\overline{G}_\tau^{*(d+1)}, A), \sigma^{(d+1)} \otimes \alpha)$ induces a sequentially split $*$ -homomorphism*

$$\overline{\eta} : A \rtimes_\alpha G \rightarrow C(\overline{G}_\tau^{*(d+1)}, A) \rtimes_{\sigma^{(d+1)} \otimes \alpha} G.$$

We end this section with a short remark concerning sequentially split $*$ -homomorphisms that will be useful to us later on.

Remark 1.18. If $\varphi : A \rightarrow B$ is a sequentially split $*$ -homomorphism between two separable C^* -algebras, then [BS16, Theorem 2.9] states that a number of properties of B are inherited by A . We list two that we will need later:

- (1) If B is an AT -algebra, then so is A .
- (2) If B is simple and has stable rank one, then A is simple and has stable rank one.

2. MAIN RESULTS

We now consider actions on separable AF-algebras. Given such an algebra A , a sequence (A_n) of finite dimensional C^* -subalgebras is called a generating nest if it is an increasing sequence whose union is dense in A .

Definition 2.1. Let A be a separable AF-algebra. An action $\alpha : G \rightarrow \text{Aut}(A)$ is said to be an inductive limit action if there is a generating nest of finite dimensional G -invariant C^* -subalgebras of A .

Recall that a C^* -dynamical system (A, G, α) is said to be unitary (or unitarily implemented) if there is a homomorphism $u : G \rightarrow \mathcal{UM}(A)$ such that $\alpha_s(a) = u_s a u_s^*$ for all $a \in A$ and $s \in G$. In other words, (A, G, α) is unitary if and only if it is exterior equivalent to the trivial system.

Henceforth, let G be a finitely generated abelian group and let \mathcal{H}_G denote the set of all finite index, free subgroups of G .

Lemma 2.2. *Let G be a finitely generated abelian group, D be a finite dimensional C^* -algebra and $\alpha : G \rightarrow \text{Aut}(D)$ be an action. Then, there exists $H \in \mathcal{H}_G$ such that $\alpha_H : H \rightarrow \text{Aut}(D)$ is unitary.*

Proof. We may assume without loss of generality that G is infinite. First assume $G = \mathbb{Z}^r$ and $D = M_k(\mathbb{C})$ for some $r \geq 1$ and $k \geq 1$. Write $\{e_1, e_2, \dots, e_r\}$ denote the standard basis for \mathbb{Z}^r as a \mathbb{Z} -module. Let $\alpha^i := \alpha_{e_i}$. Then, there exists $u_i \in \mathcal{U}(D)$ such that $\alpha^i = \text{Ad}(u_i)$. Fix $1 \leq i, j \leq r$. Since $\alpha^i \circ \alpha^j = \alpha^j \circ \alpha^i$, it follows that $u_i u_j u_i^{-1} u_j^{-1} \in Z(D)$, the center of D . Therefore, there exists $\lambda \in \mathbb{T}$ such that $u_i u_j u_i^{-1} u_j^{-1} = \lambda 1_D$. Taking determinants, we see that $\lambda^k = 1$, so $\lambda = e^{2\pi i \ell / k}$ for some $0 \leq \ell \leq k - 1$. Now, $(\alpha^i)^k = \text{Ad}(u_i^k)$ and $u_i u_j = \lambda u_j u_i$, so

$$u_i^k u_j^k = \lambda^{k^2} u_j^k u_i^k = u_j^k u_i^k.$$

Therefore, the tuple $(u_1^k, u_2^k, \dots, u_r^k)$ are tuple of mutually commuting unitaries in D . If $H := \prod_{i=1}^r k\mathbb{Z}$, then this defines a homomorphism $w : H \rightarrow \mathcal{U}(D)$ such that $\alpha_t = \text{Ad}(w_t)$ for all $t \in H$.

Now suppose D is an arbitrary finite dimensional C^* -algebra written as $D = \bigoplus_{j=1}^{\ell} D_j$, where each D_j is a full matrix algebra. Then there exists $s \in \mathbb{N}$ such that each $(\alpha^i)^s$ preserves each summand of D . We may therefore express $(\alpha^i)^s$ as

$$(\alpha^i)^s = \text{Ad}((v_{i,1}, v_{i,2}, \dots, v_{i,\ell}))$$

for some unitaries $v_{i,j} \in \mathcal{U}(D_j)$. Now fix $1 \leq j \leq \ell$ and consider $\{v_{1,j}, v_{2,j}, \dots, v_{r,j}\} \subset \mathcal{U}(D_j)$. By the first part of the proof, there exists $N_j \in \mathbb{N}$ such that $\{v_{1,j}^{N_j}, v_{2,j}^{N_j}, \dots, v_{r,j}^{N_j}\}$ form a commuting family of unitaries. If $N := s \times \text{lcm}(N_1, N_2, \dots, N_\ell)$, then we obtain a commuting family of unitaries $\{w_1, w_2, \dots, w_r\} \subset \mathcal{U}(D)$ such that $(\alpha^i)^N = \text{Ad}(w_i)$ for all $1 \leq i \leq r$. If $H := \prod_{i=1}^r N\mathbb{Z}$, this defines a homomorphism $w : H \rightarrow \mathcal{U}(D)$ such that $\alpha_t = \text{Ad}(w_t)$ for all $t \in H$.

Now suppose G is an arbitrary finitely generated abelian group, then we may write $G = \mathbb{Z}^r \oplus K$ for some $r \geq 0$ and K finite. Let $\alpha_1 : \mathbb{Z}^r \rightarrow \text{Aut}(D)$ and $\alpha_2 : K \rightarrow \text{Aut}(D)$ be the induced actions of \mathbb{Z}^r and K respectively. By the first part of the proof, there is a finite index subgroup $H_1 < \mathbb{Z}^r$ and a homomorphism $w_1 : H_1 \rightarrow \mathcal{U}(D)$ such that $(\alpha_1)_t = \text{Ad}(w_1(t))$ for all $t \in H_1$. Let $H := H_1 \times \{e\} \in \mathcal{H}_G$ and $w : H \rightarrow \mathcal{U}(D)$ be given by $w(t, e) := w_1(t)$. Then, w is a homomorphism and for all $(t, e) \in H$, $\alpha_{(t,e)} = (\alpha_1)_t = \text{Ad}(w_1(t)) = \text{Ad}(w(t, e))$. \square

Lemma 2.3. *Let G be a finitely generated abelian group, and $\alpha : G \rightarrow \text{Aut}(A)$ be an inductive limit action of G on an AF-algebra A with Rokhlin dimension d , and let (R_n) be a generating nest of G -invariant finite dimensional subalgebras. Write $\alpha^{(n)} : G \rightarrow \text{Aut}(R_n)$ for the induced action of G on R_n . Then, there is a dominating regular approximation $\tau = (H_k) \subset \mathcal{H}_G$ such that for each $n \in \mathbb{N}$, $\alpha^{(n)}|_{H_k} : H_k \rightarrow \text{Aut}(R_n)$ is unitary for all $k \geq n$.*

Proof. We construct the sequence (H_k) inductively. First choose a sequence (L_k) which forms a dominating regular approximation (which exists by [SWZ19, Proposition 3.20]). Now for R_1 , we apply Lemma 2.2 to obtain a finite index subgroup $G_1 \in \mathcal{H}_G$ such that $\alpha^{(1)}|_{G_1} : G_1 \rightarrow \text{Aut}(R_1)$ is unitary. Set $H_1 := L_1 \cap G_1$, so that $\alpha^{(1)}|_{H_1} : H_1 \rightarrow \text{Aut}(R_1)$ is unitary. Since $G_1 \in \mathcal{H}_G$, it follows that $H_1 \in \mathcal{H}_G$ as well. Now suppose we have chosen $\{H_1, H_2, \dots, H_k\}$ such that

- (1) $H_i \in \mathcal{H}_G$ for all $1 \leq i \leq k$.
- (2) $H_{i+1} \subset H_i$ for all $1 \leq i \leq k-1$ and $H_i \subset L_i$ for all $1 \leq i \leq k$.
- (3) For each $1 \leq j \leq k$, $\alpha^{(j)}|_{H_i} : H_i \rightarrow \text{Aut}(R_j)$ is unitary for all $j \leq i \leq k$.

For R_{k+1} , by Lemma 2.2, there is a finite index subgroup $G_{k+1} \in \mathcal{H}_G$ such that $\alpha^{(k+1)}|_{G_{k+1}} : G_{k+1} \rightarrow \text{Aut}(R_{k+1})$ is unitary. Define $H_{k+1} := H_k \cap L_{k+1} \cap G_{k+1}$. Then, $\{H_1, H_2, \dots, H_{k+1}\}$ satisfies the three conditions listed above. Proceeding inductively, we construct the sequence (H_k) as desired. \square

With these comments, we are now in a position to describe the structure of the crossed product C*-algebra $A \rtimes_\alpha G$ when A is an AF-algebra, G is a finitely generated abelian group and $\alpha : G \rightarrow \text{Aut}(A)$ is an inductive limit action with finite Rokhlin dimension.

Definition 2.4. For non-negative integers d and r , let $\mathcal{C}_{d,r}$ denote the class of all continuous $C(Y)$ -algebras, where Y is a compact Hausdorff space with $\dim(Y) \leq d$, and each fiber is isomorphic to a corner of an algebra of the form $C(\mathbb{T}^r) \otimes F$, where F is finite dimensional. We write $\mathcal{Q}_{d,r}$ for the class of all C*-algebras that are obtained as quotients of members of $\mathcal{C}_{d,r}$. Finally, we say that a C*-algebra B is locally in $\mathcal{Q}_{d,r}$ if for each finite set $F \subset B$ and each $\epsilon > 0$, there is a subalgebra C of B such that $C \in \mathcal{Q}_{d,r}$ and $d(a, C) < \epsilon$ for all $a \in F$.

Theorem 2.5. *Let G be a finitely generated abelian group of rank r , and $\alpha : G \rightarrow \text{Aut}(A)$ be an inductive limit action of G on a separable AF-algebra A with $d := \dim_{\text{Rok}}^c(\alpha) < \infty$. Then there is a C*-algebra B that is locally in $\mathcal{Q}_{d,r}$ which admits a sequentially split *-homomorphism $\varphi : A \rtimes_\alpha G \rightarrow B$.*

Proof. Let (R_n) be a sequence of G -invariant finite dimensional subalgebras, and let $\tau := (H_k)$ be a dominating regular approximation satisfying the conditions of Lemma 2.3. Let \overline{G}_τ denote the corresponding profinite group. Then, by Proposition 1.17, there is sequentially split *-homomorphism

$$\overline{\eta} : A \rtimes_\alpha G \rightarrow C(\overline{G}_\tau^{*(d+1)}, A) \rtimes_{\sigma^{(d+1)} \otimes \alpha} G.$$

We now show that $C(\overline{G}_\tau^{*(d+1)}, A) \rtimes_{\sigma^{(d+1)} \otimes \alpha} G$ is locally in $\mathcal{Q}_{d,r}$. Let $\sigma_k := (\sigma^{H_k})^{(d+1)}$ denote the natural action of G on $C(G/H_k^{*(d+1)})$. Then there are two inductive limits of G -algebras

$$(C(\overline{G}_\tau^{*(d+1)}, \sigma^{(d+1)}) \cong \lim_{k \rightarrow \infty} (C(G/H_k^{*(d+1)}), \sigma_k), \text{ and}$$

$$(A, \alpha) \cong \lim_{n \rightarrow \infty} (R_n, \alpha^{(n)}).$$

where $\alpha^{(n)} : G \rightarrow \text{Aut}(R_n)$ denotes the induced action on R_n . Hence,

$$\begin{aligned} C(\overline{G}_\tau^{*(d+1)}, A) \rtimes_{\sigma^{(d+1)} \otimes \alpha} G &\cong \lim_{n \rightarrow \infty} C(\overline{G}_\tau^{*(d+1)}, R_n) \rtimes_{\sigma^{(d+1)} \otimes \alpha^{(n)}} G \\ &\cong \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} C(G/H_k^{*(d+1)}, R_n) \rtimes_{\sigma_k \otimes \alpha^{(n)}} G \\ &\cong \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} (C(G/H_k^{*(d+1)}, R_n) \rtimes_{\sigma_k \otimes \alpha^{(n)}|_{H_k}} H_k) \rtimes_{\gamma_k^{(n)}, u_k^{(n)}} G/H_k \end{aligned}$$

for some twisted action $(\gamma_k^{(n)}, u_k^{(n)})$ of G/H_k on $(C(G/H_k^{*(d+1)}, R_n) \rtimes_{\sigma_k \otimes \alpha^{(n)}|_{H_k}} H_k)$. Fix $k \in \mathbb{N}$, and write $X_k := G/H_k^{*(d+1)}$ and consider the algebra $A_{k,n} := C(X_k, R_n) \rtimes_{\sigma_k \otimes \alpha^{(n)}|_{H_k}} H_k$. Note that $\sigma_k|_{H_k}$ is trivial. Therefore,

$$A_{k,n} \cong C(X_k) \otimes (R_n \rtimes_{\alpha^{(n)}|_{H_k}} H_k).$$

By [Lemma 1.3](#), this isomorphism respects the twisted action of G/H_k . Therefore,

$$B_{k,n} := A_{k,n} \rtimes_{\gamma_k^{(n)}, u_k^{(n)}} G/H_k \cong (C(X_k) \otimes (R_n \rtimes_{\alpha^{(n)}|_{H_k}} H_k)) \rtimes_{\widetilde{\gamma}_k^{(n)}, \widetilde{u}_k^{(n)}} G/H_k.$$

for some other twisted action $(\widetilde{\gamma}_k^{(n)}, \widetilde{u}_k^{(n)})$ of G/H_k on $C(X_k) \otimes (R_n \rtimes_{\alpha^{(n)}|_{H_k}} H_k)$. However, the proof of [Lemma 1.3](#) also shows that $\widetilde{\gamma}_k^{(n)}$ is of the form $\widetilde{\gamma}_k^{(n)} = \widetilde{\sigma}_k \otimes \rho$, where $\widetilde{\sigma}_k : G/H_k \rightarrow \text{Aut}(C(X_k))$ is induced by the natural free action of G/H_k on X_k . Since G/H_k is finite, this falls in the ambit of [Corollary 1.8](#). Let $Y_k := X_k/(G/H_k)$, then $\dim(Y_k) \leq \dim(X_k) \leq d$ by the second part of [Theorem 1.14](#). By [Corollary 1.8](#), $B_{k,n}$ is a continuous $C(Y_k)$ -algebra, each fiber of which is isomorphic to a corner of $M_{|G/H_k|^2}(D_{k,n})$, where $D_{k,n} := R_n \rtimes_{\alpha^{(n)}|_{H_k}} H_k$. For a given $n \in \mathbb{N}$, we may choose $K_0 \in \mathbb{N}$ so that $\alpha^{(n)}|_{H_k}$ is unitary for all $k \geq K_0$ (by [Lemma 2.3](#)). Therefore,

$$D_{k,n} \cong R_n \otimes C^*(H_k) \cong R_n \otimes C(\mathbb{T}^r)$$

since $H_k \in \mathcal{H}_G$. Hence, $B_{k,n} \in \mathcal{C}_{d,r}$. It follows that $C(\overline{G}_\tau^{*(d+1)}, A) \rtimes_{\sigma^{(d+1)} \otimes \alpha} G$ is locally in $\mathcal{Q}_{d,r}$. \square

In order to truly reap the rewards of [Theorem 2.5](#), we must now turn our attention to actions of \mathbb{Z} . Note that this amounts to understanding a single automorphism. To do this, we need some ideas introduced by Voiculescu in [[Voi86](#)].

Fix a unital, separable AF-algebra A . For two C^* -algebras $C, D \subset A$ and $\epsilon > 0$, we write $C \subset^\epsilon D$ if for every $x \in C_{\leq 1}$ there exists $y \in D_{\leq 1}$ such that $\|x - y\| < \epsilon$. We then define

$$d(C_1, C_2) := \inf\{\epsilon > 0 : C_1 \subset^\epsilon C_2 \text{ and } C_2 \subset^\epsilon C_1\}.$$

Definition 2.6. Let A be a unital, separable AF-algebra and $\alpha \in \text{Aut}(A)$ be a fixed automorphism. Then, α is called

- (1) an almost inductive limit automorphism if there exists a generating nest of finite dimensional C^* -subalgebras (A_n) of A , such that $\lim_{n \rightarrow \infty} d(\alpha(A_n), A_n) = 0$.
- (2) an inductive limit automorphism if there is a generating nest of finite dimensional α -invariant C^* -subalgebras of A .
- (3) a limit periodic automorphism if there is a generating nest of finite dimensional α -invariant C^* -algebras (A_n) of A such that $\alpha|_{A_n} \in \text{Aut}(A_n)$ has finite order for each $n \in \mathbb{N}$.
- (4) an approximately inner automorphism if it is a point-norm limit of inner automorphisms of A (equivalently, $K_0(\alpha) = \text{id}_{K_0(A)}$).

Lemma 2.7. *Let A be a unital, separable AF-algebra and $\alpha : \mathbb{Z} \rightarrow \text{Aut}(A)$ be an approximately inner action. If α has Rokhlin property then there is a unitary $u \in \mathcal{U}(A)$ such that $\text{Ad}(u) \circ \alpha$ is a limit periodic automorphism.*

Proof. By [Voi86, Proposition 2.3 and Proposition 2.5], it suffices to show that α is an almost inductive limit automorphism. This is mentioned in passing in [Voi86, Remark 3.4] and we give a detailed argument. Indeed, it suffices to show that for each finite dimensional C*-subalgebra $D \subset A$ and each $\epsilon > 0$, there is a finite dimensional C*-subalgebra B of A such that $D \subset^\epsilon B$ and $d(\alpha(B), B) < \epsilon$. To prove this, fix a finite dimensional subalgebra D and an $\epsilon > 0$. Since α is approximately inner, there is $m \in \mathbb{N}$ and finite dimensional subalgebras $\{B_1, B_2, \dots, B_m\}$ such that $D \subset B_j$ for all $1 \leq j \leq m$ and $d(\alpha(B_j), B_{j+1}) < \epsilon$ for all $1 \leq j \leq m$ with the understanding that $B_{m+1} = B_1$ (by [Voi86, Lemma 3.1]).

Set $\delta := \frac{\epsilon}{3(m+1)} > 0$. Since α has the Rokhlin property, there is a *-homomorphism $\psi : C(\mathbb{Z}/m\mathbb{Z}) \rightarrow A$ satisfying the conditions of Definition 1.12 for $F = \{1_A\} \cup \{x \in \bigcup_{j=1}^m (B_j \cup \alpha(B_j)) : \|x\| \leq 1\}$ (which is compact). Setting $e_i = \psi(\delta_{\bar{i}})$, we obtain projections $\{e_1, e_2, \dots, e_m\}$ such that

- (1) $e_i e_j = 0$ if $i \neq j$,
- (2) $x e_i \approx_\delta e_i x$ for all $x \in F$
- (3) $\sum_{i=1}^m e_i \approx_\delta 1_A$,
- (4) $\alpha(e_i) \approx_\delta e_{i+1}$ with the understanding that $e_{m+1} = e_1$.

Define $B := \sum_{j=1}^m e_j B_j e_j$. Then, B is a finite dimensional C*-subalgebra of A , and if $a \in D_{\leq 1}$, then

$$a \approx_\delta \sum_{j=1}^{m_k} a e_j \approx_{m\delta} \sum_{j=1}^m e_j a e_j \in B,$$

so $D \subset^\epsilon B$. Now let $x \in B$ be chosen such that $x = \sum_{j=1}^m e_j x_j e_j$ with $x_j \in (B_j)_{\leq 1}$ for all $1 \leq j \leq m$. Then there exist $y_j \in (B_j)_{\leq 1}$ such that $\alpha(x_j) \approx_\epsilon y_{j+1}$ for all $1 \leq j \leq m$ (again, with the understanding that $y_{m+1} = y_1$). Now set $y := \sum_{j=1}^m e_j y_j e_j \in B$. Then,

$$\alpha(x) = \sum_{j=1}^m \alpha(e_j) \alpha(x_j) \alpha(e_j) \approx_{2m\delta} \sum_{j=1}^m e_{j+1} \alpha(x_j) e_{j+1} \approx_\epsilon \sum_{j=1}^m e_{j+1} y_{j+1} e_{j+1} = y$$

by [MV23, Lemma 5.5]. It follows that $d(\alpha(B), B) < 2\epsilon$, and we conclude that α is an almost inductive limit automorphism. \square

We are now in a position to complete the proof of Theorem A.

Proof of Theorem A. By Lemma 2.7, we may assume that α is a limit periodic automorphism. Let (R_n) be a sequence of α -invariant finite dimensional subalgebras such that $\alpha^{(n)} := \alpha|_{R_n}$ has finite order for each $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$, there is a subgroup $H_n < \mathbb{Z}$ such that $\alpha_{H_n}^{(n)} : H_n \rightarrow \text{Aut}(R_n)$ is trivial. Moreover, we may choose these subgroups as we did in Lemma 2.3 so that the family $\tau := (H_k)$ forms a dominating regular approximation and for each $n \in \mathbb{N}$, $\alpha^{(n)}|_{H_k} : H_k \rightarrow \text{Aut}(R_n)$ is trivial for all $k \geq n$. Let $\overline{\mathbb{Z}}_\tau$ denote the corresponding profinite group. Then, by Proposition 1.17, there is sequentially split *-homomorphism

$$\overline{\eta} : A \rtimes_\alpha \mathbb{Z} \rightarrow C(\overline{\mathbb{Z}}_\tau, A) \rtimes_{\sigma \otimes \alpha} \mathbb{Z}.$$

where $\sigma : \mathbb{Z} \rightarrow \text{Aut}(C(\overline{\mathbb{Z}}_\tau))$ denotes the natural action as before. By Remark 1.18, it now suffices to show that $C(\overline{\mathbb{Z}}_\tau, A) \rtimes_{\sigma \otimes \alpha} \mathbb{Z}$ is an AT-algebra. Let $\sigma_k := \sigma^{H_k}$ denote the natural action of \mathbb{Z} on $C(\mathbb{Z}/H_k)$. As in Theorem 2.5, we have

$$C(\overline{\mathbb{Z}}_\tau, A) \rtimes_{\sigma \otimes \alpha} \mathbb{Z} \cong \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} B_{k,n}$$

where $B_{k,n} = C(\mathbb{Z}/H_k, R_n) \rtimes_{\sigma_k \otimes \alpha^{(n)}} \mathbb{Z}$. By Green's Theorem [Gre80, Corollary 2.8], $B_{k,n} \cong (R_n \rtimes_{\alpha_{H_k}^{(n)}} H_k) \otimes \mathcal{K}(\ell^2(\mathbb{Z}/H_k))$. Since $\alpha_{H_k}^{(n)}$ is trivial for $k \geq n$, $R_n \rtimes_{\alpha_{H_k}^{(n)}} H_k \cong R_n \otimes C(\mathbb{T})$. Therefore,

$B_{k,n}$ is a circle algebra, and it follows that $C(\overline{\mathbb{Z}}_\tau, A) \rtimes_{\sigma \otimes \alpha} \mathbb{Z}$ is an AT-algebra.

For the second part of the theorem, assume that A is simple and we wish to prove that $A \rtimes_\alpha \mathbb{Z}$ is a simple C*-algebra with stable rank one. Once again, by [Remark 1.18](#), it suffices to show that $C(\overline{\mathbb{Z}}_\tau, A) \rtimes_{\sigma \otimes \alpha} \mathbb{Z}$ is simple and has stable rank one. Note that the action of \mathbb{Z} on $\overline{\mathbb{Z}}_\tau$ is free and minimal. Since A is simple, it follows that $C(\overline{\mathbb{Z}}_\tau, A)$ has no \mathbb{Z} -invariant ideals. Moreover, $\sigma \otimes \alpha$ also has the Rokhlin property by [\[MV23, Theorem 3.2\(1\)\]](#). Therefore, $C(\overline{\mathbb{Z}}_\tau, A) \rtimes_{\sigma \otimes \alpha} \mathbb{Z}$ is a simple C*-algebra by [\[MV23, Theorem 5.6\]](#). Since $C(\overline{\mathbb{Z}}_\tau, A) \rtimes_{\sigma \otimes \alpha} \mathbb{Z}$ is an AT-algebra, it follows from [\[Dĭ92, Theorem 3.6\]](#) that it has stable rank one as well.

Finally, to prove that $A \rtimes_\alpha \mathbb{Z}$ has real rank zero, we must appeal to [\[BDR91\]](#). In other words, we need to show that the projections in $A \rtimes_\alpha \mathbb{Z}$ separate the tracial states on $A \rtimes_\alpha \mathbb{Z}$. Note that A is necessarily infinite dimensional (since α is outer) and is therefore approximately divisible by [\[BKR92, Theorem 4.1\]](#). Since A has real rank zero, the projections in A separate the tracial states on A by [\[BKR92, Theorem 1.4\]](#). Now suppose ϕ is a tracial state on $A \rtimes_\alpha \mathbb{Z}$, then the argument of [\[Kis96, Lemma 4.3\]](#) show that $\phi(aU^n) = 0$ for all $n \neq 0$ (here, U denotes the canonical unitary in $A \rtimes_\alpha \mathbb{Z}$ implementing the action). Therefore, ϕ is completely determined by its values on A . Hence, the projections in A must themselves separate the tracial states on $A \rtimes_\alpha \mathbb{Z}$. By [\[BDR91, Theorem 1.3\]](#), it follows that $A \rtimes_\alpha \mathbb{Z}$ has real rank zero. This completes the proof. \square

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