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Homotopical stable ranks for certain C*-algebras

by

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Abstract. We study the general and connected stable ranks for C*-algebras. We estimate these ranks for pullbacks of C*-algebras, and for tensor products by commutative C*-algebras. Finally, we apply these results to determine these ranks for certain commutative C*-algebras and noncommutative CW-complexes.

Introduction. Stable ranks for C*-algebras were first introduced by Rieffel [16] in his study of the nonstable K-theory of noncommutative tori. A stable rank of a C*-algebra A is a number associated to the C*-algebra, and is meant to generalize the notion of covering dimension for topological spaces. The first such notion introduced by Rieffel, called *topological stable rank*, has played an important role ever since. In particular, the structure of C*-algebras having topological stable rank 1 is particularly well understood.

Since the foundational work of Rieffel, many other ranks have been introduced for C*-algebras, including real rank, decomposition rank, nuclear dimension, etc. In this paper, we return to the original work of Rieffel, and consider two other stable ranks introduced by him: the *connected stable rank* and *general stable rank*. The general stable rank determines the stage at which stably free projective modules are forced to be free. The connected stable rank is a related notion, but its definition is less transparent. What links these two ranks, and differentiates them from the topological stable rank, is that they are homotopy invariant.

This was highlighted in a paper by Nica [13], who emphasized the relationship between these two ranks, and how they differ from topological stable rank. Furthermore, in order to compute these ranks for various examples, he showed how they behave with respect to some basic constructions (matrix algebras, quotients, inductive limits, and extensions).

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The goal of the present paper is to extend these results by examining how the connected and general stable rank (together referred to as the *homotopical* stable ranks) behave with respect to iterated pullbacks and tensor products by commutative C*-algebras, and thereby calculate these ranks for some familiar C*-algebras. We mention here that the homotopical stable ranks play an important role in K-theory ([17], [20], [23]). Although we do not dwell on that much, we believe that further investigations into these ranks will yield a much better understanding of nonstable phenomena in K-theory.

We now describe our results. Henceforth, we write tsr, gsr, and csr to denote the topological, general and connected stable rank respectively. To begin with, consider a pullback diagram of unital C*-algebras

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \delta \\ C & \xrightarrow{\gamma} & D \end{array}$$

where either γ or δ is surjective. Note that $\mathrm{tsr}(A)$ can be estimated by a theorem of Brown and Pedersen [1, Theorem 4.1]:

$$\mathrm{tsr}(A) \leq \max\{\mathrm{tsr}(B), \mathrm{tsr}(C)\}.$$

However, simple examples (see Example 2.1) show that the analogous estimate for the homotopical stable ranks cannot hold. Instead, we show

THEOREM 0.1. *Given a pullback diagram as above,*

$$\mathrm{gsr}(A) \leq \max\{\mathrm{csr}(B), \mathrm{csr}(C), \mathrm{gsr}(C(\mathbb{T}) \otimes D))\},$$
$$\mathrm{csr}(A) \leq \max\{\mathrm{csr}(B), \mathrm{csr}(C), \mathrm{csr}(C(\mathbb{T}) \otimes D))\}.$$

Furthermore, if $K_1(D) = 0$*, the first estimate may be improved to*

$$\mathrm{gsr}(A) \leq \max\{\mathrm{gsr}(B), \mathrm{gsr}(C), \mathrm{gsr}(C(\mathbb{T}) \otimes D))\}.$$

We should mention here that the precise estimates are slightly finer than those mentioned above (see Proposition 2.7), and they depend on specific information about the homotopy groups of the groups $\mathrm{GL}_n(D)$ of invertible matrices over D.

We turn to tensor products by commutative C*-algebras. If Y is a compact Hausdorff space, a projective module over $C(Y)$ corresponds to a vector bundle over Y. Furthermore, if $Y = \Sigma X$, the reduced suspension of X, then any vector bundle over Y of rank n corresponds to the homotopy class of a map from X to $\mathrm{GL}_n(\mathbb{C})$. Building on work of Rieffel [18], we describe all projective modules over C*-algebras of the form $C(\Sigma X)\otimes A$ in an analogous fashion. We conclude:

THEOREM 0.2. *For a compact Hausdorff space X and a unital C^*-algebra A,*

$$\mathrm{gsr}(C(\Sigma X) \otimes A) = \max\{\mathrm{gsr}(A), \mathrm{inj}_X(A)\}$$

where $\mathrm{inj}_X(A)$ *denotes the least* $n \geq 1$ *such that the map* $[X, \mathrm{GL}_{m-1}(A)]_* \to [X, \mathrm{GL}_m(A)]_*$ *is injective for all* $m \geq n$.

We also obtain various estimates for $\mathrm{csr}(C(X) \otimes A)$, which, once again, depend on the homotopy groups of $\mathrm{GL}_n(A)$. These estimates are particularly sharp in the case where the natural map $\mathrm{GL}_{n-1}(A) \to \mathrm{GL}_n(A)$ is a weak homotopy equivalence. In that case, we explicitly determine the homotopical stable ranks of $C(X) \otimes A$ in terms of those of A (Theorem 3.11).

Finally, we apply these results to a variety of examples. In particular, using Theorem 0.2, we determine $\mathrm{gsr}(C(\mathbb{T}^d))$ (Example 4.4), thus answering a question of Nica [13, Problem 5.8]. We also estimate the homotopical stable ranks for noncommutative CW-complexes (Theorem 4.5), which naturally fall into the ambit of this paper.

1. Preliminaries

1.1. Stable ranks. Let A be a unital C*-algebra, and $n \in \mathbb{N}$. A vector $\underline{a} = (a_1, \ldots, a_n) \in A^n$ is said to be *left unimodular* if $Aa_1 + \cdots + Aa_n = A$. Equivalently, $\underline{a}$ is left unimodular if there exists $\underline{a'} = (a'_1, \ldots, a'_n) \in A^n$ such that $\sum_{i=1}^n a'_i a_i = 1_A$. We write $\mathrm{Lg}_n(A)$ for the set of all left unimodular vectors in A^n. There is an analogous notion of a right unimodular vector, but the continuous involution on A induces a homeomorphism between the two sets. For this reason, we only focus on the left unimodular vectors.

Unimodular vectors are related to projective modules by the following observation (see [16, Lemma 10.4]): If $\underline{a} \in A^n$, then $\underline{a} \in \mathrm{Lg}_n(A)$ if and only if $\underline{a}A$ is a direct summand of A^n:

$$A^n \cong P \oplus \underline{a}A. \tag{1.1}$$

Conversely, if P is a projective right A-module such that $P \oplus A \cong A^n$, then there is a left unimodular vector $\underline{a} \in \mathrm{Lg}_n(A)$ such that (1.1) holds. The most interesting fact in all this is that the vector $\underline{a}$ also determines when P is itself a free module.

Let $\mathrm{GL}_n(A)$ denote the set of invertible elements in $M_n(A)$. Note that $\mathrm{GL}_n(A)$ acts on $\mathrm{Lg}_n(A)$ by left multiplication: $(T, \underline{a}) \mapsto T(\underline{a})$. Let $e_n \in \mathrm{Lg}_n(A)$ denote the vector $(0, \ldots, 0, 1)$. Now, if P is a projective right A-module such that (1.1) holds, then $P \cong A^{n-1}$ if and only if there exists $T \in \mathrm{GL}_n(A)$ such that $T(\underline{a}) = e_n$ (see, for instance, [9, Proposition 4.14]). This leads to the following definition:

DEFINITION 1.1. Let A be a unital C*-algebra. Then the *general stable rank* (gsr) of A is the least $n \geq 1$ such that either, and hence both, of the following hold:

- $\mathrm{GL}_m(A)$ acts transitively on $\mathrm{Lg}_m(A)$ for all $m \geq n$;
- for all $m \geq n$, if P is a projective module over A such that $P \oplus A \cong A^m$, then $P \cong A^{m-1}$.

If no such n exists, we simply write $\mathrm{gsr}(A) = +\infty$. To avoid repetition, we will adopt the same convention in the definitions of connected and topological stable rank below.

Recall that A has the *invariant basis number* (IBN) property if $A^m \cong A^n$ implies that $m = n$. This is equivalent to requiring that $[A]$ has infinite order in $K_0(A)$. Now a swindle argument (see [9, Corollary 4.22]) shows that if A does not have the IBN property, then $\mathrm{gsr}(A) = +\infty$. Thus, in this paper, we will be concerned only with C*-algebras having this property.

Given a C*-algebra A that has the IBN property, any projective module P satisfying the condition $P \oplus A^m \cong A^n$ (i.e. *stably free* projective modules) may be assigned a rank (namely $n - m$), which is independent of the isomorphism. Hence, the general stable rank of A simply determines the least rank at which stably free projective modules are forced to be free.

Now, the first condition of Definition 1.1 leads to another observation: Any subgroup of $\mathrm{GL}_n(A)$ also acts on $\mathrm{Lg}_n(A)$. Let $\mathrm{GL}_n^0(A)$ denote the connected component of the identity in $\mathrm{GL}_n(A)$, and let $\mathrm{El}_n(A)$ denote the subgroup of $\mathrm{GL}_n(A)$ generated by elementary matrices, i.e. matrices which differ from the identity matrix by at most one off-diagonal entry. Note that $\mathrm{El}_n(A)$ is a subgroup of $\mathrm{GL}_n^0(A)$. It was proved by Rieffel [16, Corollary 8.10] that, for $n \geq 2$, $\mathrm{GL}_n^0(A)$ acts transitively on $\mathrm{Lg}_n(A)$ if and only if $\mathrm{El}_n(A)$ acts transitively on $\mathrm{Lg}_n(A)$. Furthermore, he observed that the least n for which this occurs also has a topological interpretation [16, Corollary 8.5], given as the second condition below.

Definition 1.2. Let A be a unital C*-algebra. Then the *connected stable rank* (csr) of A is the least $n \geq 1$ such that either, and hence both, of the following hold:

- $\mathrm{GL}_m^0(A)$ acts transitively on $\mathrm{Lg}_m(A)$ for all $m \geq n$;
- $\mathrm{Lg}_m(A)$ is connected for all $m \geq n$.

For $n \geq 2$ this is equivalent to the condition

- $\mathrm{El}_m(A)$ acts transitively on $\mathrm{Lg}_m(A)$ for all $m \geq n$.

We now turn to the notion of stable rank that has proved to be most useful in applications.

Definition 1.3. Let A be a unital C*-algebra. Then the *topological stable rank* (tsr) of A is the least $n \geq 1$ such that $\mathrm{Lg}_n(A)$ is dense in A^n.

We mention here that if $\mathrm{Lg}_n(A)$ is dense in A^n, then $\mathrm{Lg}_m(A)$ is dense in A^m for all $m \geq n$. However, the analogous statements are not true with respect to Definitions 1.1 and 1.2. Indeed, it is possible that $\mathrm{GL}_n(A)$ acts transitively on $\mathrm{Lg}_n(A)$, but $\mathrm{GL}_{n+1}(A)$ does not act transitively on $\mathrm{Lg}_{n+1}(A)$. For instance, if A is a finite C*-algebra, then $\mathrm{GL}_1(A) = \mathrm{Lg}_1(A)$, so $\mathrm{GL}_1(A)$ clearly acts transitively on $\mathrm{Lg}_1(A)$, but it is not true that $\mathrm{gsr}(A) = 1$ when A is finite.

REMARK 1.4. If A is a nonunital C*-algebra, then the stable rank of A is simply defined as the stable rank of A^+, the C*-algebra obtained by adjoining a unit to A.

We now enumerate some basic properties of these ranks that are known or are easily observed from the definitions. While the original proofs are scattered through the literature, [13] is an immediate reference for all these facts.

(1) $\mathrm{gsr}(A \oplus B) = \max\{\mathrm{gsr}(A), \mathrm{gsr}(B)\}$. Analogous statements hold for csr and tsr.
(2) $\mathrm{gsr}(A) \leq \mathrm{csr}(A) \leq \mathrm{tsr}(A) + 1$. Strict inequalities are possible in both cases. In fact, it is possible that $\mathrm{tsr}(A) = +\infty$ while $\mathrm{csr}(A) < \infty$.
(3) For any $n \in \mathbb{N}$,
$$\mathrm{csr}(M_n(A)) \leq \left\lceil \frac{\mathrm{csr}(A) - 1}{n} \right\rceil + 1, \qquad \mathrm{gsr}(M_n(A)) \leq \left\lceil \frac{\mathrm{gsr}(A) - 1}{n} \right\rceil + 1.$$
Here, $\lceil x \rceil$ is the least integer greater than or equal to x.
(4) If $\pi : A \to B$ is surjective, then
$$\mathrm{csr}(B) \leq \max\{\mathrm{csr}(A), \mathrm{tsr}(A)\}, \qquad \mathrm{gsr}(B) \leq \max\{\mathrm{gsr}(A), \mathrm{tsr}(A)\}.$$
(5) Furthermore, if $\pi : A \to B$ is a *split epimorphism* (i.e. there is a morphism $s : B \to A$ such that $\pi \circ s = \mathrm{id}_B$), then
$$\mathrm{csr}(B) \leq \mathrm{csr}(A), \qquad \mathrm{gsr}(B) \leq \mathrm{gsr}(A).$$
(6) If $0 \to J \to A \to B$ is an exact sequence of C*-algebras, then
$$\mathrm{csr}(A) \leq \max\{\mathrm{csr}(J), \mathrm{csr}(B)\}, \qquad \mathrm{gsr}(A) \leq \max\{\mathrm{gsr}(J), \mathrm{csr}(B)\}.$$
It is worth mentioning here that when J is an ideal of A, there is, a priori, no relation between the homotopical stable ranks of A and those of J.
(7) If $\{A_i : i \in J\}$ is an inductive system of C*-algebras with $A := \lim A_i$, then
$$\mathrm{csr}(A) \leq \liminf_{i \in J} \mathrm{csr}(A_i), \qquad \mathrm{gsr}(A) \leq \liminf_{i \in J} \mathrm{gsr}(A_i).$$
(8) If $\mathrm{gsr}(A) = 1$ (and hence if $\mathrm{csr}(A) = 1$), then A is stably finite. Conversely, if $\mathrm{gsr}(A) \leq 2$ and A is finite, then $\mathrm{gsr}(A) = 1$.
(9) If $\mathrm{csr}(A) = 1$, then $K_1(A) = 0$. The converse is true if $\mathrm{tsr}(A) = 1$.
(10) If $\mathrm{tsr}(A) = 1$, then A has cancellation of projections, so $\mathrm{gsr}(A) = 1$.

Finally, we turn to the most interesting property shared by gsr and csr, viz. homotopy invariance. Two morphisms $\phi_0, \phi_1 : A \to B$ are said to be *homotopic* (in symbols, $\varphi_0 \simeq \varphi_1$) if there is a $*$-homomorphism $h : A \to C([0,1], B)$ such that $\varphi_0 = p_0 \circ h$ and $\varphi_1 = p_1 \circ h$, where $p_t(\zeta) := \zeta(t)$. We say that A *homotopically dominates* B if there are morphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $\varphi \circ \psi \simeq \mathrm{id}_B$. If, in addition, $\psi \circ \varphi \simeq \mathrm{id}_A$, then we say that A and B are *homotopy equivalent* (in symbols, $A \simeq B$). In the commutative case, $C(X) \simeq C(Y)$ if and only if X and Y are homotopy equivalent as topological spaces (once again, we then write $X \simeq Y$). The following result is due to Nistor [14, Lemma 2.8] for the connected stable rank and Nica [13, Theorem 4.1] for the general stable rank:

Theorem 1.5. *If A homotopically dominates B, then*

$$\mathrm{csr}(A) \geq \mathrm{csr}(B), \qquad \mathrm{gsr}(A) \geq \mathrm{gsr}(B).$$

In particular, if $A \simeq B$, then $\mathrm{csr}(A) = \mathrm{csr}(B)$ *and* $\mathrm{gsr}(A) = \mathrm{gsr}(B)$.

We now turn to the problem of computing these ranks. An important tool in such an investigation is the following (see [17, Section 1]): For $m \geq 2$, the orbit of $e_m \in \mathrm{Lg}_m(A)$ under the action of $\mathrm{GL}_m(A)$ is called the space of *last columns* of A, and is denoted by $\mathrm{Lc}_m(A)$. As was first proved by Corach and Larotonda [2], the natural map $t : \mathrm{GL}_m(A) \to \mathrm{Lc}_m(A)$ defines a principal, locally trivial fiber bundle on $\mathrm{Lc}_m(A)$, with structural group $\mathrm{TL}_m(A)$, the set of matrices of the form

$$\begin{pmatrix} x & 0 \\ c & 1 \end{pmatrix}$$

where $x \in \mathrm{GL}_{m-1}(A)$ and $c \in A^{m-1}$. Now, $\mathrm{TL}_m(A)$ is homotopy equivalent to $\mathrm{GL}_{m-1}(A)$, so the long exact sequence of homotopy groups arising from the fibration $\mathrm{TL}_m(A) \to \mathrm{GL}_m(A) \xrightarrow{t} \mathrm{Lc}_m(A)$ takes the form

$$\cdots \to \pi_{n+1}(\mathrm{Lc}_m(A)) \to \pi_n(\mathrm{GL}_{m-1}(A)) \to \pi_n(\mathrm{GL}_m(A)) \to \pi_n(\mathrm{Lc}_m(A)) \to \cdots, \tag{1.2}$$

which ends in a sequence of pointed sets $\pi_0(\mathrm{GL}_{m-1}(A)) \to \pi_0(\mathrm{GL}_m(A)) \to \pi_0(\mathrm{Lc}_m(A))$. This will be of fundamental importance to us.

1.2. Notational conventions. We fix some notation we will use repeatedly: We write $\mathbb{S}^n$ for the n-dimensional sphere, $\mathbb{D}^n$ for the n-dimensional disk, $\mathbb{I}^k$ for the k-fold product of the unit interval $\mathbb{I} = [0,1]$, and $\mathbb{T}^k$ for the k-fold product of the circle $\mathbb{T}$. Given a C*-algebra A and a compact Hausdorff space X, we identify $C(X) \otimes A$ with $C(X, A)$, the space of continuous functions taking values in A. If $X = \mathbb{T}^k$, we simply write $\mathbb{T}^k A$ for $C(\mathbb{T}^k, A)$.

We write θ_A^n for the map $\mathrm{GL}_{n-1}(A) \to \mathrm{GL}_n(A)$ given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

If there is no ambiguity, we simply write θ_A for this map. Given a unital $*$-homomorphism $\varphi : A \to B$, we write φ_n for the induced maps in a variety of situations, such as $M_n(A) \to M_n(B)$, $\mathrm{GL}_n(A) \to \mathrm{GL}_n(B)$, $\mathrm{Lg}_n(A) \to \mathrm{Lg}_n(B)$, etc. Furthermore, when there is no ambiguity, we once again drop the subscript and denote the map by φ. Also, when dealing with modules over a C*-algebra, we will implicitly be referring to finitely generated *right* modules.

Given topological spaces X and Y, we will write $[X, Y]$ for the set of free homotopy classes of continuous maps between them. If X and Y are pointed spaces, then we write $[X, Y]_*$ for the set of based homotopy classes of continuous functions based at those distinguished points. Here, we will be concerned with three pointed spaces associated to a unital C*-algebra A: $\mathrm{GL}_n(A)$, as a subspace of $M_n(A)$ with base point I_n; $\mathrm{Lg}_m(A)$, as a subspace of A^m with base point e_m; and $\mathrm{Lc}_m(A)$, as a subspace of A^m with base point e_m.

2. Homotopical stable ranks of pullbacks. Given unital $*$-homomorphisms $\gamma : C \to D$ and $\delta : B \to D$, we consider the pullback

$$A := B \oplus_D C = \{(b, c) \in B \oplus C : \delta(b) = \gamma(c)\}.$$

As is customary, A is best described by a pullback diagram, which we fix throughout the section:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\delta} \\ C & \xrightarrow{\gamma} & D \end{array} \tag{2.1}$$

where α and β are the projection maps. Furthermore, we assume that either γ or δ is surjective. As pointed out in [1, example following Theorem 4.1], this is quite a natural assumption when considering stable ranks. The goal then is to determine $\mathrm{gsr}(A)$ and $\mathrm{csr}(A)$ in terms of those of B and C. To put things in perspective, we recall that the topological stable rank of A may be estimated by [1, Theorem 4.1]:

$$\mathrm{tsr}(A) \leq \max\{\mathrm{tsr}(B), \mathrm{tsr}(C)\}.$$

The next example shows that the corresponding estimate for homotopical stable ranks cannot hold.

Example 2.1. Consider the pullback diagram

$$\begin{array}{ccc} C(\mathbb{S}^n) & \longrightarrow & C(\mathbb{D}^n) \\ \big\downarrow & & \big\downarrow \delta \\ C(\mathbb{D}^n) & \xrightarrow{\ \gamma\ } & C(\mathbb{S}^{n-1}) \end{array}$$

where γ and δ are the natural restriction maps. Since $\mathbb{D}^n$ is contractible, we have $\mathrm{gsr}(C(\mathbb{D}^n)) = 1$, but $\mathrm{gsr}(C(\mathbb{S}^n)) > 1$ if $n \geq 5$ (see Example 3.1).

2.1. General stable rank. To determine $\mathrm{gsr}(A)$, we now describe a recipe due to Milnor [11] to construct projective modules over A. Given a unital ring homomorphism $f : R \to S$ and a right R-module M, we write $f_{\#}(M)$ for the right S-module $S \otimes_R M$, and denote by f_* the canonical map $M \to f_{\#}(M)$ given by $m \mapsto 1 \otimes_R m$. Note that if M is a free R-module with basis $\{b_\alpha\}$, then $f_{\#}(M)$ is a free S-module with basis $\{f_*(b_\alpha)\}$.

Now consider a pullback diagram as above. Let P and Q be projective modules over B and C respectively, and suppose we are given a D-isomorphism

$$h : \delta_{\#}(P) \to \gamma_{\#}(Q).$$

Consider

$$M := \{(p, q) \in P \oplus Q : h \circ \delta_*(p) = \gamma_*(q)\}.$$

Then M has a natural right A-module structure given by $(p, q) \cdot a := (p \cdot \alpha(a), q \cdot \beta(a))$. We denote this module by $M(P, Q, h)$. Milnor now proves the following

Theorem 2.2 ([11, Theorems 2.1–2.3]).

(1) *The module $M = M(P, Q, h)$ is projective over A. Furthermore, if P and Q are finitely generated over B and C respectively, then M is finitely generated over A.*
(2) *Every projective A-module is isomorphic to $M(P, Q, h)$ for some suitably chosen P, Q and h.*
(3) *The modules P and Q are naturally isomorphic to $\alpha_{\#}(M)$ and $\beta_{\#}(M)$ respectively.*

Furthermore, one has the following result. Recall that, for our purposes, we are only interested in C*-algebras that have the IBN property.

Proposition 2.3 ([9, Corollary 13.11]). *Given a pullback diagram as above, suppose that B or C has the IBN property. Let $h_1, h_2 \in \mathrm{GL}_n(D)$. Then $M(B^n, C^n, h_1) \cong M(B^n, C^n, h_2)$ if and only if $h_1 = \delta(S_1)h_2\gamma(S_2)$ for some $S_1 \in \mathrm{GL}_n(B)$, $S_2 \in \mathrm{GL}_n(C)$.*

For $h_1, h_2 \in \mathrm{GL}_n(D)$, we write $h_1 \sim h_2$ if there exist $S_1 \in \mathrm{GL}_n(B)$, $S_2 \in \mathrm{GL}_n(C)$ such that $h_1 = \delta(S_1)h_2\gamma(S_2)$. Note that this is an equivalence relation, whose equivalence classes are the double cosets

$$\delta(\mathrm{GL}_n(B))\backslash\mathrm{GL}_n(D)/\gamma(\mathrm{GL}_n(C)).$$

Given two elements $h_1, h_2 \in \mathrm{GL}_n(D)$, we write $h_1 \sim_{\mathrm{h}} h_2$ if there is a path $f : \mathbb{I} \to \mathrm{GL}_n(D)$ such that $f(0) = h_1$ and $f(1) = h_2$. The following observation now guides our investigation:

LEMMA 2.4. *Consider the pullback diagram as above, and suppose h_1, h_2 in $\mathrm{GL}_n(D)$ are such that $h_1 \sim_{\mathrm{h}} h_2$. Then $M(B^n, C^n, h_1) \cong M(B^n, C^n, h_2)$.*

Proof. Without loss of generality, assume that γ is surjective. Since $h_2^{-1}h_1 \in \mathrm{GL}_n^0(D)$, there exists $S_2 \in \mathrm{GL}_n(C)$ such that $h_2^{-1}h_1 = \gamma(S_2)$, and so $h_1 = \delta(I_{B^n})h_2\gamma(S_2)$. The result follows by Proposition 2.3. ∎

Consider the sequence of groups

$$\{1_D\} = \mathrm{GL}_0(D) \hookrightarrow D^\times = \mathrm{GL}_1(D) \hookrightarrow \mathrm{GL}_2(D) \hookrightarrow \cdots .$$

For a compact Hausdorff space X, this induces a sequence of homotopy groups (of maps based at the identity)

$$[X, \mathrm{GL}_0(D)]_* \to [X, \mathrm{GL}_1(D)]_* \to [X, \mathrm{GL}_2(D)]_* \to \cdots .$$

We define

- $\mathrm{inj}_X(D)$ to be the least $n \geq 1$ such that the map $(\theta_D)_* : [X, \mathrm{GL}_{m-1}(D)]_* \to [X, \mathrm{GL}_m(D)]_*$ is injective for all $m \geq n$;
- $\mathrm{surj}_X(D)$ to be the least $n \geq 1$ such that $(\theta_D)_* : [X, \mathrm{GL}_{m-1}(D)]_* \to [X, \mathrm{GL}_m(D)]_*$ is surjective for all $m \geq n$;
- $\mathrm{inj}_n(D)$ and $\mathrm{surj}_n(D)$ to be $\mathrm{inj}_X(D)$ and $\mathrm{surj}_X(D)$ respectively for $X = \mathbb{S}^n$.

REMARK 2.5. Let A be a unital C*-algebra, and X a compact Hausdorff space. Then the natural map $\mathrm{Lg}_m(C(X) \otimes A) \hookrightarrow C(X, \mathrm{Lg}_m(A))$ given by evaluation is a homeomorphism (see [17, Lemma 2.3]). It follows that

$$\pi_0(\mathrm{Lg}_m(C(X) \otimes A)) = [X, \mathrm{Lg}_m(A)]$$

where the right hand side denotes the free homotopy classes of maps from X to $\mathrm{Lg}_m(A)$. Furthermore, evaluation at a point gives a split epimorphism $C(X) \otimes A \to A$. Hence, it follows from Remark 1.4(5) that

$$\mathrm{csr}(C(X) \otimes A) \geq \mathrm{csr}(A), \qquad \mathrm{gsr}(C(X) \otimes A) \geq \mathrm{gsr}(A).$$

Before we begin, note that $\mathrm{GL}_n(A)$ is an open subset of a locally path connected space, so connected components in $\mathrm{GL}_n(A)$ coincide with path components. The same is true for $\mathrm{Lg}_m(A)$ and $\mathrm{Lc}_m(A)$ (see [17, Section 1]) as well, and we will use this fact implicitly henceforth.

LEMMA 2.6. *Let X be a compact Hausdorff space, A a unital C^*-algebra, and $m \in \mathbb{N}$. Let $F : [X, \mathrm{Lg}_m(A)]_* \to [X, \mathrm{Lg}_m(A)]$ be the forgetful map. If $m \geq \mathrm{csr}(A)$, then F is surjective. If $m \geq \mathrm{gsr}(\mathbb{T}A)$, then F is injective.*

Proof. Suppose $m \geq \mathrm{csr}(A)$. Then $\mathrm{Lc}_m(A) = \mathrm{Lg}_m(A)$ and $\mathrm{GL}_m^0(A)$ acts transitively on $\mathrm{Lg}_m(A)$. Let $x_0 \in X$ be a fixed base point, and $f : X \to \mathrm{Lg}_m(A)$ a continuous function. Then there exists $T \in \mathrm{GL}_m^0(A)$ such that $T(f(x_0)) = e_m$. Let $g : X \to \mathrm{Lg}_m(A)$ be given by $g(x) := T(f(x))$. If $h : \mathbb{I} \to \mathrm{GL}_m(A)$ is a path such that $h(0) = I_n$ and $h(1) = T$, then the homotopy $H : \mathbb{I} \times X \to \mathrm{Lg}_m(A)$ given by $H(t,x) := h(t)(f(x))$ is such that $H(0,x) = f(x)$ and $H(1,x) = g(x)$ for all $x \in X$. Hence, $[f] = [g]$ in $[X, \mathrm{Lg}_m(A)]$. Since $g(x_0) = e_m$, we see that F is surjective.

Now suppose $m \geq \mathrm{gsr}(\mathbb{T}A)$. Then $m \geq \mathrm{gsr}(A)$, so $\mathrm{Lc}_m(A) = \mathrm{Lg}_m(A)$. Let f and g be continuous functions from X to $\mathrm{Lg}_m(A)$ such that $f(x_0) = g(x_0) = e_m$, and suppose that there is a free homotopy $H : \mathbb{I} \times X \to \mathrm{Lg}_m(A)$ such that

$$H(0,x) = f(x), \qquad H(1,x) = g(x)$$

for all $x \in X$. Consider $\gamma : \mathbb{I} \to \mathrm{Lg}_m(A)$ given by $\gamma(t) := H(t, x_0)$. Then $\gamma(0) = \gamma(1) = e_m$, so γ induces a map $\overline{\gamma} : \mathbb{T} \to \mathrm{Lg}_m(A)$. Since $m \geq \mathrm{gsr}(\mathbb{T}A)$, we have $\mathrm{Lc}_m(\mathbb{T}A) = \mathrm{Lg}_m(\mathbb{T}A)$, and the map $t : \mathrm{GL}_m(\mathbb{T}A) \to \mathrm{Lc}_m(\mathbb{T}A)$ is surjective. Identifying $\mathrm{Lg}_m(\mathbb{T}A)$ with $C(\mathbb{T}, \mathrm{Lg}_m(A))$ and $\mathrm{GL}_m(\mathbb{T}A)$ with $C(\mathbb{T}, \mathrm{GL}_m(A))$, we see that the map t induces a surjective map $t : C(\mathbb{T}, \mathrm{GL}_m(A)) \to C(\mathbb{T}, \mathrm{Lg}_m(A))$. Hence, there exists $h : \mathbb{T} \to \mathrm{GL}_m(A)$ such that

$$h(z)e_m = \overline{\gamma}(z)$$

for all $z \in \mathbb{T}$. In particular, $h(1)e_m = e_m$ so that $h(1)^{-1}e_m = e_m$. Define $\overline{h} : \mathbb{T} \to \mathrm{GL}_m(A)$ by $\overline{h}(z) := h(z)h(1)^{-1}$. Then $\overline{h}(1) = I_n$ and $\overline{h}(z)e_m = \overline{\gamma}(z)$ for all $z \in \mathbb{T}$. The map $\overline{h}$ induces a map $\widetilde{h} : \mathbb{I} \to \mathrm{GL}_m(A)$ such that $\widetilde{h}(0) = \widetilde{h}(1) = I_n$ and $\widetilde{h}(t)e_m = \gamma(t)$ for all $t \in \mathbb{I}$. Now define a homotopy $\widetilde{H} : \mathbb{I} \times X \to \mathrm{Lg}_m(A)$ by

$$\widetilde{H}(t,x) := \widetilde{h}(t)^{-1}(H(t,x)).$$

Then $\widetilde{H}(0,x) = f(x)$ and $\widetilde{H}(1,x) = g(x)$ for all $x \in X$. Furthermore, $\widetilde{H}(t,x_0) = e_m$ for all $t \in \mathbb{I}$, so $\widetilde{H}$ defines a base-point preserving homotopy from f to g. Thus, the map $F : [X, \mathrm{Lg}_m(A)]_* \to [X, \mathrm{Lg}_m(A)]$ is injective. ∎

We will be most interested in the following quantities, which appear in the estimates for the homotopical stable ranks of a pullback:

Proposition 2.7. *For a unital C^*-algebra D,*

$$\begin{aligned} \mathrm{surj}_0(D) &\leq \mathrm{csr}(D), \\ \mathrm{inj}_0(D) &\leq \mathrm{gsr}(\mathbb{T}D), \\ \max\{\mathrm{inj}_0(D), \mathrm{surj}_1(D)\} &\leq \mathrm{csr}(\mathbb{T}D). \end{aligned}$$

Proof. Consider the long exact sequence of homotopy groups arising from the fibration $\mathrm{TL}_m(D) \to \mathrm{GL}_m(D) \to \mathrm{Lc}_m(D)$ (see (1.2)),

$$\cdots \to \pi_{n+1}(\mathrm{Lc}_m(D)) \to \pi_n(\mathrm{GL}_{m-1}(D)) \\ \to \pi_n(\mathrm{GL}_m(D)) \to \pi_n(\mathrm{Lc}_m(D)) \to \cdots,$$

which ends in a sequence of pointed sets $\pi_0(\mathrm{GL}_{m-1}(D)) \to \pi_0(\mathrm{GL}_m(D)) \to \pi_0(\mathrm{Lc}_m(D))$.

For the first inequality, suppose $m \geq \mathrm{csr}(D)$; then $m \geq \mathrm{gsr}(D)$, so $\mathrm{Lc}_m(D) = \mathrm{Lg}_m(D)$ is connected. Hence, $\pi_0(\mathrm{GL}_{m-1}(D)) \to \pi_0(\mathrm{GL}_m(D))$ is surjective.

For the second inequality, suppose $m \geq \mathrm{gsr}(\mathbb{T}D)$; then the natural map $t : \mathrm{GL}_m(\mathbb{T}D) \to \mathrm{Lc}_m(\mathbb{T}D)$ is surjective. As mentioned in Lemma 2.6, it follows that, for any loop $\overline{\gamma} : \mathbb{T} \to \mathrm{Lg}_m(A)$ based at e_m, there exists a loop $\overline{h} : \mathbb{T} \to \mathrm{GL}_m(A)$ such that $\overline{h}(1) = I_m$ and $\overline{h}(z)e_m = \overline{\gamma}(z)$ for all $z \in \mathbb{T}$. Hence, the map

$$\pi_1(\mathrm{GL}_m(D)) \to \pi_1(\mathrm{Lg}_m(D))$$

is surjective. By exactness of the above sequence, this implies that the map $\pi_0(\mathrm{GL}_{m-1}(D)) \to \pi_0(\mathrm{GL}_m(D))$ is injective.

For the third inequality, if $m \geq \mathrm{csr}(\mathbb{T}D)$, then $m \geq \mathrm{csr}(D)$, so $\mathrm{Lc}_m(D) = \mathrm{Lg}_m(D)$ is connected. Furthermore, $\mathrm{Lg}_m(\mathbb{T}D) = C(\mathbb{T}, \mathrm{Lg}_m(D))$, so that

$$0 = \pi_0(\mathrm{Lg}_m(C(\mathbb{T}, D))) = \pi_0(C(\mathbb{T}, \mathrm{Lg}_m(D))) = [\mathbb{T}, \mathrm{Lg}_m(D)].$$

By Remark 1.4(2), $m \geq \mathrm{gsr}(\mathbb{T}D)$, so $\pi_1(\mathrm{Lg}_m(D))$ is trivial by Lemma 2.6. The exactness of the above sequence now implies that $\pi_1(\mathrm{GL}_{m-1}(D)) \to \pi_1(\mathrm{GL}_m(D))$ is surjective, and $\pi_0(\mathrm{GL}_{m-1}(D)) \to \pi_0(\mathrm{GL}_m(D))$ is injective. ■

We are now ready to prove an estimate for the general stable rank of the pullback as in (2.1).

THEOREM 2.8. *Given a pullback diagram as above with either γ or δ surjective,*

$$\mathrm{gsr}(A) \leq \max\{\mathrm{csr}(B), \mathrm{csr}(C), \mathrm{inj}_0(D)\}.$$

Furthermore, if $K_1(D) = 0$, then

$$\mathrm{gsr}(A) \leq \max\{\mathrm{gsr}(B), \mathrm{gsr}(C), \mathrm{inj}_0(D)\}.$$

Proof. Let $m \geq \max\{\mathrm{csr}(B), \mathrm{csr}(C), \mathrm{inj}_0(D)\}$, and M be a projective A-module such that $M \oplus A \cong A^m$. Then write $M = M(P, Q, h)$ for some P, Q and h as in Theorem 2.2. Then

$$M(P \oplus B, Q \oplus C, h \oplus I_D) \cong A^m = M(B^m, C^m, I_{D^m}).$$

Hence,

$$P \oplus B \cong \alpha_\#(A^m) \cong B^m.$$

Since $m \geq \mathrm{csr}(B) \geq \mathrm{gsr}(B)$, it follows that $P \cong B^{m-1}$. Similarly, $Q \cong C^{m-1}$. Hence, we may think of h as belonging to $\mathrm{GL}_{m-1}(D)$. Now consider the diagram

$$\begin{array}{ccccc}
\mathrm{GL}_{m-1}(B) & \xrightarrow{\delta_{m-1}} & \mathrm{GL}_{m-1}(D) & \xleftarrow{\gamma_{m-1}} & \mathrm{GL}_{m-1}(C) \\
\downarrow{\scriptstyle \theta_B} & & \downarrow{\scriptstyle \theta_D} & & \downarrow{\scriptstyle \theta_C} \\
\mathrm{GL}_m(B) & \xrightarrow{\delta_m} & \mathrm{GL}_m(D) & \xleftarrow{\gamma_m} & \mathrm{GL}_m(C)
\end{array}$$

By Proposition 2.3, $\theta_D(h) \sim I_m$, so there exist $b \in \mathrm{GL}_m(B)$ and $c \in \mathrm{GL}_m(C)$ such that $\theta_D(h) = \delta_m(b)\gamma_m(c)$. Since $m \geq \mathrm{csr}(B)$, Proposition 2.7 implies that $m \geq \mathrm{surj}_0(B)$, so there exists $b' \in \mathrm{GL}_{m-1}(B)$ such that $b \sim_{\mathrm{h}} \theta_B(b')$. Hence,

$$\delta_m(b) \sim_{\mathrm{h}} \delta_m(\theta_B(b')) = \theta_D(\delta_{m-1}(b')).$$

Similarly, there exists $c' \in \mathrm{GL}_{m-1}(C)$ such that $\gamma_m(c) \sim_{\mathrm{h}} \theta_D(\gamma_{m-1}(c'))$, so

$$\theta_D(h) \sim_{\mathrm{h}} \theta_D(\delta_{m-1}(b')\gamma_{m-1}(c')).$$

Since $m \geq \mathrm{inj}_0(D)$, we have $h \sim_{\mathrm{h}} \delta_{m-1}(b')\gamma_{m-1}(c')$, and so by Lemma 2.4 and Proposition 2.3,

$$M \cong M(B^{m-1}, C^{m-1}, h) \cong A^{m-1}.$$

Hence, $m \geq \mathrm{gsr}(A)$ as required.

Now consider the special case where $K_1(D) = 0$. We follow the proof of the first part of the theorem until we obtain $h \in \mathrm{GL}_{m-1}(D)$. Note that, until this point, we only used the fact that $m \geq \max\{\mathrm{gsr}(B), \mathrm{gsr}(C)\}$. Since $m \geq \mathrm{inj}_0(D)$ and $K_1(D) = 0$, it follows that $h \sim_{\mathrm{h}} I_{D^{m-1}}$, so that $M \cong M(B^{m-1}, C^{m-1}, h) \cong A^{m-1}$ by Lemma 2.4. We conclude that $m \geq \mathrm{gsr}(A)$ as required. ∎

Note that Example 2.1 shows that the term $\mathrm{inj}_0(D)$ on the right hand side cannot be dropped, and furthermore that equality can hold in the above estimate, since $\mathrm{inj}_0(C(\mathbb{S}^4)) = \mathrm{gsr}(C(\mathbb{S}^5)) = 4$.

2.2. Connected stable rank. Given a pullback diagram $A = B \oplus_D C$ as before, we wish to estimate $\mathrm{csr}(A)$ in terms of $\mathrm{csr}(B)$ and $\mathrm{csr}(C)$. To begin with, we provide an alternate proof of the estimate of $\mathrm{gsr}(A)$. This will help guide us to a proof for the corresponding estimate for $\mathrm{csr}(A)$ as well.

Lemma 2.9. *Suppose $S_1 \in \mathrm{GL}_n(B)$, $S_2 \in \mathrm{GL}_n(C)$ satisfy $\delta(S_1) = \gamma(S_2)$. Then there exists a unique $T \in \mathrm{GL}_n(A)$ such that $\alpha(T) = S_1$, $\beta(T) = S_2$.*

Proof. Since $M_n(\mathbb{C})$ is nuclear, we obtain a pullback diagram

$$\begin{array}{ccc} M_n(A) & \xrightarrow{\alpha} & M_n(B) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\delta} \\ M_n(C) & \xrightarrow{\gamma} & M_n(D) \end{array}$$

by [15, Theorem 3.9]. Hence, there exists a unique $T \in M_n(A)$ such that $\alpha(T) = S_1$, $\beta(T) = S_2$. We show that $T \in \mathrm{GL}_n(A)$: To see this, let $S_1' \in \mathrm{GL}_n(B)$ be such that $S_1'S_1 = S_1S_1' = I_{B^n}$, and similarly $S_2' \in \mathrm{GL}_n(C)$ such that $S_2S_2' = S_2'S_2 = I_{C^n}$. Then

$$\delta(S_1')\delta(S_1) = \delta(S_1')\gamma(S_2) = I_{D^n} = \gamma(S_2')\gamma(S_2).$$

Hence, $\delta(S_1') = \gamma(S_2')$, so there exists $T' \in M_n(A)$ such that $\alpha(T') = S_1'$, $\beta(T') = S_2'$. Now note that

$$\alpha(T)\alpha(T') - \alpha(I_{A^n}) = S_1S_1' - I_{B^n} = 0,$$

so $TT' - I_{A^n} \in \ker(\alpha)$. Similarly, $TT' - I_{A^n} \in \ker(\beta)$. As $\ker(\alpha) \cap \ker(\beta) = \{0\}$, it follows that $TT' = I_{A^n}$, and similarly $T'T = I_{A^n}$. Hence, $T \in \mathrm{GL}_n(A)$. ∎

We now prove a fact that is probably well-known, but we have not found an exact reference. Since it is crucial to our arguments, we include a proof. Note that, for a nonunital C*-algebra A, we write A^+ for its unitization.

PROPOSITION 2.10. *If $\gamma : C \to D$ is a unital, surjective $*$-homomorphism, then it has the path-lifting property for invertibles: Given $n \in \mathbb{N}$ and a path $g : [0,1] \to \mathrm{GL}_n(D)$ and $S \in \mathrm{GL}_n(C)$ such that $g(1) = \gamma(S)$, there is a path $f : [0,1] \to \mathrm{GL}_n(C)$ such that $f(1) = S$ and $\gamma \circ f = g$.*

Proof. Consider the cone over $M_n(D)$, $\mathcal{C}M_n(D) = C_0[0,1) \otimes M_n(D)$, and observe that

$$\mathcal{C}M_n(D)^+ = \{g : [0,1] \to M_n(D) : g(1) \in \mathbb{C}I_n\}.$$

Since $C_0[0,1) \otimes M_n(\mathbb{C})$ is exact, the induced map $\gamma : \mathcal{C}M_n(C)^+ \to \mathcal{C}M_n(D)^+$ is surjective. Suppose $g : [0,1] \to \mathrm{GL}_n(D)$ and $S \in \mathrm{GL}_n(C)$ are such that $g(1) = \gamma(S)$. Replacing g by $g(\cdot)\gamma(S^{-1})$ we may assume without loss of generality that $g(1) = I$. Then $H : \mathbb{I} \times \mathbb{I} \to \mathrm{GL}_n(D)$ given by $H(s,t) = g(1 - s(1-t))$ is a continuous map such that $H(0,t) = I$, $H(1,t) = g(t)$, and $H(s,1) = I$. Thus, H defines a path in $\mathrm{GL}(\mathcal{C}M_n(D)^+)$ such that $H(0) = I$, $H(1) = g$. Hence, $g \in \mathrm{GL}_1^0(\mathcal{C}M_n(D)^+)$. So there exists $f \in \mathrm{GL}(\mathcal{C}M_n(C)^+)$ such that $\gamma(f) = g$. Since $f(1) \in \mathbb{C}I_n$ and γ is linear, it follows that $f(1) = I_n$. Thus, f satisfies the required conditions. ∎

As a warm-up for the estimate of the connected stable rank, we now provide a second proof of the estimate of the general stable rank from Theorem 2.8.

THEOREM 2.11. *Given a pullback diagram as above with either γ or δ surjective,*

$$\mathrm{gsr}(A) \leq \max\{\mathrm{csr}(B), \mathrm{csr}(C), \mathrm{inj}_0(D)\}.$$

Proof. Suppose $n \geq \max\{\mathrm{csr}(B), \mathrm{csr}(C), \mathrm{inj}_0(D)\}$. We wish to prove that $\mathrm{GL}_n(A)$ acts transitively on $\mathrm{Lg}_n(A)$. To this end, fix $v \in \mathrm{Lg}_n(A)$, so that $\alpha(v) \in \mathrm{Lg}_n(B)$. Since $n \geq \mathrm{csr}(B)$, there exists $S_1 \in \mathrm{GL}_n^0(B)$ such that $S_1\alpha(v) = e_n$. Hence, $\delta(S_1)w = e_n$ where $w = \delta(\alpha(v)) = \gamma(\beta(v))$. Similarly, there exists $S_2 \in \mathrm{GL}_n^0(C)$ such that $S_2\beta(v) = e_n$. Then $\gamma(S_2), \delta(S_1) \in \mathrm{GL}_n^0(D)$, so $\gamma(S_2) \sim_{\mathrm{h}} \delta(S_1)$.

Consider $S := \delta(S_1)\gamma(S_2)^{-1}$. Then $S \in \mathrm{GL}_n^0(D)$ and $Se_n = e_n$. Hence, S has the form

$$S = \begin{pmatrix} S' & 0 \\ c & 1 \end{pmatrix}$$

where $c \in D^{n-1}$ and $S' \in \mathrm{GL}_{n-1}(D)$. Since $S \sim_{\mathrm{h}} I_{D^n}$, it follows that

$$\begin{pmatrix} S' & 0 \\ 0 & 1 \end{pmatrix} \sim_{\mathrm{h}} S \sim_{\mathrm{h}} I_{D^n}$$

where the first homotopy linearly sends 0 to c. Since $n \geq \mathrm{inj}_0(D)$, we have $S' \sim_{\mathrm{h}} I_{D^{n-1}}$ via a path $g : \mathbb{I} \to \mathrm{GL}_{n-1}(D)$ such that $g(0) = S'$, $g(1) = I$. By the path-lifting property of γ, there is a path $h : \mathbb{I} \to \mathrm{GL}_{n-1}(C)$ such that $h(1) = I$ and $\gamma \circ h = g$. Let $c' \in C^{n-1}$ be such that $\gamma(c') = c$, and consider the element

$$S_2' := \begin{pmatrix} h(0) & 0 \\ c' & 1 \end{pmatrix} S_2 \in \mathrm{GL}_n^0(C).$$

Then

$$\gamma(S_2') = \begin{pmatrix} \gamma(h(0)) & 0 \\ c & 1 \end{pmatrix} \gamma(S_2) = \begin{pmatrix} S' & 0 \\ c & 1 \end{pmatrix} \gamma(S_2) = S\gamma(S_2) = \delta(S_1).$$

By Lemma 2.9, there exists $T \in \mathrm{GL}_n(A)$ such that $\alpha(T) = S_1, \beta(T) = S_2'$. Furthermore,

$$S_2'(\beta(v)) = \begin{pmatrix} h(0) & 0 \\ c' & 1 \end{pmatrix} S_2(\beta(v)) = \begin{pmatrix} h(0) & 0 \\ c' & 1 \end{pmatrix} e_n = e_n.$$

Hence, $Tv - e_n \in A^n$ has the property that $\beta(Tv - e_n) = 0$, and $\alpha(Tv - e_n) = 0$. Since $\ker(\alpha) \cap \ker(\beta) = \{0\}$, it follows that $Tv = e_n$, so $\mathrm{gsr}(A) \leq n$. ∎

We now wish to determine conditions under which the operator T produced in the above proof may be chosen to be in $\mathrm{GL}_n^0(A)$. We begin with a lemma. Given a C*-algebra C and $n \in \mathbb{N}$, we write $\iota : M_{n-1}(C) \to M_n(C)$ for the natural inclusion map, and define

$$X_n(C) := \{f : [0,1] \to M_n(C) : f(0) \in \iota(M_{n-1}(C)),\ f(1) = 0\}.$$

For a C*-algebra A, we write $\mathcal{C}A$ for the cone $C_0[0,1) \otimes A$.

Lemma 2.12. *Let $\gamma : C \to D$ be a unital, surjective $*$-homomorphism, and $n \in \mathbb{N}$ be fixed. Then the induced map $\gamma : X_n(C) \to X_n(D)$ is also surjective.*

Proof. Note that $X_n(C)$ is a pullback

$$\begin{array}{ccc} X_n(C) & \longrightarrow & M_{n-1}(C) \\ \big\downarrow & & \big\downarrow \iota \\ \mathcal{C}M_n(C) & \xrightarrow{\ \rho\ } & M_n(C) \end{array}$$

where $\rho(f) = f(0)$. We now wish to appeal to [15, Theorem 9.1]. To do this, consider the commuting diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{C}\ker(\gamma_n) & \longrightarrow & \mathcal{C}M_n(C) & \longrightarrow & \mathcal{C}M_n(D) & \longrightarrow & 0 \\ & & \big\downarrow \overline{\rho} & & \big\downarrow \rho & & \big\downarrow \widetilde{\rho} & & \\ 0 & \longrightarrow & \ker(\gamma_n) & \longrightarrow & M_n(C) & \longrightarrow & M_n(D) & \longrightarrow & 0 \\ & & \big\uparrow \overline{\iota} & & \big\uparrow \iota & & \big\uparrow \widetilde{\iota} & & \\ 0 & \longrightarrow & \ker(\gamma_{n-1}) & \longrightarrow & M_{n-1}(C) & \longrightarrow & M_{n-1}(D) & \longrightarrow & 0 \end{array}$$

where $\gamma_k : M_k(C) \to M_k(D)$ is the map induced by γ, and the vertical maps are induced by ρ and ι. In order to conclude that the map $\gamma : X_n(C) \to X_n(D)$ is surjective, we must verify that $E = F$ where

$$E := \rho(\mathcal{C}M_n(C)) \cap \iota(M_{n-1}(C)) \cap \ker(\gamma_n),$$
$$F := \overline{\rho}(\mathcal{C}\ker(\gamma_n)) \cap \iota(M_{n-1}(C)) + \rho(\mathcal{C}M_n(C)) \cap \overline{\iota}(\ker(\gamma_{n-1})).$$

Note that $\iota(M_{n-1}(C)) \cap \ker(\gamma_n) = \overline{\iota}(\ker(\gamma_{n-1}))$, so

$$E = \rho(\mathcal{C}M_n(C)) \cap \iota(M_{n-1}(C)) \cap \ker(\gamma_n) = \rho(\mathcal{C}M_n(C)) \cap \overline{\iota}(\ker(\gamma_{n-1})) \subset F$$

Furthermore, $\overline{\rho}$ is the restriction of ρ to $\mathcal{C}\ker(\gamma_n)$, therefore $\overline{\rho}(\mathcal{C}\ker(\gamma_n)) \subset \rho(\mathcal{C}M_n(C)) \cap \ker(\gamma_n)$. Hence,

$$\overline{\rho}(\mathcal{C}\ker(\gamma_n)) \cap \iota(M_{n-1}(C)) \subset \rho(\mathcal{C}M_n(C)) \cap \ker(\gamma_n) \cap \iota(M_{n-1}(C)) = E$$

and $F \subset E$ also holds. Thus, [15, Theorem 9.1] applies, and $\gamma : X_n(C) \to X_n(D)$ is surjective. ∎

We conclude that the induced map $\gamma : X_n(C)^+ \to X_n(D)^+$ is also surjective, and note that

$$X_n(C)^+ = \{f : [0,1] \to M_n(D) : \text{there exists } \lambda \in \mathbb{C} \text{ such that } f(0) \in \iota(M_{n-1}(D)) + \lambda I_n,\ f(1) = \lambda I_n\}.$$

In what follows, $\theta_C^n : \mathrm{GL}_{n-1}(C) \to \mathrm{GL}_n(C)$ denotes the natural inclusion of groups.

Lemma 2.13. *Let $\gamma : C \to D$ be a unital, surjective $*$-homomorphism, and suppose $n \in \mathbb{N}$ is such that $(\theta_D^n)_* : \pi_1(\mathrm{GL}_{n-1}(D)) \to \pi_1(\mathrm{GL}_n(D))$ is surjective. If $g : \mathbb{I} \to \mathrm{GL}_n(D)$ is a path such that $g(0) = g(1) = I_n$, then there exists $h : \mathbb{I} \to \mathrm{GL}_n(C)$ such that $h(0) \in \theta_C^n(\mathrm{GL}_{n-1}(C))$, $h(1) = I_n$, and $\gamma \circ h = g$.*

Proof. Consider $\overline{g} : \mathbb{T} \to \mathrm{GL}_n(D)$ induced by g. As $(\theta_D^n)_* : \pi_1(\mathrm{GL}_{n-1}(D)) \to \pi_1(\mathrm{GL}_n(D))$ is surjective, there exists $f : \mathbb{T} \to \mathrm{GL}_{n-1}(D)$ such that $f(1) = I_n$, and a homotopy $H : \mathbb{I} \times \mathbb{T} \to \mathrm{GL}_n(D)$ such that $H(t,1) = I_n$ for all $t \in \mathbb{I}$, and

$$H(0,z) = \overline{g}(z), \qquad H(1,z) = (\theta_D^n \circ f)(z)$$

for all $z \in \mathbb{T}$. Think of f as a path $f : \mathbb{I} \to \mathrm{GL}_{n-1}(D)$ such that $f(0) = f(1) = I_n$. By the path-lifting property, there exists $\overline{f} : \mathbb{I} \to \mathrm{GL}_{n-1}(C)$ such that $\overline{f}(1) = I_n$ and $\gamma \circ \overline{f} = f$.

Define $\widetilde{f} : \mathbb{I} \to \mathrm{GL}_n(C)$ by $\widetilde{f} := \theta_C^n \circ \overline{f}$, so that $\gamma \circ \widetilde{f} = \theta_D^n \circ f$. We may think of H as a map $H : \mathbb{I} \times \mathbb{I} \to \mathrm{GL}_n(D)$ such that $H(t,0) = H(t,1) = I_n$. Hence, H defines a path

$$H : \mathbb{I} \to \mathrm{GL}(X_n(D)^+)$$

such that $H(0) = g$ and $H(1) = \gamma \circ \widetilde{f}$. By the previous lemma and the path-lifting property of Proposition 2.10, there exists $\overline{H} : \mathbb{I} \to \mathrm{GL}(X_n(C)^+)$ such that $\gamma \circ \overline{H} = H$ and $\overline{H}(1) = \widetilde{f}$. Now $h := \overline{H}(0) \in \mathrm{GL}(X_n(C)^+)$ is a path $h : \mathbb{I} \to \mathrm{GL}_n(C)$ such that

$$\gamma \circ h = \gamma \circ \overline{H}(0) = H(0) = g.$$

Since $h(1) \in \mathbb{C}I_n$ and γ is linear, it follows that $h(1) = g(1) = I_n$. Hence, $h(0) \in (\iota(M_{n-1}(C)) + I_n) \cap \mathrm{GL}_n(C)$, which implies $h(0) \in \theta_C^n(\mathrm{GL}_{n-1}(C))$. ∎

We are now in a position to prove the estimate on the connected stable rank of A defined as a pullback as in (2.1). Together with Proposition 2.7 and Theorem 2.8, this completes the proof of Theorem 0.1.

Theorem 2.14. *Given a pullback diagram as above with either γ or δ surjective,*

$$\mathrm{csr}(A) \leq \max\{\mathrm{csr}(B), \mathrm{csr}(C), \mathrm{inj}_0(D), \mathrm{surj}_1(D)\}.$$

Proof. Let $n \geq \max\{\mathrm{csr}(B), \mathrm{csr}(C), \mathrm{inj}_0(D), \mathrm{surj}_1(D)\}$. Then we wish to prove that $\mathrm{GL}_n^0(A)$ acts transitively on $\mathrm{Lg}_n(A)$. To this end, fix $v \in \mathrm{Lg}_n(A)$, so that $\alpha(v) \in \mathrm{Lg}_n(B)$ and $\beta(v) \in \mathrm{Lg}_n(C)$. By the proof of Theorem 2.11, there exist $S_1 \in \mathrm{GL}_n^0(B)$ and $S_2 \in \mathrm{GL}_n^0(C)$ such that $S_1\alpha(v) = e_n$, $S_2\beta(v) = e_n$, and $\delta(S_1) = \gamma(S_2)$. Hence, we obtain $T \in \mathrm{GL}_n(A)$ such that $\alpha(T) = S_1$, $\beta(T) = S_2$, and $T(v) = e_n$.

Now fix a path $\overline{S_1} : \mathbb{I} \to \mathrm{GL}_n(B)$ such that $\overline{S_1}(0) = S_1$ and $\overline{S_1}(1) = I_n$, and a path $\overline{S_2} : \mathbb{I} \to \mathrm{GL}_n(C)$ such that $\overline{S_2}(0) = S_2$ and $\overline{S_2}(1) = I_n$. Consider $g : \mathbb{I} \to \mathrm{GL}_n(D)$ given by

$$g(t) = \delta(\overline{S_1}(t))\gamma(\overline{S_2}(t))^{-1}.$$

Then $g(0) = g(1) = I_n$. So by the previous lemma, there exists $h : \mathbb{I} \to \mathrm{GL}_n(C)$ such that $h(0) \in \theta_C^n(\mathrm{GL}_{n-1}(C))$, $h(1) = I_n$ and $\gamma \circ h = g$. Now define

$$S_2' := h(0)S_2.$$

Then $S_2'\beta(v) = e_n$ because $h(0)e_n = e_n$, and $\gamma(S_2') = g(0)\gamma(S_2) = \delta(S_1)$. Hence, by Lemma 2.9, there exists $T' \in \mathrm{GL}_n(A)$ such that $\alpha(T') = S_1$, $\beta(T') = S_2'$, and $T'(v) = e_n$. We wish to show that $T' \in \mathrm{GL}_n^0(A)$.

Define $\overline{S_2'} : \mathbb{I} \to \mathrm{GL}_n(C)$ by $\overline{S_2'}(t) := h(t)\overline{S_2}(t)$. Since $\gamma \circ h = g$, we have

$$\gamma \circ \overline{S_2'} = \delta \circ \overline{S_1}.$$

Since $C[0,1] \otimes M_n(\mathbb{C})$ is nuclear, by [15, Theorem 3.9] we have a pullback

$$\begin{array}{ccc} C[0,1] \otimes M_n(A) & \xrightarrow{\alpha} & C[0,1] \otimes M_n(B) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\delta} \\ C[0,1] \otimes M_n(C) & \xrightarrow{\gamma} & C[0,1] \otimes M_n(D) \end{array}$$

so we obtain a path $f : \mathbb{I} \to M_n(A)$ such that $\alpha \circ f = \overline{S_1}$ and $\beta \circ f = \overline{S_2'}$. For each $t \in \mathbb{I}$ we have $\beta(f(t)) \in \mathrm{GL}_n(C)$ and $\alpha(f(t)) \in \mathrm{GL}_n(B)$, so $f(t) \in \mathrm{GL}_n(A)$ by Lemma 2.9. Hence, f defines a path in $\mathrm{GL}_n(A)$. Furthermore, $\alpha(f(0)) = S_1$ and $\beta(f(0)) = S_2'$, so by the uniqueness in Lemma 2.9, $f(0) = T'$. Similarly, $f(1) = I_n$, so $T' \in \mathrm{GL}_n^0(A)$. Hence, $\mathrm{GL}_n^0(A)$ acts transitively on $\mathrm{Lg}_n(A)$, whence $\mathrm{csr}(A) \leq n$. ∎

3. Tensor products by commutative C^*-algebras. We now wish to calculate the homotopical stable ranks for algebras of the form $C(X) \otimes A$. Once again, we consider the general and connected stable ranks separately.

3.1. General stable rank. To compute $\mathrm{gsr}(C(X) \otimes A)$, we wish to describe all projective modules over $C(X) \otimes A$. If $A = \mathbb{C}$, by the Serre–Swan theorem, this amounts to describing all vector bundles over X. This is prohibitively difficult, of course, so we consider the potentially simpler situation when X is itself a suspension.

Let X be a compact Hausdorff space, and $x_0 \in X$ be a fixed base point. The *reduced suspension* of X is $\Sigma X = (X \times \mathbb{I})/\sim$ where

$$(0,x) \sim (0,x_0), \quad (1,x) \sim (1,x_0), \quad (s,x_0) \sim (0,x_0) \quad \forall x \in X,\ s \in \mathbb{I}.$$

Now we observe that vector bundles of rank n over ΣX correspond to homotopy classes of maps from X into $\mathrm{GL}_n(\mathbb{C})$ based at the identity. Hence, $\mathrm{gsr}(C(\Sigma X))$ is the least $n \geq 1$ such that the map $[X, \mathrm{GL}_{m-1}(\mathbb{C})]_* \to [X, \mathrm{GL}_m(\mathbb{C})]_*$ induced by $\theta_{\mathbb{C}}$ is injective for all $m \geq n$. In our notation, this simply gives

$$\mathrm{gsr}(C(\Sigma X)) = \mathrm{inj}_X(\mathbb{C}). \tag{3.1}$$

This is precisely the observation used by Nica to give the first nontrivial calculation of the general stable rank.

Example 3.1 ([13, Proposition 5.5]).

$$\mathrm{gsr}(C(\mathbb{S}^d)) = \begin{cases} \lceil d/2 \rceil + 1 & \text{if } d > 4 \text{ and } d \notin 4\mathbb{Z}, \\ \lceil d/2 \rceil & \text{if } d > 4 \text{ and } d \in 4\mathbb{Z}, \\ 1 & d \leq 4. \end{cases}$$

The goal of this section is to expand on this idea, by describing projective modules over $C(\Sigma X) \otimes A$, which allows us to prove an analogue of (3.1). To begin with, we fix a unital C*-algebra A, and we identify functions $f : \Sigma X \to A$ with functions $f : \mathbb{I} \times X \to A$ such that

$$f(0,x) = f(1,x) = f(s,x_0) \quad \forall x \in X,\ s \in \mathbb{I}. \tag{3.2}$$

We now follow the work of Rieffel [18] to construct projective modules over $C(\Sigma X) \otimes A$. If V is a projective right A-module, then $\mathrm{Aut}_A(V)$ is equipped with the point-norm topology, and has the base point id_V. Let $C_{x_0}(X, \mathrm{Aut}_A(V))$ be the space of continuous functions $u : X \to \mathrm{Aut}_A(V)$ such that $u(x_0) = \mathrm{id}_V$. Given a projective right A-module V and $u \in C_{x_0}(X, \mathrm{Aut}_A(V))$, we define

$$M(u) = \{\varphi : \mathbb{I} \times X \to V : \varphi(0,x) = \varphi(s,x_0), \\ \varphi(1,x) = u(x)\varphi(0,x)\ \forall x \in X,\ s \in \mathbb{I}\}.$$

Note that $M(u)$ is a right $C(\Sigma X) \otimes A$-module with the action given by

$$(\varphi \cdot f)(t,x) := \varphi(t,x) f(t,x).$$

Lemma 3.2. *If u_0, u_1 are path connected in $C_{x_0}(X, \mathrm{Aut}_A(V))$, then $M(u_0) \cong M(u_1)$.*

Proof. Let $H : \mathbb{I} \to C_{x_0}(X, \mathrm{Aut}_A(V))$ be a path such that $H(0) = u_0$, $H(1) = u_1$. Then we may think of H as a map $H : \mathbb{I} \times X \to \mathrm{Aut}_A(V)$. Define $F : M(u_0) \to M(u_1)$ by

$$F(\varphi)(s,x) := H(s,x) u_0(x)^{-1} \varphi(s,x)$$

so $F(\varphi)(1,x)=H(1,x)u_0(x)^{-1}\varphi(1,x)=u_1(x)\varphi(0,x)$ and $F(\varphi)(0,x)=\varphi(0,x)$. Hence, F is well-defined. Also, F is clearly a module homomorphism because the action of $C(\Sigma X)\otimes A$ is on the right. To show that F is an isomorphism, we define $G : M(u_1) \to M(u_0)$ by

$$G(\psi)(s,x) := u_0(x)H(s,x)^{-1}\psi(s,x).$$

Then G is a well-defined module homomorphism such that $G \circ F = \mathrm{id}_{M(u_0)}$ and $F \circ G = \mathrm{id}_{M(u_1)}$. ■

Given projective right A-modules V_1 and V_2, $u_1 \in C_{x_0}(X, \mathrm{Aut}_A(V_1))$, and $u_2 \in C_{x_0}(X, \mathrm{Aut}_A(V_2))$, we write $u_1 \oplus u_2 \in C_{x_0}(X, \mathrm{Aut}_A(V_1 \oplus V_2))$ for the map

$$(u_1 \oplus u_2)(x)(v_1, v_2) := (u_1(x)(v_1), u_2(x)(v_2)).$$

The proof of the next two lemmas is entirely obvious from the definition.

Lemma 3.3. *If $u_1 \in C_{x_0}(X, \mathrm{Aut}_A(V_1))$, $u_2 \in C_{x_0}(X, \mathrm{Aut}_A(V_2))$, where V_1 and V_2 are projective right A-modules, then*

$$M(u_1 \oplus u_2) \cong M(u_1) \oplus M(u_2).$$

Lemma 3.4. *If $\iota_{A^n} \in C_{x_0}(X, \mathrm{GL}_n(A))$ denotes the identity automorphism on A^n, then*

$$M(\iota_{A^n}) \cong (C(\Sigma X) \otimes A)^n.$$

Lemma 3.5. *Let $u, v \in C_{x_0}(X, \mathrm{Aut}_A(V))$ be such that $M(u) \cong M(v)$. Then there exists $w \in C(X, \mathrm{Aut}_A(V))$ such that v is path connected to wuw^{-1} in $C_{x_0}(X, \mathrm{Aut}_A(V))$.*

Proof. Note that $M(u)$ and $M(v)$ are both section algebras of locally trivial bundles over ΣX with fibers V, so if $M(u) \cong M(v)$, then the isomorphism is implemented by a map $\widehat{g} : \Sigma X \to \mathrm{Aut}_A(V)$. As in (3.2), we identify $\widehat{g}$ with a function $g : \mathbb{I} \times X \to \mathrm{Aut}_A(V)$ such that

$$g(0,x) = g(1,x) = g(s,x_0) = \mathrm{id}_V \quad \forall x \in X,\ s \in \mathbb{I}.$$

Then for any $\varphi \in M(u)$,

$$\begin{aligned} v(x)g(0,x)\varphi(0,x) = v(x)(g(\varphi))(0,x) = g(\varphi)(1,x) = g(1,x)\varphi(1,x) \\ = g(1,x)u(x)\varphi(0,x). \end{aligned}$$

Hence, $v(x)g(0,x) = g(1,x)u(x)$ for all $x \in X$, so that

$$v(x) = g(1,x)u(x)g(0,x)^{-1} \quad \forall x \in X.$$

Let $w : X \to \mathrm{Aut}_A(V)$ be given by $w(x) := g(0,x)$, and let $H : \mathbb{I} \times X \to C_{x_0}(X, \mathrm{Aut}_A(V))$ be given by $H(t,x) := g(t,x)u(x)g(0,x)^{-1}$. Then

$$H(0,x) = w(x)u(x)w(x)^{-1}, \quad H(1,x) = v(x).$$

Furthermore, $H(s,x_0) = g(s,x_0)u(x_0)g(0,x_0)^{-1} = u(x_0) = \mathrm{id}_V$ for all $s \in \mathbb{I}$. Hence H implements a homotopy $v \sim_{\mathrm{h}} wuw^{-1}$ in $C_{x_0}(X, \mathrm{Aut}_A(V))$. ■

LEMMA 3.6. *Every projective $C(\Sigma X)\otimes A$-module is isomorphic to $M(u)$ for some projective A-module V and some $u \in C_{x_0}(X, \mathrm{Aut}_A(V))$.*

Proof. Observe that a projective $C(\Sigma X)\otimes A$-module M is isomorphic to $P((C(\Sigma X)\otimes A)^n)$ for some projection $P \in M_n(C(\Sigma X)\otimes A)$. We identify P with a map $P : \mathbb{I} \times X \to M_n(A)$ satisfying (3.2). Let $p := P(0, x_0)$ and $V := p(A^n)$. If we think of P as a path $P : \mathbb{I} \to C(X) \otimes M_n(A)$, then there is a path of unitaries $U : \mathbb{I} \to \mathrm{GL}(C(X) \otimes M_n(A))$ such that

$$P(t,x) = U(t,x)^{-1}P(0,x)U(t,x) = U(t,x)^{-1}pU(t,x).$$

Furthermore, we have $U(0,x) = \mathrm{id}_{A^n} = U(s,x_0)$ for all $x \in X$ and $s \in \mathbb{I}$. Hence,

$$U(1,x)^{-1}pU(1,x) = U(1,x)^{-1}P(0,x)U(1,x) = P(1,x) = P(0,x) = p,$$

thus $U(1,x)p = pU(1,x)$, so we define $u(x) := U(1,x)p \in \mathrm{Aut}_A(V)$ and $u \in C_{x_0}(X, \mathrm{Aut}_A(V))$. Finally, if $f \in P((C(\Sigma X) \otimes A)^n)$, then we think of f as a function $f : \mathbb{I} \times X \to A^n$ satisfying (3.2) and $P(t,x)f(t,x) = f(t,x)$. Hence, we may define $\varphi : \mathbb{I} \times X \to V$ by

$$\varphi(t,x) := U(t,x)f(t,x),$$

and this is well-defined because

$$\begin{aligned} p\varphi(t,x) &= P(0,x)U(t,x)f(t,x) = U(t,x)P(t,x)f(t,x) = U(t,x)f(t,x) \\ &= \varphi(t,x). \end{aligned}$$

Furthermore, $\varphi(1,x) = U(1,x)f(1,x) = U(1,x)f(0,x)$ and $\varphi(0,x) = f(0,x)$. Hence, $\varphi \in M(u)$. It is then easy to check that the map that sends f to φ is an isomorphism from $P((C(\Sigma X) \otimes A)^n)$ to $M(u)$. ∎

We are now ready to prove the main theorem of this section. Recall that $\mathrm{inj}_X(A)$ is the least $n \geq 1$ such that the map $(\theta_A)_* : [X, \mathrm{GL}_{m-1}(A)]_* \to [X, \mathrm{GL}_m(A)]_*$ is injective for all $m \geq n$.

THEOREM 3.7.

$$\mathrm{gsr}(C(\Sigma X) \otimes A) = \max\{\mathrm{gsr}(A), \mathrm{inj}_X(A)\}.$$

Proof. For simplicity of notation, write $B := C(\Sigma X) \otimes A$. Let $n \geq \max\{\mathrm{gsr}(A), \mathrm{inj}_X(A)\}$, and let P be a projective module over B such that $P \oplus B \cong B^n$. By Lemma 3.6, there exists a projective A-module V and a map $u \in C_{x_0}(X, \mathrm{Aut}_A(V))$ such that $P \cong M(u)$. The map $\pi : B \to A$ given by evaluation at $[(0, x_0)] \in \Sigma X$ is a ring homomorphism, so

$$\pi_\#(P) \oplus A \cong A^n.$$

But $\pi_\#(P) \cong V$ and $\mathrm{gsr}(A) \leq n$ so $V \cong A^{n-1}$. Hence, we may think of u as belonging to $C_{x_0}(X, \mathrm{GL}_{n-1}(A))$. Now note that

$$M(u \oplus \iota_A) \cong M(\iota_{A^n}),$$

so by Lemmas 3.5 and 3.4, $u \oplus \iota_A \sim_{\mathrm{h}} \iota_{A^n}$ in $C_{x_0}(X, \mathrm{GL}_n(A))$. Since $n \geq \mathrm{inj}_X(A)$, it follows that $u \sim_{\mathrm{h}} \iota_{A^{n-1}}$ in $C_{x_0}(X, \mathrm{GL}_{n-1}(A))$, whence $P \cong B^{n-1}$ by Lemma 3.2. Hence, $\mathrm{gsr}(B) \leq n$ as required.

For the reverse inequality, let $n \geq \mathrm{gsr}(B)$. Then by Remark 2.5, $n \geq \mathrm{gsr}(A)$. Now suppose $u \in C_{x_0}(X, \mathrm{GL}_{n-1}(A))$ is such that $u \oplus \iota_A \sim_{\mathrm{h}} \iota_{A^n}$ in $C_{x_0}(X, \mathrm{GL}_n(A))$; then let $P = M(u)$. By Lemmas 3.3 and 3.4,

$$P \oplus B \cong M(u \oplus \iota_A) \cong M(\iota_{A^n}) \cong B^n.$$

By hypothesis, $P \cong B^{n-1} = M(\iota_{A^{n-1}})$. By Lemma 3.5, $u \sim_{\mathrm{h}} \iota_{A^{n-1}}$ in $C_{x_0}(X, \mathrm{GL}_{n-1}(A))$, and so $\mathrm{inj}_X(A) \leq n$. ■

3.2. Connected stable rank. Let A be a C*-algebra, and X a compact Hausdorff space. We wish to determine estimates for $\mathrm{csr}(C(X) \otimes A)$ in terms of $\dim(X)$ and other parameters that depend on A. To this end, we notice that if X is a CW-complex of dimension at most n, we may write $X = X_0 \cup_{\varphi} \mathbb{D}^n$, where X_0 is a CW-complex of dimension at most n, and $\varphi : \mathbb{S}^n \to X_0$ is the attaching map. By [12, Lemma 1.4], we have a pullback diagram

$$\begin{array}{ccc} C(X) \otimes A & \longrightarrow & C(X_0) \otimes A \\ \downarrow & & \downarrow{\scriptstyle \varphi^*} \\ C(\mathbb{D}^n) \otimes A & \xrightarrow{\gamma} & C(\mathbb{S}^{n-1}) \otimes A \end{array}$$

where γ is the restriction map. By Theorem 2.14, we get

$$\begin{aligned} \mathrm{csr}(C(X) \otimes A) \leq \max\{&\mathrm{csr}(C(X_0) \otimes A), \mathrm{csr}(C(\mathbb{D}^n) \otimes A), \\ &\mathrm{inj}_0(C(\mathbb{S}^{n-1}) \otimes A), \mathrm{surj}_1(C(\mathbb{S}^{n-1}) \otimes A)\}. \end{aligned}$$

In order to estimate $\mathrm{inj}_0(C(\mathbb{S}^{n-1}) \otimes A)$, we observe that

LEMMA 3.8. *If X is a compact Hausdorff space and A a unital C*-algebra, then*

$$\mathrm{inj}_0(C(X) \otimes A) = \mathrm{inj}_X(A).$$

Proof. First note that $\mathrm{GL}_k(C(X) \otimes A) = C(X, \mathrm{GL}_k(A))$, so $[X, \mathrm{GL}_k(A)] = \pi_0(\mathrm{GL}_k(C(X) \otimes A)$. Hence, $\mathrm{inj}_0(C(X) \otimes A)$ is the least $m \geq 1$ such that

$$(\theta_A)_* : [X, \mathrm{GL}_{k-1}(A)] \to [X, \mathrm{GL}_k(A)]$$

is injective for all $k \geq m$. Let now $x_0 \in X$ be a fixed base point and let $C_{x_0}(X, \mathrm{GL}_k(A))$ be the set of all continous maps $f : X \to \mathrm{GL}_k(A)$ such that $f(x_0) = I_n$. Suppose that $n \geq \mathrm{inj}_0(C(X) \otimes A)$, $k \geq n$ and $f, g \in C_{x_0}(X, \mathrm{GL}_{k-1}(A))$ are such that $\theta_A(f) \sim_{\mathrm{h}} \theta_A(g)$ in $C(X, \mathrm{GL}_k(A))$. Since $n \geq \mathrm{inj}_0(C(X) \otimes A)$, the above comments imply that there is a free homotopy $H : \mathbb{I} \times X \to \mathrm{GL}_{k-1}(A)$ such that $H(0, x) = f(x)$ and $H(1, x) = g(x)$ for all $x \in X$. Then $\widetilde{H}(t, x) := H(t, x) H(t, x_0)^{-1}$ defines a base-point preserving homotopy from f to g. Hence, $n \geq \mathrm{inj}_X(A)$.

Conversely, let $n \geq \text{inj}_X(A)$, $k \geq n$ and let $f, g : X \to \text{GL}_{k-1}(A)$ be such that $\theta_A(f) \sim_{\text{h}} \theta_A(g)$ in $C(X, \text{GL}_k(A))$. Then there is a homotopy $H : \mathbb{I} \times X \to \text{GL}_k(A)$ connecting $\theta_A(f)$ to $\theta_A(g)$. Hence $\widetilde{H}$, as defined above, is a base-point preserving homotopy from $\theta_A(\alpha(f))$ to $\theta_A(\alpha(g))$ where $\alpha(h)(x) := h(x)h(x_0)^{-1}$. Since $n \geq \text{inj}_X(A)$, $\alpha(f)$ is homotopic to $\alpha(g)$ in $C_{x_0}(X, \text{GL}_{k-1}(A))$. If $G : \mathbb{I} \times X \to \text{GL}_{k-1}(A)$ is a base-point preserving homotopy such that $G(0, x) = \alpha(f)(x)$ and $G(1, x) = \alpha(g)(x)$ for all $x \in X$, then $\widehat{G}(t, x) := G(t, x)G(t, x_0)$ is a homotopy connecting f to g in $C(X, \text{GL}_{k-1}(A))$. Hence, $n \geq \text{inj}_0(C(X) \otimes A)$. ∎

In order to estimate the term $\text{surj}_1(C(\mathbb{S}^{n-1}) \otimes A)$ that occurs in the above inequality, we turn to the work of Thomsen [21], where he defines an axiomatic homology theory that will be relevant to us. Recall [19] that a *homology theory* is a sequence $\{h_n\}$ of covariant functors from an admissible category $\mathcal{D}$ of C*-algebras to abelian groups which satisfies the following axioms:

- Homotopy Axiom: If $\varphi_0, \varphi_1 : A \to B$ are homotopic morphisms (in the sense described in the discussion which precedes Theorem 1.5), then $(\varphi_0)_* = (\varphi_1)_* : h_n(A) \to h_n(B)$ for all $n \in \mathbb{N}$.
- Exactness Axiom: Let $0 \to J \to A \to B \to 0$ be a short exact sequence in $\mathcal{D}$; then there is a map $\partial : h_n(B) \to h_{n-1}(J)$ and a long exact sequence $\cdots \to h_n(J) \to h_n(A) \to h_n(B) \xrightarrow{\partial} h_{n-1}(J) \to h_{n-1}(A) \to \cdots$. The map ∂ is natural with respect to morphisms of short exact sequences.

These two axioms imply that any homology theory is additive: If $0 \to J \to A \to B$ is a split exact sequence in $\mathcal{D}$, then there is a natural isomorphism $h_n(A) \cong h_n(J) \oplus h_n(B)$ for all $n \in \mathbb{N}$.

Now, let A be a C*-algebra (not necessarily unital), A^+ the C*-algebra obtained by adjoining a unit to A, and consider A as an ideal of A^+. For $m \in \mathbb{N}$, define

$$\text{GL}_m(A) := \{x \in \text{GL}_m(A^+) : x - I_{(A^+)^m} \in M_m(A)\}.$$

Then Thomsen [21, Theorem 2.5] proves that, for a fixed $m \in \mathbb{N}$, the functor

$$h_n(A) := \pi_{n+1}(\text{GL}_m(A))$$

defines a homology theory from the category of C*-algebras to the category of abelian groups.

LEMMA 3.9. *Let $D = C(\mathbb{S}^{n-1}) \otimes A$. Then*

$$\max\{\text{inj}_0(D), \text{surj}_1(D)\} \leq \max\{\text{surj}_1(A), \text{surj}_n(A), \text{inj}_{n-1}(A)\}.$$

Proof. Let $m \geq \max\{\text{surj}_1(A), \text{surj}_n(A), \text{inj}_{n-1}(A)\}$. It follows from Lemma 3.8 that $\text{inj}_{n-1}(A) = \text{inj}_0(D)$, so $m \geq \text{inj}_0(D)$. We have a split exact sequence

$$0 \to C_0(\mathbb{R}^{n-1}) \otimes A \to C(\mathbb{S}^{n-1}) \otimes A \to A \to 0.$$

Furthermore, by [21, Lemma 2.3],

$$\pi_n(\mathrm{GL}_k(A)) \cong \pi_1(\mathrm{GL}_k(C_0(\mathbb{R}^{n-1}) \otimes A)).$$

By additivity of the functor $A \mapsto \pi_1(\mathrm{GL}_k(A))$, there is a natural isomorphism

$$\pi_1(\mathrm{GL}_k(C(\mathbb{S}^{n-1} \otimes A))) \cong \pi_n(\mathrm{GL}_k(A)) \oplus \pi_1(\mathrm{GL}_k(A))$$

for $k \in \{m, m-1\}$. As $m \geq \max\{\mathrm{surj}_1(A), \mathrm{surj}_n(A)\}$, also $m \geq \mathrm{surj}_1(D)$. ∎

THEOREM 3.10. *Let X be a compact Hausdorff space of dimension at most n. Then*

$$\mathrm{csr}(C(X) \otimes A) \leq \max\{\mathrm{csr}(A), \mathrm{surj}_k(A), \mathrm{inj}_{k-1}(A) : 1 \leq k \leq n\}.$$

Proof. If X is a compact Hausdorff space of dimension at most n, then X is an inverse limit of compact metric spaces (X_i) such that $\dim(X_i) \leq n$ [10]. Since $C(X) \otimes A \cong \lim C(X_i) \otimes A$, it follows from Remark 1.4(7) that $\mathrm{csr}(C(X) \otimes A) \leq \liminf \mathrm{csr}(C(X_i) \otimes A)$. Furthermore, if X is a compact metric space of dimension at most n, then X is an inverse limit of finite CW-complexes (Y_i), such that $\dim(Y_i) \leq n$ [3]. Once again, $\mathrm{csr}(C(X) \otimes A)) \leq \liminf C(Y_i) \otimes A$. Hence, it suffices to assume that X is itself a finite CW-complex with $\dim(X) \leq n$.

By induction, we may assume that $X = X_0 \cup_\varphi \mathbb{D}^n$ where X_0 is a finite CW-complex of dimension at most $n-1$ and $\varphi : \mathbb{S}^{n-1} \to X_0$ is a continuous function. As mentioned at the start of this section, it follows that

$$\begin{aligned}\mathrm{csr}(C(X) \otimes A) \leq \max\{&\mathrm{csr}(C(X_0) \otimes A), \mathrm{csr}(C(\mathbb{D}^n) \otimes A),\\ &\mathrm{inj}_0(C(\mathbb{S}^{n-1}) \otimes A), \mathrm{surj}_1(C(\mathbb{S}^{n-1}) \otimes A)\}.\end{aligned}$$

By homotopy invariance, $\mathrm{csr}(C(\mathbb{D}^n) \otimes A) = \mathrm{csr}(A)$, so the result now follows by induction and Lemma 3.9. ∎

We now turn our attention to a particularly tractable class of C*-algebras. Let $\mathcal{F}$ be the class of C*-algebras A such that the map $\theta_A^m : \mathrm{GL}_{m-1}(A) \to \mathrm{GL}_m(A)$ induces a weak homotopy equivalence for all $m \geq 2$. The following algebras are known to be in $\mathcal{F}$:

- [5] If $\mathcal{Z}$ denotes the Jiang–Su algebra, then $A \otimes \mathcal{Z} \in \mathcal{F}$ for any C*-algebra A. In particular, if A is a separable, approximately divisible C*-algebra, then $A \cong A \otimes \mathcal{Z}$, so $A \in \mathcal{F}$.
- [17] If A is an irrational rotation algebra, then $A \in \mathcal{F}$.
- [21] If $\mathcal{O}_n$ denotes the Cuntz algebra, then $A \otimes \mathcal{O}_n \in \mathcal{F}$ for any C*-algebra A.
- [21] If A is an infinite-dimensional simple AF-algebra, then $A \otimes B \in \mathcal{F}$ for any C*-algebra B.

- [24] If A is a purely infinite, simple C*-algebra, and p any nonzero projection of A, then $pAp \in \mathcal{F}$.

Note that $A \in \mathcal{F}$ if and only if $\pi_n(\mathrm{Lc}_m(A)) = 0$ for all $n \geq 0$, and $m \geq 2$. Furthermore, if $A \in \mathcal{F}$, then, for all $n \geq 0$ and $m \geq 1$,

$$\pi_n(\mathrm{GL}_m(A)) \cong \begin{cases} K_1(A) & \text{for } n \text{ even}, \\ K_0(A) & \text{for } n \text{ odd}. \end{cases}$$

In particular, the natural map $\mathrm{GL}_1(A)/\mathrm{GL}_1^0(A) \to K_1(A)$ is an isomorphism.

THEOREM 3.11. *Let $A \in \mathcal{F}$, and let X be a compact Hausdorff space. Then*

$$\mathrm{gsr}(C(X) \otimes A) = \mathrm{gsr}(A),$$

$$\mathrm{csr}(C(X) \otimes A) = \begin{cases} \mathrm{csr}(A) & \textit{if } \mathrm{csr}(A) \geq 2, \\ 1 \textit{ or } 2 & \textit{if } \mathrm{csr}(A) = 1. \end{cases}$$

Proof. We first consider the connected stable rank: Since $A \in \mathcal{F}$, we have $\pi_n(\mathrm{Lc}_m(A)) = 0$ for all $n \geq 0$ and $m \geq 2$. Since $\mathrm{Lc}_m(A)$ is an open subset of a normed linear space [17, Section 1], it is homotopy equivalent to a CW-complex [8, Chapter IV, Corollary 5.5]. By Whitehead's theorem [4, Theorem 4.5], it follows that $\mathrm{Lc}_m(A)$ is contractible. Therefore, if $m \geq \max\{2, \mathrm{gsr}(A)\}$, then $\mathrm{Lg}_m(A) = \mathrm{Lc}_m(A)$ is contractible. Identifying $\mathrm{Lg}_m(C(X) \otimes A)$ with $C(X, \mathrm{Lg}_m(A))$, we see that $\pi_0(\mathrm{Lg}_m(C(X) \otimes A)) = [X, \mathrm{Lg}_m(A)]$ is trivial. Thus,

$$\mathrm{csr}(C(X) \otimes A) \leq \max\{2, \mathrm{gsr}(A)\}.$$

Now the result follows from the fact that $\mathrm{gsr}(A) \leq \mathrm{csr}(A) \leq \mathrm{csr}(C(X) \otimes A)$.

For the general stable rank: By the first part of the argument, we have

$$\mathrm{gsr}(A) \leq \mathrm{gsr}(C(X) \otimes A) \leq \mathrm{csr}(C(X) \otimes A) \leq \max\{2, \mathrm{gsr}(A)\}.$$

If $\mathrm{gsr}(A) \geq 2$, there is nothing to prove. If $\mathrm{gsr}(A) = 1$, then A must be stably finite, and hence $C(X) \otimes A$ is finite. Since $\mathrm{gsr}(C(X) \otimes A) \leq 2$, it must be that $\mathrm{gsr}(C(X) \otimes A) = 1$ by Remark 1.4(8). ∎

EXAMPLE 3.12. Some examples illustrate our results:

(1) If $A \in \mathcal{F}$ and $\mathrm{csr}(A) = 1$, then it is possible that $\mathrm{csr}(C(X) \otimes A) = 2$, depending on X. For instance, if A is a simple, infinite-dimensional, unital AF-algebra, then $\mathrm{csr}(A) = 1$. Taking $X = \mathbb{T}$, we see that $K_1(\mathbb{T}A) \cong K_0(A) \oplus K_1(A) \neq 0$ because A is stably finite. Hence, $\mathrm{csr}(\mathbb{T}A) = 2$ by Remark 1.4(9).

(2) If A is an irrational rotation algebra, then $\mathrm{tsr}(A) = 1$ and $K_1(A) \neq 0$, so $\mathrm{gsr}(A) = 1$ and $\mathrm{csr}(A) = 2$ by Remark 1.4(9)–(10). Since A, and hence $C(X) \otimes A$, is finite, it follows that

$$\mathrm{gsr}(C(X) \otimes A) = 1, \quad \mathrm{csr}(C(X) \otimes A) = 2$$

for any compact Hausdorff space X. This was proved by Rieffel [17, Propositions 2.5, 2.7] in the case where $X = \mathbb{T}^k$. In fact, these were crucial in proving that $A \in \mathcal{F}$.

(3) If A is a Kirchberg algebra, then $A \in \mathcal{F}$ by [24]. Furthermore, it was proved by Xue [22] that $\mathrm{gsr}(A) = \mathrm{csr}(A) = 2$ if and only if A has the IBN property (otherwise $\mathrm{gsr}(A) = \mathrm{csr}(A) = +\infty$). So if A is a Kirchberg algebra with the IBN property, we can conclude that

$$\mathrm{gsr}(C(X) \otimes A) = \mathrm{csr}(C(X) \otimes A) = 2$$

for any compact Hausdorff X. In particular, this is true for $A = \mathcal{O}_\infty$.

(4) If A is a C*-algebra of real rank zero, then it has been proved in [6, Lemma 2.2] that $\mathrm{inj}_0(A) = 1$. Hence, it follows from Theorem 3.7 that $\mathrm{gsr}(\mathbb{T}A) = \mathrm{gsr}(A)$. This is precisely the argument in [23, Proposition 3.1].

4. Examples and calculations. We now turn to a few examples that have informed this investigation.

4.1. Commutative C*-algebras. If X and Y are two compact Hausdorff spaces and $X \vee Y$ denotes their wedge sum, then $C(X \vee Y) \cong C(X) \oplus_{\mathbb{C}} C(Y)$ where the maps $C(X) \to \mathbb{C}$ and $C(Y) \to \mathbb{C}$ are the evaluation maps at the common base point. Hence, we get the following corollary to Theorems 2.8 and 2.14:

COROLLARY 4.1. *For any compact Hausdorff spaces X and Y,*

$$\mathrm{gsr}(C(X \vee Y)) = \max\{\mathrm{gsr}(C(X)), \mathrm{gsr}(C(Y))\},$$
$$\mathrm{csr}(C(X \vee Y)) = \max\{\mathrm{csr}(C(X)), \mathrm{csr}(C(Y))\}.$$

Proof. For the general stable rank: The inclusion map $\iota : X \hookrightarrow X \vee Y$ induces a surjection $\iota^* : C(X \vee Y) \to C(X)$. Furthermore, the 'pinching' map $P : X \vee Y \to X$ that pinches Y to the base point has the property that $\iota^* \circ P^* = \mathrm{id}_{C(X)}$. So it follows from Remark 1.4(5) that $\mathrm{gsr}(C(X \vee Y)) \geq \mathrm{gsr}(C(X))$. By symmetry, the same true for Y, thus

$$\mathrm{gsr}(C(X \vee Y)) \geq \max\{\mathrm{gsr}(C(X)), \mathrm{gsr}(C(Y))\}.$$

Now observe that $K_1(\mathbb{C}) = 0$ and $\mathrm{inj}_0(\mathbb{C}) = 1$, so the result follows from Theorem 2.8.

For the connected stable rank: The same argument as above shows that

$$\begin{aligned}\max\{\mathrm{csr}(C(X)), \mathrm{csr}(C(Y))\} &\leq \mathrm{csr}(C(X \vee Y)) \\ &\leq \max\{\mathrm{csr}(C(X)), \mathrm{csr}(C(Y)), 2\}\end{aligned}$$

where the second inequality follows from Theorem 2.14 and the fact that $\mathrm{surj}_1(\mathbb{C}) = 2$. Thus, if $\max\{\mathrm{csr}(C(X)), \mathrm{csr}(C(Y))\} \geq 2$, then the conclusion follows. Suppose $\mathrm{csr}(C(X)) = \mathrm{csr}(C(Y)) = 1$. We must then prove that

$\operatorname{csr}(C(X\vee Y)) = 1$. By the above inequality, we know that $\operatorname{csr}(C(X\vee Y)) \leq 2$. Hence, it suffices to show that $\mathrm{Lg}_1(C(X\vee Y))$ is connected. However,

$$\pi_0(\mathrm{Lg}_1(C(X\vee Y))) = \pi_0(C(X\vee Y, \mathrm{Lg}_1(\mathbb{C}))) \cong [X\vee Y, \mathbb{T}].$$

Since $\operatorname{csr}(C(X)) = \operatorname{csr}(C(Y)) = 1$, we know that $[X, \mathbb{T}]$ and $[Y, \mathbb{T}]$ are both trivial. Since $\operatorname{gsr}(C(\mathbb{T})) = \operatorname{csr}(\mathbb{C}) = 1$, it follows from Lemma 2.6 that $[X, \mathbb{T}]_*$ and $[Y, \mathbb{T}]_*$ are both trivial as well. If $f : X\vee Y \to \mathbb{T}$ is a map based at the identity, then $f\circ\iota : X \to \mathbb{T}$ must be null-homotopic. Similarly, if $j : Y \hookrightarrow X\vee Y$ denotes the inclusion map, then $f\circ j$ is also null-homotopic. Furthermore, the homotopies may be chosen to preserve the common base point, so we may paste the two homotopies together to conclude that f is null-homotopic. Hence, $[X\vee Y, \mathbb{T}]_*$ is trivial. Once again, by Lemma 2.6, we conclude that $[X\vee Y, \mathbb{T}]$ is also trivial. Hence, $\mathrm{Lg}_1(C(X\vee Y))$ is connected, whence $\operatorname{csr}(C(X\vee Y)) = 1$ as required. ∎

Our next goal is to determine $\operatorname{gsr}(C(\mathbb{T}^d))$. To begin with, we have the following observation:

Corollary 4.2. *If X is a compact Hausdorff space, then*

$$\operatorname{gsr}(C(\mathbb{T}\times X)) = \max\{\operatorname{gsr}(C(X)), \operatorname{gsr}(C(\Sigma X))\}.$$

Proof. Note that $C(\mathbb{T}\times X) = \mathbb{T}A$ where $A = C(X)$, so by Theorem 3.11,

$$\operatorname{gsr}(C(\mathbb{T}\times X)) = \max\{\operatorname{gsr}(C(X)), \operatorname{inj}_0(C(X))\}.$$

By Lemma 3.8, $\operatorname{inj}_0(C(X)) = \operatorname{inj}_X(\mathbb{C})$, so the result follows from (3.1). ∎

Recall that a space X is said to homotopically dominate Y if there are maps $P : X \to Y$ and $f : Y \to X$ such that $P\circ f \simeq \mathrm{id}_Y$. If this happens, then $C(X)$ homotopically dominates $C(Y)$, so it follows from Theorem 1.5 that $\operatorname{gsr}(C(X)) \geq \operatorname{gsr}(C(Y))$.

Lemma 4.3. *If $X = \prod_{i=1}^k \mathbb{S}^{n_i}$, then ΣX homotopically dominates $\mathbb{S}^{n+1}$ where $n = \sum_{i=1}^k n_i$. In particular, $\Sigma\mathbb{T}^n$ homotopically dominates $\mathbb{S}^{n+1}$.*

Proof. We claim that

$$\Sigma X \simeq \mathbb{S}^{n+1}\vee M$$

for some manifold M of dimension $\leq n$. To see this, we proceed by induction on k. It is clearly true if $k = 1$, so let $Y = \prod_{i=1}^{k-1}\mathbb{S}^{n_i}$ and assume $\Sigma Y \simeq \mathbb{S}^{\ell+1}\vee N$ where $\ell = \sum_{i=1}^{k-1} n_i$ and N is a manifold of dimension $\leq \ell$. Then by [4, Proposition 4I.1],

$$\begin{aligned}\Sigma X = \Sigma(Y\times\mathbb{S}^{n_k}) &\simeq \Sigma Y\vee\mathbb{S}^{n_k+1}\vee\Sigma(Y\wedge\mathbb{S}^{n_k})\\ &\simeq \mathbb{S}^{\ell+1}\vee N\vee\mathbb{S}^{n_k+1}\vee\Sigma^{n_k}(\mathbb{S}^{\ell+1}\vee N)\\ &\simeq \mathbb{S}^{\ell+1}\vee N\vee\mathbb{S}^{n_k+1}\vee\Sigma^{n_k}(N)\vee\mathbb{S}^{\ell+n_k+1} \simeq M\vee\mathbb{S}^{n+1}\end{aligned}$$

where $M = \mathbb{S}^{\ell+1} \vee N \vee \mathbb{S}^{n_k+1} \vee \Sigma^{n_k}(N)$. Note that

$$\dim(M) \leq \max\{\ell+1, \ell, n_k+1, n_k+\ell\} \leq n_k + \ell = n.$$

This proves the claim. So we get a map $P : \Sigma X \to M \vee \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$ that 'pinches' M to a point, and a map $f : \mathbb{S}^{n+1} \to \mathbb{S}^{n+1} \vee M \to \Sigma X$ by composing the homotopy equivalence with the natural map $\mathbb{S}^{n+1} \to \mathbb{S}^{n+1} \vee M$. Note that $P_* : H_{n+1}(\Sigma X) \to H_{n+1}(\mathbb{S}^{n+1})$ is an isomorphism because $\dim(M) \leq n$, and $f_* : H_{n+1}(\mathbb{S}^{n+1}) \to H_{n+1}(\Sigma X)$ is also an isomorphism. Hence,

$$(P \circ f)_* : H_{n+1}(\mathbb{S}^{n+1}) \to H_{n+1}(\mathbb{S}^{n+1})$$

is an isomorphism. Since both P and f are orientation-preserving, it follows that $P \circ f$ has degree 1, and so $P \circ f \simeq \mathrm{id}_{\mathbb{S}^{n+1}}$ as required. ■

The following is an answer to a question posed by Nica [13, Problem 5.8]. Before we begin, we observe that if X is a compact Hausdorff space whose covering dimension is ≤ 4, then Nica has shown [13, Proposition 5.5] that $\mathrm{gsr}(C(X)) = 1$. The point of this next example, thus, is to use the previous lemma to compare $\mathrm{gsr}(C(\mathbb{T}^d))$ and $\mathrm{gsr}(C(\mathbb{S}^d))$ for $d \geq 5$.

To put this in perspective, if X is a compact Hausdorff space of covering dimension $\leq n$, then $\mathrm{csr}(C(X)) \leq \lceil n/2 \rceil + 1$ by [14, Corollary 2.5] (see also Corollary 4.6). Furthermore, Nica has shown [13, Theorem 5.3] that this upper bound is attained provided the top cohomology group $H^{\mathrm{odd}}(X)$ is nonvanishing. In particular, this implies that, for all $d \geq 1$,

$$\mathrm{csr}(C(\mathbb{T}^d)) = \lceil d/2 \rceil + 1.$$

Example 4.4.

$$\mathrm{gsr}(C(\mathbb{T}^d)) = \begin{cases} 1 & \text{if } d \leq 4, \\ \lceil d/2 \rceil + 1 & \text{if } d > 4. \end{cases}$$

Proof. For $d \leq 4$, the result follows from the preceding discussion. For $d \geq 5$, we know that

$$\mathrm{gsr}(C(\mathbb{T}^d)) \leq \mathrm{csr}(C(\mathbb{T}^d)) \leq \lceil d/2 \rceil + 1,$$

so it suffices to prove the reverse inequality. We proceed by induction on d. For $d = 5$, by Corollary 4.2 and Lemma 4.3,

$$\mathrm{gsr}(C(\mathbb{T}^5)) \geq \mathrm{gsr}(C(\Sigma\mathbb{T}^4)) \geq \mathrm{gsr}(C(\mathbb{S}^5))$$

and $\mathrm{gsr}(C(\mathbb{S}^5)) = 4$ by Example 3.1. For $d \geq 6$, by induction

$$\begin{aligned} \mathrm{gsr}(C(\mathbb{T}^d)) &= \max\{\mathrm{gsr}(C(\mathbb{T}^{d-1})), \mathrm{gsr}(C(\Sigma\mathbb{T}^{d-1}))\} \\ &\geq \max\left\{\left\lceil \frac{d-1}{2} \right\rceil + 1, \mathrm{gsr}(C(\mathbb{S}^d))\right\}. \end{aligned}$$

Once again the result follows from Example 3.1. ■

4.2. Noncommutative CW-complexes. As observed in Subsection 3.2, a commutative C*-algebra whose spectrum is a finite CW-complex can be expressed as an (iterated) pullback. Noncommutative CW-complexes (NCCW complexes), first studied by Pedersen [15], are meant to generalize this idea: An NCCW-complex A_0 of dimension 0 is a finite-dimensional C*-algebra. An NCCW-complex A_k of dimension k is described by a pullback

$$\begin{array}{ccc} A_k & \longrightarrow & A_{k-1} \\ \downarrow & & \downarrow \\ C(\mathbb{D}^k)\otimes F_k & \xrightarrow{\gamma} & C(\mathbb{S}^{k-1})\otimes F_k \end{array}$$

where F_k is a finite-dimensional C*-algebra, A_{k-1} is an NCCW-complex of dimension $k-1$, and γ is the restriction map. If F is a finite-dimensional C*-algebra, then it follows from Remark 1.4 that $\mathrm{csr}(F) = 1$. Hence, $\mathrm{csr}(A_0) = 1$ and $\mathrm{csr}(C(\mathbb{D}^k)\otimes F_k) = \mathrm{csr}(F_k) = 1$ by homotopy invariance. If $D = C(\mathbb{S}^{k-1})\otimes F_k$, then by Lemma 3.9,

$$\max\{\mathrm{inj}_0(D), \mathrm{surj}_1(D)\} \le \max\{\mathrm{surj}_1(F_k), \mathrm{surj}_k(F_k), \mathrm{inj}_{k-1}(F_k)\}.$$

Write $F_k = \bigoplus_{i=1}^{n_k} M_{\ell_i}(\mathbb{C})$. Then $\mathrm{surj}_1(F_k) = 2$, so computing the right hand side boils down to asking whether, for all $1 \le i \le n_k$, the map

$$\pi_k(\mathrm{GL}_{\ell_i(m-1)}(\mathbb{C})) \to \pi_k(\mathrm{GL}_{\ell_i m}(\mathbb{C})) \text{ is surjective, and}$$
$$\pi_{k-1}(\mathrm{GL}_{\ell_i(m-1)}(\mathbb{C})) \to \pi_{k-1}(\mathrm{GL}_{\ell_i m}(\mathbb{C})) \text{ is injective.}$$

By Bott periodicity, these maps are isomorphisms if $k \le 2\ell_i(m-1) - 1$ (see, for instance, [7, pp. 251–254]). Furthermore, if $k = 2\ell_i(m-1)$, then both conditions are satisfied because the second map is an isomorphism, and $\pi_k(\mathrm{GL}_{\ell_i m}(\mathbb{C})) = 0$. So if $d_k = \min\{\ell_i : 1 \le i \le j_k\}$, then

$$\max\{\mathrm{inj}_0(D), \mathrm{surj}_1(D)\} \le \left\lceil \frac{k}{2d_k} \right\rceil + 1.$$

The following estimate is thus a corollary of Theorem 2.14:

THEOREM 4.5. *Let A_n be an NCCW-complex of topological dimension at most n whose structure can be described as above. Then*

$$\mathrm{csr}(A_n) \le \max_{1\le k\le n}\left\{\left\lceil \frac{k}{2d_k} \right\rceil + 1\right\} \le \left\lceil \frac{n}{2} \right\rceil + 1.$$

A special case of this theorem is that of a commutative C*-algebra whose spectrum is a finite CW-complex. As in the proof of Theorem 3.10, by passing to inductive limits we obtain yet another proof of a result due to Nistor.

COROLLARY 4.6 ([14, Corollary 2.5]). *If X is a compact Hausdorff space of dimension at most n, then* $\mathrm{csr}(C(X)) \le \lceil n/2 \rceil + 1$.

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