AMENABLE GROUPS: PROJECT DESCRIPTION

1. Overview

The following theorem was proved in 1924 by Banach and Tarski: Given a ball in 3-dimensional space, there is a way of decomposing it into finitely many disjoint pieces that can be rearranged to form two balls of the same size as the original one. This counter-intuitive statement is really a statement in measure theory - it says that there is no finitely additive measure on $\mathbb{R}^3$ which is defined for all subsets of $\mathbb{R}^3$, that is invariant under isometries, and that gives the unit ball non-zero measure. Consequently, one cannot extend the Lebesgue measure to all subsets of $\mathbb{R}^3$. On the other hand, there is no analogue for the Banach-Tarski paradox for lower dimensions. Indeed there do exist such measures on $\mathbb{R}$ and $\mathbb{R}^2$. The reason for this dichotomy is that the isometry groups of $\mathbb{R}$ and $\mathbb{R}^2$ are *amenable*, while that of $\mathbb{R}^3$ is not.

The goal of this project is to study discrete amenable groups, beginning with a proof of the Banach-Tarski paradox. Reformulating it in measure theoretic language gives Tarski's theorem, which provides us with one definition of amenable groups. We then characterize amenable groups in terms of their representation theory, and other decomposition properties. We then explore the reduced C*-algebra of an amenable group, and show that the reduced C*-algebra is nuclear if and only if the group is amenable.

2. Tentative Schedule

(1) Banach-Tarski Paradox (from [Knudby])
   (a) Prove Tarski's theorem
   (b) Amenable groups
   (c) Hereditary properties. Solvable groups are amenable.
   (d) Banach-Tarski paradox does not work on $\mathbb{R}$ or $\mathbb{R}^2$

`[by September]`

(2) Equivalent conditions for Amenability (from [Notes])
   (a) Hahn-Banach separation theorem and consequences (Goldstein's theorem)
   (b) Amenable $\Leftrightarrow$ Følner condition $\Leftrightarrow$ weak containment of trivial representation in left regular representation

`[by November]`

(3) Group algebras (from [Brown-Ozawa], [Davidson], and [Notes])
   (a) C*-algebras, Completely positive maps, Nuclearity.
   (b) Reduced C* algebra of a group, $C_r^*(G)$.
   (c) For a discrete group G, amenability $\Leftrightarrow$ nuclearity of $C_r^*(G)$.

`[by March]`

References

[Knudby] S. Knudby, *The Banach-Tarski Paradox* (2009, UG Thesis, Univ. Copenhagen)
[Davidson] K. Davidson, *C* Algebras by Example*
[Brown-Ozawa] N.P. Brown, N. Ozawa, *C* Algebras and Finite Dimensional Approximations*
[Notes] Notes on Amenable Groups, `https://home.iiserb.ac.in/~prahlad/documents/expository/amenable.pdf`