COMPACT AND FREDHOLM OPERATORS : PROJECT DESCRIPTION

1. Overview

Given an operator on a finite dimensional complex vector space, one can naturally associate to it a matrix, and one can calculate its eigen-values by solving a polynomial equation. However, such notions break down (or, at the very least, become more difficult) if one replaces the underlying vector space by an infinite dimensional Hilbert space.

We begin the project by looking at a class of operators, called *compact* operators, which naturally generalize finite dimensional operators. We study the spectrum of such an operator, and prove the Fredholm alternative - a theorem that helps us understand equations of the form

$$(I - K)x = 0$$

where K is a compact operator.

We then introduce some Banach algebra theory, which will allow us to study Fredholm operators, and their index. We will prove the basic properties of the index function, and look at some examples.

The last part of the project will be the study of Toeplitz operators with continuous symbol on the Hardy space $H^2(\mathbb{D})$. We goal will be to prove that the index of such an operator is given by the winding number of its symbol.

2. Tentative Schedule

(1) Preliminaries :
   (a) Hilbert Spaces
   (b) Linear Operators and their Adjoints
   (c) Compact Operators
   Ref: [Conway, §I.1-§II.4] `[by 30/8/20]`

(2) The Fredholm Alternative
   (a) Riesz Theory for Compact Operators
   (b) Fredholm Alternative
   Ref: [Arveson, §3.2] `[by 30/9/20]`

(3) Fredholm Operators
   (a) Fredholm Operators
   (b) Definition of Index and multiplicativity
   (c) Banach Algebras
   (d) Invertible elements

   (e) Quotients, and the Calkin Algebra
   (f) Atkinson's Theorem
   (g) $\mathcal{F}(H)$, the set of Fredholm operators, is open
   Ref: [Arveson, §1 and §3] `[by 30/11/20]`

(4) Index Theory
   (a) Definition and Examples
   (b) Continuity and Homotopy invariance of index
   (c) Index : $\pi_0(\mathcal{F}(H)) \to \mathbb{Z}$ is an isomorphism
   Ref: [Arveson, §3.4] `[by 15/1/21]`

(5) Toeplitz Operators
   (a) Definition and Characterization
   (b) The Toeplitz algebra and Short exact sequence
   (c) Index of Toeplitz Operator with Continuous Symbol
   Ref: [Arveson, §2.1, §4.2 – 4.4] `[by 30/3/21]`

References

[Conway] J.B. Conway, *A Course in Functional Analysis (2nd Ed.)*
[Arveson] W. Arveson, *A Short Course on Spectral Theory*
[Murphy] G.J. Murphy, *C* Algebras and Operator Theory*
[Douglas] R.G. Douglas, *Banach Algebra techniques in Operator Theory (2nd Ed.)*
[Schechter] M. Schechter, *Principles of Functional Analysis (2nd Ed.)*

3. Important Comments

(1) We will have weekly meetings in which you are expected to give a progress report and discuss questions. You must treat these meetings with utmost seriousness, and must cancel them only in case of unavoidable emergencies. (Taking the GATE/GRE do not count as emergencies)
(2) Every few weeks, you will be expected to give a short board presentation on what you have learnt.
(3) You should start writing your project report immediately after the first seminar. I expect to see the first draft of Chapter I by the beginning of the second semester.
(4) Read the DUGC guidelines for the project, and pay attention to the deadlines given therein. If you have any questions, make sure that you ask them *now*.
(5) Please make sure that you complete all the paperwork on time, and ensure that you do not rush your PEC members by doing things last-minute.