FREDHOLM OPERATORS : PROJECT DESCRIPTION

1. Overview

Given a fixed continuous function σ on a closed interval $[a, b]$, one considers the problem of solving the following differential equation

$$f''(x) + f(x) = \sigma(x)f(x), \quad a \leq x \leq b$$

such that

$$f(a) = 1 \text{ and } f'(a) = 0$$

In doing so, one is naturally led (See [Schechter, §1.1]) to an *integral equation*: Given a continuous function k on a square $[a, b] \times [a, b]$ and a $u \in C[a, b]$ fixed, we wish to determine all functions f which satisfy the equation

$$f(x) = u(x) + \int_a^x k(x, t)f(t)dt$$

Treating the integral above as an *operator* on the function f, one is reduced to solving an equation of the form

$$f = u + Kf$$

where K is some bounded linear operator on the appropriate function space. In particular, we wish to know if the operator

$$(I - K)$$

is surjective. In 1903, Fredholm proved that $(I - K)$ is surjective if and only if it is injective. This is called the *Fredholm Alternative*, and will be the starting point of the project.

Examining Fredholm's approach leads to two interesting concepts : The class of *Fredholm operators*, and the notion of *index* of such an operator. The goal of this project is to understand these concepts, and finally apply it to the special case of *Toeplitz operators* on the Hardy space $H^2(S^1, d\sigma)$. The index of a Toeplitz operator with continuous symbol φ is given by the negative of the winding number of φ. These types of theorems, where an analytical invariant (index) is related to a topological one (winding number), play an important role in modern mathematics.

2. Tentative Schedule

(1) Preliminaries :
   (a) Hilbert Spaces
   (b) Banach Spaces
   (c) Linear Operators and their Adjoints
   Ref: [Conway, §I.1-I.4, §II.1-2, §III.1-III.5] `[by 15/7/15]`

(2) **[1st Talk] The Fredholm Alternative**
   (a) Compact Operators
   (b) Fredholm Operators

(c) Definition of Index and multiplicativity
(d) Fredholm Alternative
(e) Applications to integral operators

Ref: [Murphy, §1.4], [Douglas, §5] `[by 15/8/15]`

(3) **[2nd Talk] Atkinson's Theorem**
(a) Banach Algebras
(b) Invertible elements
(c) Quotients, and the Calkin Algebra
(d) Atkinson's Theorem
(e) $\mathcal{F}(H)$, the set of Fredholm operators, is open

Ref: [Murphy, §1], [Arveson, §3] `[by 15/10/15]`

(4) **[3rd Talk] Index Theory**
(a) Definition and Examples
(b) Continuity and Homotopy invariance of index
(c) Index : $\pi_0(\mathcal{F}(H)) \to \mathbb{Z}$ is an isomorphism

Ref: [Murphy, §1.4], [Conway, §IX.5], [Douglas, §5] `[by 15/12/15]`

(5) **[4th Talk] Toeplitz Operators**
(a) Definition and Characterization
(b) The Toeplitz algebra and Short exact sequence
(c) Index of Toeplitz Operator with Continuous Symbol

Ref: [Arveson, §2.1, §4.2 – 4.4], [Murphy, §3.5] `[by 15/3/16]`

References

[Conway] J.B. Conway, *A Course in Functional Analysis (2nd Ed.)*
[Arveson] W. Arveson, *A Short Course on Spectral Theory*
[Murphy] G.J. Murphy, *C* Algebras and Operator Theory*
[Douglas] R.G. Douglas, *Banach Algebra techniques in Operator Theory (2nd Ed.)*
[Schechter] M. Schechter, *Principles of Functional Analysis (2nd Ed.)*

3. Important Comments

(1) We will have weekly meetings in which you are expected to give a progress report and discuss questions. You must treat these meetings with utmost seriousness, and must cancel them only in case of unavoidable emergencies. (Taking the GATE/GRE do not count as emergencies)
(2) Every other week, you will be expected to give a short board presentation on what you have learnt.
(3) You should start writing your project report immediately after the first seminar. I expect to see the first draft of Chapter I by the time of the second seminar.
(4) Read the DUGC guidelines for the project, and pay attention to the deadlines given therein. If you have any questions, make sure that you ask them *now*.
(5) Please make sure that you complete all the paperwork on time, and ensure that you do not rush your PEC members by doing things last-minute.