PHY201 Waves and Optics
Assignment 7

Handed out: October 11, 2018 Discussion: October 11, 2018

Problem 1

We have seen in the lecture that the Laplacian of a spherically symmetric wavefunction $\psi(\vec{r}) = \psi(r, \theta, \phi) = \psi(r)$ is

$$\nabla^2 \psi(r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) \tag{1}$$

We can obtain this result starting with the Cartesian form of the Laplacian; operate on the spherically symmetrical wavefunction $\psi(r)$; and convert each term to polar coordinates. Using the relation

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial r} \frac{\partial r}{\partial x} \tag{2}$$

show that

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{x^2}{r^2} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \left(1 - \frac{x^2}{r^2} \right) \frac{\partial \psi}{\partial r} \tag{3}$$

and

$$\nabla^2 \psi(r) = \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r} \tag{4}$$

Problem 2

A He-Ne laser beam has a wavelength $\lambda = 632.8\ nm$, an area $A = 0.10\ cm^2$ and a power $P = 1\,W$. Calculate the intensity (Wm^-2) and the amplitude of the corresponding electric field and that of the magnetic field. What should be the power of an incandescence lamp of efficiency 10% to produce the same light intensity at 1 m from the lamp? What should the radius of a particle of mass density $10^3\ kgm^{-3}$ be in order for it to be suspended by this upward laser beam? Assume that the particle totally absorbs light.

Problem 3

What is the equation of motion of a (point-like) particle with the charge q and the mass m in an electromagnetic field ($\boldsymbol{E}$; $\boldsymbol{B}$) (the emission of radiation by the moving charge is to be neglected)? Determine the temporal change of the energy W of the particle in the external field.

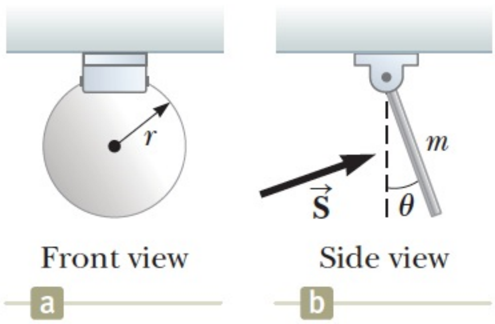
Problem 4

Pressure from a Laser Pointer When giving presentations, many people use a laser pointer to direct the attention of the audience to information on a screen. If a 3.0-mW pointer creates a spot on a screen that is 2.0 mm in diameter, determine the radiation pressure on a screen that reflects 70% of the light that strikes it. The power 3.0 mW is a time-averaged value.

Problem 5

Consider, an example of energy carried by an electromagnetic wave, the energy we receive from the sun. Above the atmosphere, its intensity is about $S = 1.3\ kW/m^2$. The atmosphere absorbs and reflects a fraction of this flux, depending on the weather conditions, on the latitude, etc. On average, half of the flux reaches the soil. Typically, on a surface of 100 m^2, you can have an incident power of about 50 kW on a clear sunny day at intermediate latitudes. If you want to know the usable power, you must multiply this by the efficiency of your device (say 10-20%).
(a) Find the magnitude of the electric field of the wave.
(b) Find the magnitude of the magnetic field of the wave. For comparison, it is known that the earth's magnetic field is an order of magnitude larger.

Problem 6



A uniform circular disk of mass $m = 24.0\ g$ and radius $r = 40.0\ cm$ hangs vertically from a fixed, frictionless, horizontal hinge at a point on its circumference as shown in Figure. A beam of electromagnetic radiation with intensity 10.0 MW/m^2 is incident on the disk in a direction perpendicular to its surface. The disk is perfectly absorbing, and the resulting radiation pressure makes the disk rotate. Assuming the radiation is always perpendicular to the surface of the disk, find the angle θ through which the disk rotates from the vertical as it reaches its new equilibrium position shown in Figure.

Problem 7

In the new field of laser cooling and trapping, the forces associated with radiation pressure are used to slow down atoms from thermal speeds of hundreds of meters per second at room temperature to speeds of just a few meters per second or slower. An isolated atom will absorb radiation only at specific resonant frequencies. If the frequency of the laser beam radiation is one of the resonant frequencies of the target atom, then the radiation is absorbed via a process called resonant absorption. The effective cross-sectional area of the atom for resonant absorption is approximately equal to λ^2, where λ is the wavelength of the laser beam. Estimate the acceleration of a rubidium atom (atomic mass 85 g/mol) in a laser beam whose wavelength is 780 nm and intensity is 10 W/m^2.

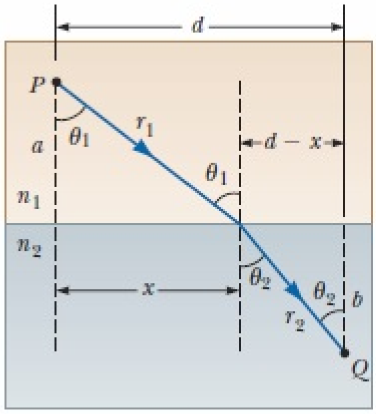
Problem 8

An electromagnetic wave travels through a homogeneous dielectric medium with a frequency of $\omega = 2.10 \times 10^{15}\ rad/s$ and $k = 1.10 \times 10^{7}\ rad/m$. The $\boldsymbol{E}$-field of the wave is

$$\boldsymbol{E} = (180V/m)\boldsymbol{e}_y e^{i(kx-\omega t)} \tag{5}$$

Determine (a) the direction of $\boldsymbol{B}$, (b) the speed of the wave, (c) the associated $\boldsymbol{B}$-field, (d) the index of refraction, (e) the permittivity, and (f) the irradiance of the wave.

Problem 9



Pierre de Fermat (1601-1665) showed that whenever light travels from one point to another, its actual path is the path that requires the smallest time interval. This statement is known as Fermats principle. The simplest example is for light propagating in a homogeneous medium. It moves in a straight line because a straight line is the shortest distance between two points. Derive Snells law of refraction from Fermats principle. Proceed as follows. In Figure, a light ray travels from point P in medium 1 to point Q in medium 2. The two points are, respectively, at perpendicular distances a and b from the interface. The displacement from P to Q has the component d parallel to the interface, and we let x represent the coordinate of the point where the ray enters the second medium. Let $t = 0$ be the instant the light starts from P.

(a) Show that the time at which the light arrives at Q is

$$t = \frac{r_1}{v_1} + \frac{r_2}{v_2} = \frac{n_1\sqrt{a^2+x^2}}{c} + \frac{n_2\sqrt{b^2+(d-x)^2}}{c} \tag{6}$$

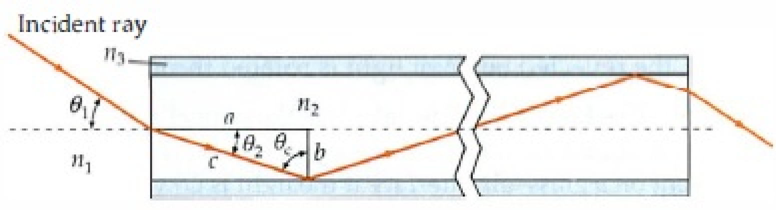
(b) To obtain the value of x for which t has its minimum value, differentiate t with respect to x and set the derivative equal to zero. Show that the result implies

$$\frac{n_1 x}{\sqrt{a^2+x^2}} = \frac{n_2(d-x)}{\sqrt{b^2+(d-x)^2}} \tag{7}$$

(c) Show that this expression in turn gives Snells law,

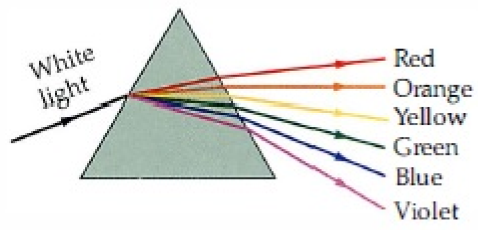
$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{8}$$

Problem 10

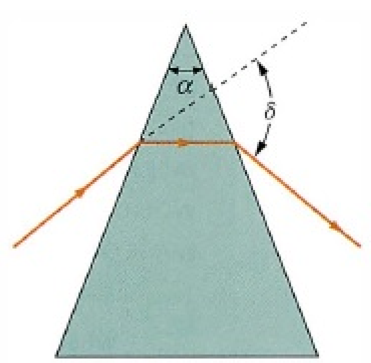


An optical fiber allows rays of light to propagate long distances through total internal reflection. As shown in Figure, the fiber consists of a core material with index of refraction n_2 and radius b, surrounded by a cladding material of index $n_3 < n_2$. The numerical aperture of the fiber is defined as $\sin\theta_1$, where θ_1 is the angle of incidence of a ray of light impinging the end of the fiber that reflects off the core-cladding interface at the critical angle. Using the figure as a guide, show that the numerical aperture is given by $\sqrt{n_2^2 - n_3^2}$ assuming the ray is incident from air.

Problem 11



A beam of white light incident on a glass prism is dispersed into its component colors. The index of refraction decreases as the wavelength increases so that the longer wavelengths (red) are bent less than the shorter wavelengths (violet).



Light passes symmetrically through a prism with an apex angle α as shown in Figure.

(a) Show that the angle of deviation δ is given by

$$\sin\frac{\alpha+\delta}{2} = n\sin\frac{\alpha}{2} \tag{9}$$

(b) If the refractive index for red light is 1.48 and the refractive index for violet light is 1.52, what is the angular separation of visible light for a prism with an apex angle of 60 degrees?

Problem 12

Different colors (frequencies) of light travel at different speeds (a phenomena referred to as dispersion). This can cause problems in fiber-optic communications systems where pulses of light must travel very long distances in glass. Assuming a fiber is made of silicate crown glass, calculate the difference in time needed for two short pulses of light to travel 15 km of fiber if the first pulse has a wavelength of 700 nm and the second pulse has a wavelength of 500 nm.

Problem 13

The wave $U(\boldsymbol{r},t)$, at fixed $\boldsymbol{r}$, treated as a function of t defined for all t, will be called a wavetrain passing through $\boldsymbol{r}$. Two wavetrains are coherent or correlated if their random variations with time are statistically correlated (one with the other). In more qualitative terms, imagine that we know the amplitude $U_1(t)$ of one wavetrain at some time t_1. A second wavetrain is coherent with the first if we could then predict the value of $U_2(t_1)$ at the same time t_1 and this predictability exists for any choice of t_1. Usually, we are interested in near-monochromatic light, such as a single spectral line, and then the wave possesses a phase. For this special case, two wavetrains are coherent if they have a phase difference that is independent of time t.

(a) **Ideally monochromatic waves and coherence**
Consider a plane wave whose amplitude is represented by the analytic signal $U(\boldsymbol{r},t) = U_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$, where U_0 is a constant. This expression defines the wave amplitude at all times t and all locations $\boldsymbol{r}$, i.e., it specifies the entire wavetrain. The only frequency present is $\frac{\omega}{2\pi}$, so that the wave is exactly monochromatic.
(1) Show that $U_2 = U(\boldsymbol{r}+\boldsymbol{s},t) = U(\boldsymbol{r},t)e^{i\boldsymbol{k}\cdot\boldsymbol{s}}$, in which we shall take $\boldsymbol{s}$ as constant. Show that wavetrain U_2 is fully coherent with wavetrain $U(\boldsymbol{r},t)$.
(2) Show that $U_3 = U(\boldsymbol{r},t+\tau) = U(\boldsymbol{r},t)e^{-i\omega\tau}$, in which we shall take τ as constant. Show that wavetrain U_3 is fully coherent with wavetrain $U(\boldsymbol{r},t)$.

(b) **Ideally monochromatic waves and interference**
Let two waves have amplitudes $Ae^{i(\boldsymbol{k_1}\cdot\boldsymbol{r}-\omega t)}$, $Be^{i(\boldsymbol{k_2}\cdot\boldsymbol{r}-\omega t-\alpha)}$, where A and B are real. If these waves are superposed at locations $\boldsymbol{r}$, show that the amplitude U_{total} is

$$U_{total} = e^{i(\boldsymbol{k_1}\cdot\boldsymbol{r}-\omega t)}(A + Be^{i[(\boldsymbol{k_2}-\boldsymbol{k_1})\cdot\boldsymbol{r}-\alpha]}) \tag{10}$$

and the intensity is proportional to $|U_{total}|^2$

$$|U_{total}|^2 = A^2 + B^2 + 2AB\cos\left[(\boldsymbol{k_2}-\boldsymbol{k_1})\cdot\boldsymbol{r}-\alpha\right] \tag{11}$$

The term A^2, B^2 represent the intensities of the individual waves. The cosine term is recognizable as interference: something additional (positive or negative), happening to the intensity, that depends upon the relative phases of the contributing waves.

(c) **Add intensities when waves are incoherent**

(1) Let two waves have amplitudes $Ae^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$, $Be^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t-\alpha)}$, where A and B are real and the phase difference α is constant. If these waves are superposed at locations $\boldsymbol{r}$, show that the amplitude U_{total} is

$$U_{total} = e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}(A + Be^{i-\alpha}) \tag{12}$$

and the intensity is proportional to $|U_{total}|^2$

$$|U_{total}|^2 = A^2 + B^2 + 2AB\cos\alpha \tag{13}$$

(2) Now measure the intensity, not once but many times, each time with a different value of α, and with the αs distributed randomly over a range of 2π. Show that the average of all these measurements is

$$< \left|U_{total}^2\right| >= A^2 + B^2 \tag{14}$$

By taking an average, we have rendered invisible the cross-terms $2AB\cos\alpha$ containing the random phase α; equivalently, the interference effects have been averaged away. It is as if intensities should be added instead of amplitudes.

(3) Let's make things less abstract. Imagine that we do an interference experiment by superposing two wavetrains of light and observe the result by eye. One wave is monochromatic, $U_1 = Ae^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$; the other is $U_2 = Be^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t-\alpha)}$, where B is constant but α changes randomly over a timescale of a microsecond or so (very long compared with an optical period but very short compared with the timescale an eye can notice). The changes of α mean that this wave is modulated and cannot be strictly monochromatic. Argue for yourself that each 'measurement' made by the eye is an average over an ensemble of about 10^5 different values of α. Note that it is a matter of common experience that an eye can detect intensity changes up to a frequency of order 10 Hz.

(d) **Interference of two parallel plane waves**

Let two waves have amplitudes $Ae^{i(kz-\omega_1 t-\alpha_1)}$, $Be^{i(kz-\omega_2 t-\alpha_2)}$, where A and B are real. If these waves are superposed at location z, show that the intensity as measured in a time interval T is proportional to

$$< |U_{total}|^2 >_T = A^2 + B^2 + 2AB < \cos\left[(\omega_1 - \omega_2)t - (\alpha_1 - \alpha_2)\right] >_T \tag{15}$$

Show that the interference term is zero if $T >> |\omega_1 - \omega_2|^{-1}$.