Approximation Algorithms for the Partial Degree Bounded Edge Packing (PDBEP) Problem

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- There exist polynomial time algorithms to find the maximum matching in graph G

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- Introduced by Peng Zhang in 2012

Example

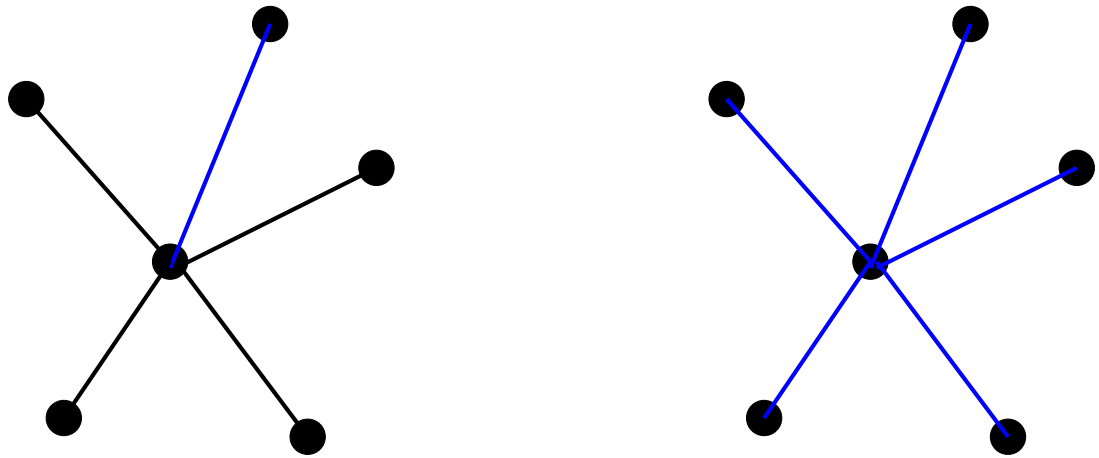

Figure: Matching Vs PDBEP

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- Hence, PDBEP too is NP-hard [Zhang 2012]

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- Since $OPT \leq |V|$, we get $|F| \geq OPT/2$

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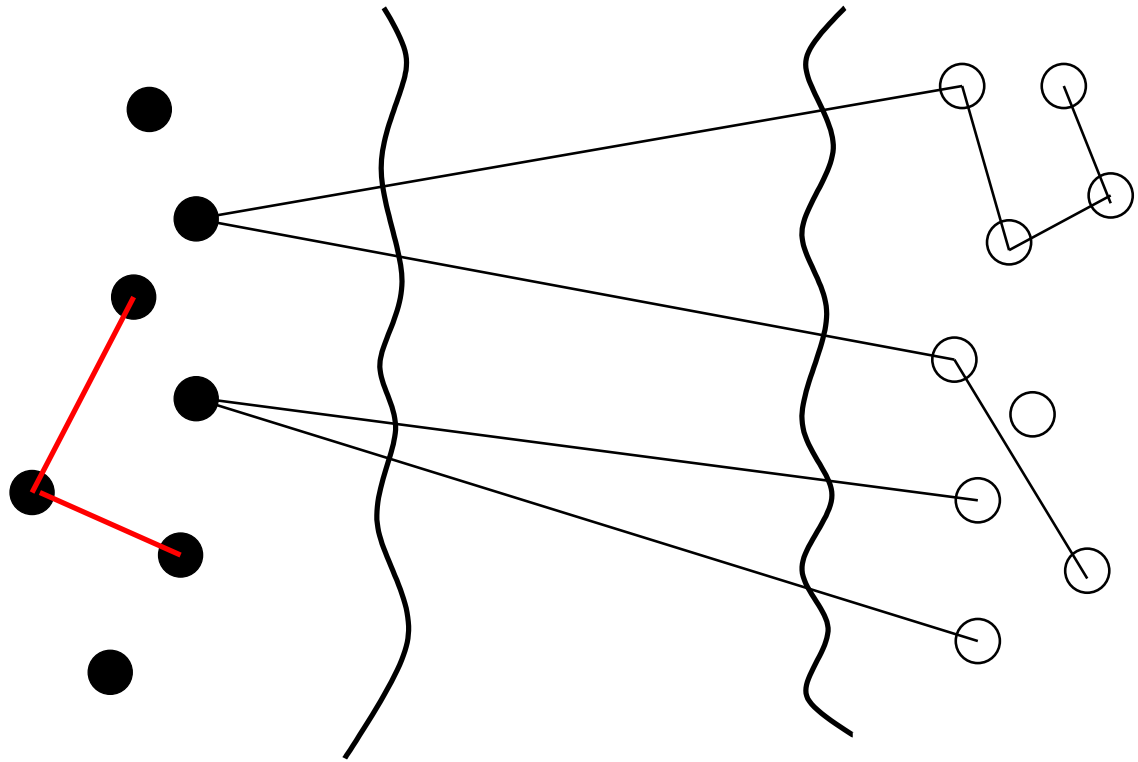
  - Find $F \subseteq E$ s.t. $\sum_{e=\{w,u\} \in E} d(e) \leq c(u)$ **OR** $\sum_{e=\{w,v\} \in E} d(e) \leq c(v)$ for all $\{u, v\} \in F$

$c : V \rightarrow \mathbb{N}$ $d, w : E \rightarrow 1$ [A.,Singh,Mehta]

Upper Bound on OPT



$OPT \leq \sum_{w \in W} c(w) \leq \sum_{v \in V} c(v)$, $W = \{v \in V | deg(v) \leq c(v)\}$

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  - $E_{max} = E^1_{max} \cup E^2_{max}$

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- G' is a forest

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- That gives, $OPT \leq w(E^1_{max}) + 2w(E^2_{max})$

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- Label the vertices in G' with distance from the root

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- Return $\max\{\overrightarrow{\delta}(X), \overrightarrow{\delta}(Y)\} \geq (w(E^1_{\max}) + 2w(E^2_{\max}))/2 \geq OPT/2$

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The Knapsack Problem

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- There exists an algorithm that can find a solution that is at least $(1 - \epsilon)OPT$ in time polynomial in n and $1/\epsilon$

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- So we want to maximize the total weight of the edges that can be packed such that the total demand does not exceed the capacity

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- Using a simple randomized algorithm we can compute a DICUT of expected weight $w(E')/4$

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- This leads to a $(4+\epsilon)$-approximate solution

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- We get a $(2 + \epsilon)$-approximate solution

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- Does there exist a 2-approximation algorithm for the most general case?
- We will need better upper bounds

Thank you!
Questions?