MTH 202: Probability and Statistics

HW 2 (due date: 23/01/2015)

Problems for submission

1. Suppose three identical fair coins are tossed once. For $i = 1, 2, 3$, let A_i be the event that the i^{th} coin shows a head. Show that the events A_1, A_2, A_3 are mutually independent (that is, they are pairwise independent).

2. Experience shows that 20 percent of the people reserving tables at a certain restaurant never show up. If the restaurant has 100 tables and takes 102 reservations, what is the probability that it will be able to accommodate everyone who shows up? (Hint: Use Bayes rule)

3. A circular target of unit radius is divided into four annular zones with outer radii $0.25, 0.5, 0.75$, and 1. Suppose 20 shots are fired independently and at random into the target.
(a) Compute the probability that at most two shots land in the zone bounded by the circles of radii 0.25 and 1.
(b) If 8 shots land inside the disk of radius 0.75, find the probability that at least one is in the disk of radius 0.5.

4. Suppose 10 couples arrive at a party. The boys and girls are then paired off at random. What is the probability that exactly 5 specified couples are paired?

5. We proved that for two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
Prove the generalized statement by induction on the number n of events.
Let $A_1, \cdots, A_n$ be events. Then

$$P(A_1 \cup \cdots A_n) = S_1 - S_2 + S_3 - \cdots + (-1)^{n-1} S_n,$$

where,

$$\begin{aligned} S_1 &= \sum_{i=1}^{n} P(A_i) \\ S_2 &= \sum_{1 \le i_1 < i_2 \le n} P(A_{i_1} \cap A_{i_2}) \\ S_3 &= \sum_{1 \le i_1 < i_2 < i_3 \le n} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) \\ &\vdots \\ S_n &= P(A_1 \cap \cdots \cap A_n). \end{aligned}$$

Problems not for submission

1. Prove that $1 - x \le e^{-x}$, for all $x \ge 0$.

2. Prove that if the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{N \to \infty} \sum_{n=N}^{\infty} a_n = 0$.

3. Let $A_1, A_2, \cdots$ be a sequence of events such that $\sum_{n=1}^{\infty} P(A_n) < \infty$. Prove that $P(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n)$.

4. A man has 30 keys, exactly one of which fits the lock. He tries the keys one at a time. At each trial, he chooses a key at random, which is not tried earlier. Find the probability that the 14th key is the correct key.

5. The Morse code consists of a sequence of dots and dashes with repetitions permitted.
(a) How may Morse codes of length 20 are there?
(b) How many Morse codes of length at most 20 are there?

6. Suppose every box of cereal contains a toy. Suppose there are r types of toys that a cereal box can have, and a given box is equally likely to contain any one of these r toys. If n boxes are purchased, find the probability of
(a) having collected all of the r toys.
(b) of not collecting exactly k of the r toys.

7. A box contains 40 good and 10 defective fuses. If 10 fuses are selected at random, what is the probability that all of them will be good?

8. Construct an example of a sequence of events $A_1, A_2, \cdots$ such that the probability of infinitely many of the events $A_1, A_2, \cdots$ occurring equals 0.5.

9. Let $A_1, A_2, \cdots$ be a sequence of events. Define

$$\liminf A_n := \cup_{n\geq 1} \cap_{k\geq n} A_k, \quad \limsup A_n := \cap_{n\geq 1} \cup_{k\geq n} A_k.$$

Show that $\emptyset \subseteq \liminf A_n \subseteq \limsup A_n \subseteq \Omega$.
(In words, $\liminf A_n$ denote the event that all except finitely many of the events $A_1, A_2, \cdots$ occur; and $\limsup A_n$ denote the event that infinitely many of the events $A_1, A_2, \cdots$ occur).