THE EQUIVALENCE OF MEASURED FOLIATIONS AND MEASURED LAMINATIONS

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1. INTRODUCTION

It is a well known fact that measured foliations and measured laminations are basically the same objects. Measured foliations are more closely related to analysis since they arise naturally in complex analysis, from quadratic differentials. Measured laminations are more topological and more transparently a completion of weighted curve systems. In this project, we prove that there is a bijective correspondence between space of measured foliations $\mathcal{MF}(S)$ and the space of measured laminations $\mathcal{ML}(S)$. Consequently, the spaces are homeomorphic as topological spaces.

Let S be a orientable hyperbolic surface.

Definition 1.1. A *foliation* $\mathcal{F}$ of S is a local product structure. That is, at each X there exists a neighbourhood U and a diffeomorphism $U \to \mathbb{R}^k \times \mathbb{R}^\ell$ such that the overlap maps take each $p \times \mathbb{R}^\ell$ to some $q \times \mathbb{R}^\ell$.

The equivalence classes generated by the relation of lying in the same $p \times \mathbb{R}^\ell$ are the *leaves* of the foliation.

Definition 1.2. A *singular foliation* of S is a foliation with 1-dimensional leaves except at isolated singular points of valency $p \geq 3$.
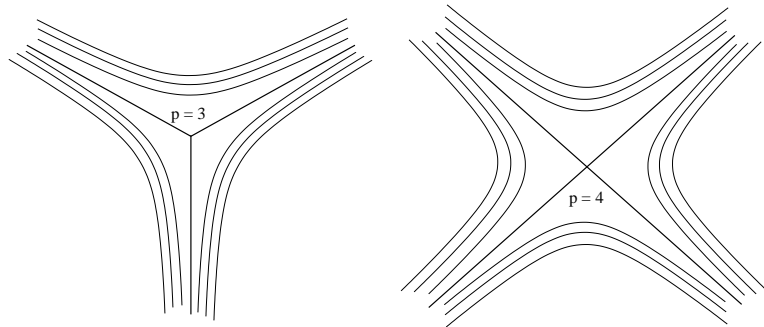

FIGURE 1. Singular foliations on the surface S

Example 1.3. We obtain a singular foliation of the hyperbolic polygon, by joining its center to the vertices with edges. This triangulates the polygon

into triangular regions. Now, in each triangular region, we join its center to the vertices with vertices with edges. This divides the polygonal region to polygonal subregions. Finally, we foliate each subtriangular region so that the leaves of the foliation are parallel to the edge of the polygon in the triangle. It is important to note at this point that there might be different ways of triangulating and foliating the polygonal regions. However, all the resulting foliations are Whitehead equivalent.
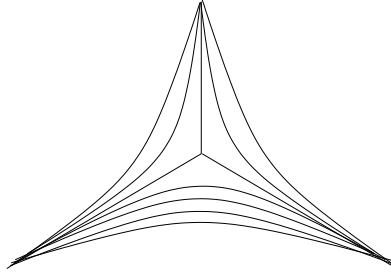

FIGURE 2. A foliated triangular region

Example 1.4. We obtain a foliation with compact leaves on the annulus in the following manner.
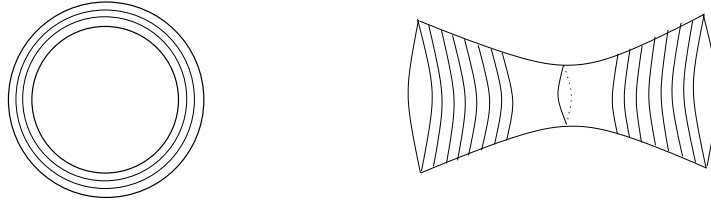

FIGURE 3. A foliated annulus

Example 1.5. We obtain a singular foliation on the pair of pants in the following manner.
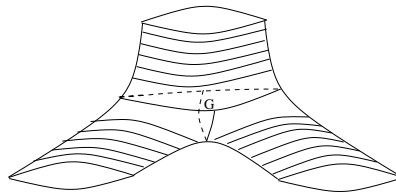

FIGURE 4. A foliated pair of pants

We introduce a graph G (a singular set) whose complement is union of three annular regions. We foliate each annular region with leaves parallel to the boundary.

Definition 1.6. A *measured foliation* $(\mathcal{F}, \mu)$ of S is a singular foliation $\mathcal{F}$ with an invariant measure μ that is nonzero on transverse arcs. That is, a measure μ that :
  (1) Assigns a nonnegative real number for each arc that is additive for unions of arcs.
  (2) Assigns 0 to an arc if and only if the arc lies in a leaf.
  (3) Remains invariant if a transverse arc is moved, keeping it transverse and keeping its endpoints in the same leaves.
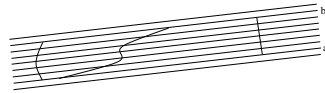

FIGURE 5. Think of the leaves as the lines $y = c$. Then the measure of the transverse arc is $|b - a|$.

Definition 1.7. Two measured foliations are called *Whitehead equivalent* if they are related by a sequence of
  (a) isotopies, and
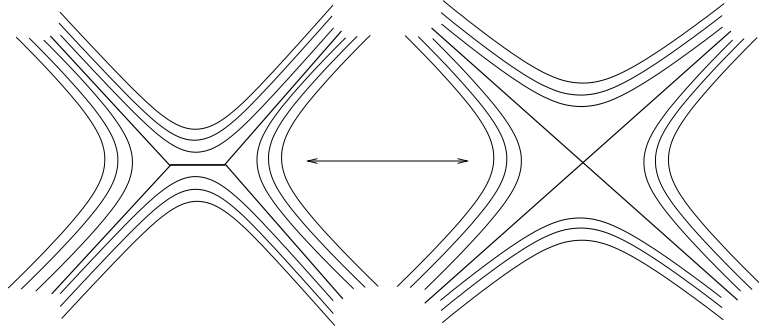 (b) collapses or expansions of this type.

FIGURE 6. Whitehead move on a singular foliation

We will denote by $\mathcal{MF}(S)$ the topological space of all measured foliations of S modulo Whitehead equivalence.

Remark 1.8. Thurston showed that [2] $\mathcal{MF}(S)$ is homeomorphic to $S^{2m-1} \times (0, \infty)$ where the Teichmüller space of S denoted by $\mathcal{T}(S)$ is homeomorphic to $\mathbb{R}^{2m}$.

Definition 1.9. A *lamination* $\mathcal{L}$ of S is a closed subset A of S with a product structure for A.

Remark 1.10. A lamination is like a foliation of a closed subset of S. Leaves of a lamination are defined just as for a foliation.

Definition 1.11. A *measured lamination* $(\mathcal{L}, \mu)$ of S is a lamination with an invariant measure μ that is nonzero on transverse arcs.

Definition 1.12. A geodesic lamination $\mathcal{L}$ is said to be *complete* if every connected component of $S \setminus \mathcal{L}$ is isometric to the interior of a hyperbolic ideal triangle.

A measured lamination whose leaves are geodesics is a called a *measured geodesic lamination*. We will denote by $\mathcal{ML}(S)$ the space of all measured geodesic laminations on S that are equivalent up to isotopy.

Definition 1.13. Suppose that μ is a measure coming from a measured foliation or a measured lamination on S. Then, for an isotopy class α of imbedded loops in S, we define the *measure of α under μ* to be the positive number $L_\mu(\alpha)$ defined by $L_\mu(\alpha) = \inf\limits_{c \in \alpha} L_\mu(\alpha)$.

Definition 1.14. A *weighted train track* [3] is a 1-complex in S with
  (1) All vertices are of valency ≥ 3.
  (2) All edges share a tangent direction at each vertex.
  (3) The edges have an assignment of positive weights that satisfy the *switch condition*, that is the sums of the weights in the two directions in each vertex are equal.
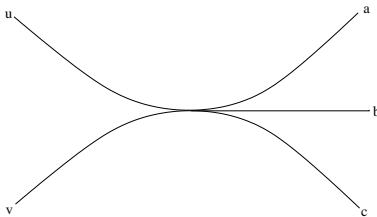

FIGURE 7. The switch condition in this train track is $u + v = a + b + c$

Definition 1.15. Let $d_1, \ldots, d_n$ be disjoint nonisotopic curves in S with assigned real weights $w_1, \ldots, w_n$. Then $\mathbb{D}_w = \{(d_1, w_1), \ldots, (d_n, w_n)\}$ is called a *weighted curve system* on S.

Remark 1.16. A weighted curve system $\mathbb{D}_w = \{(d_1, w_1), \ldots, (d_n, w_n)\}$ determines a measures geodesic lamination $(\mathcal{L}, \mu)$ of S by choosing the corresponding geodesic representatives from the classes of w_i instead of w_i.

Definition 1.17. Suppose that $\mathbb{D}_w = \{(d_1, w_1), \ldots, (d_n, w_n)\}$ is a weighted curve system in S. Then, for an isotopy class α of embedded loops in S, we define the *length of α under $\mathbb{D}_w$* as the positive number $L_{\mathbb{D}_w}(\alpha)$ defined by $L_{\mathbb{D}_w}(\alpha) = \sum\limits_{i=1}^{n} w_i \cdot i(\alpha, d_i)$, where $i(\alpha, d_i)$ is the minimum of the intersection numbers of curves c in α with $\cup d_i$.

Notation 1.18. We will denote the real number $\displaystyle\sum_{i=1}^{n} w_i \cdot i(\alpha, d_i)$ by $i(\alpha, \mathbb{D}_w)$.

Lemma 1.19. *Every measured foliation $(\mathcal{F}, \mu)$ on S with compact leaves determines a weighted curve system $\mathbb{D}_w$ with $L_\mu(\alpha) = i(\alpha, \mathbb{D}_w)$ for every isotopy class of curves α. Conversely, every weighted curve system $\mathbb{D}_w$ in S determines a foliation measured foliation $(\mathcal{F}, \mu)$ with compact leaves with $L_\mu(\alpha) = i(\alpha, \mathbb{D}_w)$.*

Proof. Suppose that $(\mathcal{F}, \mu)$ is a measured foliation with compact leaves of S. We start splitting apart along singular leaves. Since the leaves are compact, each singular leaf has both end points at singular points, so the splitting process ends with foliated submanifolds of S with no singular points. That is, a true foliation by circles. Since S is orientable, the submanifolds are tori or annuli.

If the submanifold is a torus, it can have no singular leaves. If the leaf is a (p, q) curve $C_{p,q}$, then $L_\mu(\alpha) = w \cdot i(\alpha, C_{p,q}) = i(\alpha, \{(C_{p,q}, w)\})$ where w is the measure of a transverse circle that crosses $C_{p,q}$ once and $\{(C_{p,q}, w)\} \subseteq \mathbb{D}_w$.

If the submanifold is an annulus, for each isotopy class of a leaf, add up the transverse measure of the leaves in that isotopy class. This gives a $(d_i, w_i) \in \mathbb{D}_w$ where d_i can be taken to be any curve in the isotopy class of the leaf.

Conversely, suppose that we have a weighted curve system $\mathbb{D}_w = \{(d_1, w_1), \ldots, (d_n, w_n)\}$ in S. We start with foliated annular neighborhoods of d_i with transverse measure w_i. We obtain a partial foliation $(\mathcal{F}_I, \mu_0)$ on a union of annuli in S with $L_{\mu_0}(\alpha) = \sum_{i=1}^{n} w_i \cdot i(\alpha, d_i) = i(\alpha, \mathbb{D}_w)$.

Now just expand these annuli. When they meet, they fuse into singular leaves giving a singular foliation $\mathcal{F}$ of S up to Whitehead equivalence. To define the measure on this foliation we take any transverse arc α. Suppose the p and q are the end points of the arc α that lie on leaves l and m. We consider the projection of the annulus to the close interval $[0, 1]$ by sending each leaf of the foliation to a point. Consider the subannulus determined by union of the the leaves between l and m. This is sent under the projection to a subinterval $[l, m]$.
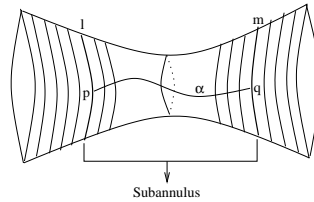

FIGURE 8. The subannulus determined by the end points of arc α

We define the measure of the arc α to be the measure of the interval $[l, m]$, that is $|l - m|$. This defines an invariant measure $\overline{\mu}$ on the singular foliation $\mathcal{F}$.

$\square$

Definition 1.20. A weighted train track *carries* a measures lamination if there is a suitable set of weights that produces the measured lamination

Lemma 1.21 (Thurston, [1]). *Every measured lamination in S is carried by a suitable train track.*

Remark 1.22. Rationally related weights correspond to measured geodesic laminations with all leaves compact, that is a weighted curve system. Irrationally related weights give a measured geodesic lamination with noncompact leaves.

2. THE MAIN THEOREM

Theorem 2.1. *Let S be a orientable hyperbolic surface. Then, there is a bijective correspondence between $\mathcal{MF}(S)$ and $\mathcal{ML}(S)$. Consequently, $\mathcal{MF}(S) \cong \mathcal{ML}(S)$.*

Proof. Suppose that we have a measured foliation $(\mathcal{F}, \mu)$ of S. If the leaves of the foliation are compact, then by Lemma 1.19 the foliation $(\mathcal{F}, \mu)$ determines a weighted curve system and hence a geodesic measured lamination $\mathcal{F}, \overline{\mu}$ with compact leaves by Remark 1.16.

If the leaves are not noncompact, then we can collapse the singular foliation to a weighted train track. By Remark 1.22 we get a measured geodesic lamination $(\mathcal{L}, \overline{\mu})$ carried by the train track.

Conversely, suppose that we have a measured geodesic lamination $(\mathcal{L}, \mu)$ of S. If $\mathcal{L}$ is a finite union of closed geodesics with compact leaves, then it forms weighted curve system under the measure μ. By Lemma 1.19 we have a corresponding measured foliation $(\mathcal{F}, \overline{\mu})$ and we are done.

Suppose that L does not form a weighted curve system. Then $S \setminus \mathcal{L}$ is a finite disjoint union of connected regions.

If one of the complementary regions is a hyperbolic subsurface, then add singular sets (or graphs) as in the foliation of the pair of pants in Example 1.5 to obtain a foliation on the complements of the graphs (which are annuli). The measure on each annulus is defined as in proof of Lemma 1.19. Therefore we obtain a measured foliation $(\mathcal{F}, \overline{\mu})$ on the entire surface.

If the region is a hyperbolic triangle, we consider the thickened weighted train track that carried the measured lamination $(\mathcal{L}, \mu)$. We employ a similar technique to the one used in Example 1.3 to foliate the triangle so that the leaves are parallel to the weighted edges of the train track as shown in the figure below.

Thus, we obtain a foliation $\mathcal{F}$ on the entire surface S. To define a measure on $\mathcal{F}$, we consider a transverse arc α. We consider a map from the subtriangular region containing α to an interval $[0, z]$ of taking each singular leaf
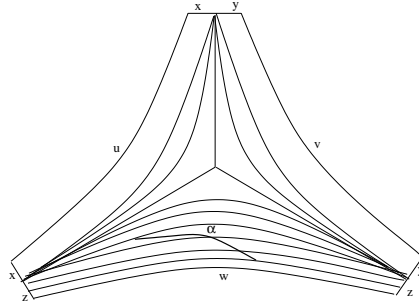
FIGURE 9. Foliaiton of the triangle with leaves parallel to the edges of the train track$\mathcal{L}$

to a point. Suppose the p and q are the end points of the arc α that lie on leaves l and m. Then the union of the leaves between l and m is mapped to a subinterval of length $\frac{|l-m|}{z}$. We define the measure of the arc α to be the measure of the interval $[l, m]$, that is $\frac{|l-m|}{z}$. This defines an invariant measure $\overline{\mu}$ on the singular foliation $\mathcal{F}$.

In general, if the complementary region is a hyperbolic polygon, we can consider the train track that carries the lamination. We use a similar procedure to the one used in the triangular case to obtain a measured foliation on the surface S.

$\square$

REFERENCES

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