MTH 507 Midterm Solutions

1. Let \( X \) be a path connected, locally path connected, and semilocally simply connected space. Let \( H_0 \) and \( H_1 \) be subgroups of \( \pi_1(X, x_0) \) (for some \( x_0 \in X \)) such that \( H_0 \leq H_1 \). Let \( p_i : X_{H_i} \to X \) (for \( i = 0, 1 \)) be covering spaces corresponding to the subgroups \( H_i \). Prove that there is a covering space \( f : X_{H_0} \to X_{H_1} \) such that \( p_i \circ f = p_0 \).

**Solution.** Choose \( \tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0) \) so that \( p_i : (X_{H_i}, \tilde{x}_i) \to (X, x_0) \) and \( p_* : (X_{H_i}, \tilde{x}_i) = H_i \) for \( i = 0, 1 \). Since \( H_0 \leq H_1 \), by the Lifting Criterion, there exists a lift \( f : X_{H_0} \to X_{H_1} \) of \( p_0 \) such that \( p_1 \circ f = p_0 \). It remains to show that \( f : X_{H_0} \to X_{H_1} \) is a covering space.

Let \( y \in X_{H_1} \), and let \( U \) be a path-connected neighbourhood of \( x = p_1(y) \) that is evenly covered by both \( p_0 \) and \( p_1 \). Let \( V \subset X_{H_1} \) be the slice of \( p_1^{-1}(U) \) that contains \( y \). Denote the slices of \( p_0^{-1}(U) \) by \( \{V'_z : z \in p_0^{-1}(x)\} \). Let \( C \) denote the subcollection \( \{V'_z : z \in f^{-1}(y)\} \) of \( p_1^{-1}(U) \). Every slice \( V_z \) is mapped by \( f \) into a single slice of \( p_1^{-1}(U) \), as these are path-connected. Also, since \( f(z) \in p_1^{-1}(y) \), \( f(V'_z) \subset V \) iff \( f(z) = y \). Hence \( f^{-1}(V) \) is the union of the slices in \( C \).

Finally, we have that for \( V'_z \in C \), \( (p_1|_V)^{-1} \circ (p_0|_{V'_z}) = f|_{V'_z} \). Hence \( f|_{V'_z} \) is a homeomorphism, and \( V \) is an evenly covered neighborhood of \( y \).

2. A homomorphism between two covering spaces \( p_1 : \tilde{X}_1 \to X \) and \( p_2 : \tilde{X}_2 \to X \) is a map \( f : \tilde{X}_1 \to \tilde{X}_2 \) so that \( p_1 = p_2 \circ f \).

(a) Classify all the covering spaces of \( S^1 \) up to isomorphism.

(b) Find all homomorphisms between these covering spaces.

**Solution.** (a) Let \( x_0 = (1, 0) \), then by Classification of Covering Spaces, the basepoint-preserving isomorphism classes of covering spaces of \((S^1, x_0)\) correspond to the subgroups of \( \pi_1(S^1, x_0) \cong \mathbb{Z} \). Each non-trivial subgroup \( m\mathbb{Z} \leq \mathbb{Z} \) corresponds to the covering space \( p_m : (S^1, \tilde{x}_0) \to (S^1, x_0) \), where \( p_m(z) = z^m \) and \( \tilde{x}_0 \) is an \( m \)th root of unity in \( S^1 \subset \mathbb{C} \). The trivial subgroup corresponds to the universal cover \( p : \mathbb{R} \to S^1 \) given by \( p(s) = e^{i(2\pi s)} \).

(b) Let \( p_m : (S^1, \tilde{x}_0) \to (S^1, x_0) \) and \( p_n : (S^1, \tilde{x}_1) \to (S^1, x_0) \) be covering spaces. Any homomorphism \( f : (S^1, \tilde{x}_0) \to (S^1, \tilde{x}_1) \) must induce a homomorphism \( f_* : \mathbb{Z}_m \to \mathbb{Z}_n \). Such a homomorphism exists iff \( m | n \),
and when \( m \mid n \), \( f_* \) is the natural injection \( \mathbb{Z}_m \hookrightarrow \mathbb{Z}_n \). There can both be no homomorphism between universal cover \( p : \mathbb{R} \to S^1 \) to any of the finite-sheeted covering spaces mentioned above, as its fundamental group is trivial. Finally any homomorphism \( f \) from the universal cover to itself should be an isomorphism (by the Lifting Criterion) which satisfies \( p \circ f = p \). Since \( p(s) = e^{i(2\pi s)} \), \( f \) has to be of the form \( s \mapsto s + k \), where \( k \) is an integer.

3. Let \((X, x_0)\) and \((Y, y_0)\) be topological spaces.

   (a) Show that \( \pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0) \). [Hint: Use the projection maps \( p_1 : X \times Y \to X \) and \( p_2 : X \times Y \to Y \).]

   (b) Compute the fundamental group of the solid torus.

Solution. (a) Please see Theorem 60.1 (Page 371 ) in Munkres.

(b) The solid torus \( X \approx D^2 \times S^1 \). Hence \( \pi_1(X) \approx \pi_1(D^2) \times \pi_1(S^1) \), from part(a). Since \( D^2 \) is simply connected, we have that \( \pi_1(X) \cong \mathbb{Z} \).

4. Find all 2-sheeted covering spaces of the torus \( S^1 \times S^1 \) up to isomorphism.

Solution. By the Classification of Covering Spaces, the isomorphism class of any 2-sheeted covering space will correspond to a subgroup to \( \pi_1(S^1 \times S^1) \cong \mathbb{Z}^2 \) of index 2. The two obvious nontrivial subgroups are \( 2\mathbb{Z} \times \mathbb{Z} \) and \( \mathbb{Z} \times 2\mathbb{Z} \), which correspond to the coverings \( z^2 \times w : S^1 \times S^1 \to S^1 \times S^1 \) and \( z \times w^2 : S^1 \times S^1 \to S^1 \times S^1 \) respectively.

There is only one other nontrivial subgroup. Before we describe this subgroup, note that any such subgroup will be isomorphic to the kernel of a homomorphism \( \mathbb{Z}^2 \to \mathbb{Z}_2 \). The third nontrivial subgroup is isomorphic to the kernel of the homomorphism that maps both of the standard generators of \( \mathbb{Z}^2 \) to 1 \( \in \mathbb{Z}_2 \). Explicitly, this subgroup can be described as \( \{(x, y) \in \mathbb{Z}^2 \mid x + y \pmod{2} = 0 \} \). Can you describe the covering space this subgroup corresponds to?

5. Consider the real projective \( n \)-space \( \mathbb{R}P^n \) obtained by identifying each point \( x \in S^n \) with its antipode \( -x \).

   (a) Compute its fundamental group.

   (b) Find all its covering spaces up to isomorphism.
(c) Show that every map from $\mathbb{R}P^2 \to S^1$ is nullhomotopic.

**Solution.** (a) We know from class that the quotient map $q : S^n \to \mathbb{R}P^n$ for ($n \geq 2$) is a 2-fold universal covering space. (Note that we are assuming here that $S^n$ is simply connected.) Therefore, the lifting correspondence is bijective, and consequently, $\pi_1(\mathbb{R}P^n)$ is a group that has exactly 2 elements. Therefore, $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$.

(b) Up to isomorphism, any covering space of $\mathbb{R}P^n$ will correspond to a subgroup of $\mathbb{Z}_2$. As the only subgroups of $\mathbb{Z}_2$ are itself and the trivial group, they will correspond to the covering spaces of $\text{id} : \mathbb{R}P^2 \to \mathbb{R}P^2$ and $q : S^n \to \mathbb{R}P^n$ (which in this case is the universal cover) respectively.

(c) Suppose that $f : \mathbb{R}P^2 \to S^1$ is a continuous map. Then it induces a homomorphism $h_* : \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2 \to \pi_1(S^1) \cong \mathbb{Z}$, which has be trivial. By the Lifting Criterion, $f$ lifts to a map $\tilde{f} : \mathbb{R}P^2 \to \mathbb{R}$ such that $p \circ \tilde{f} = f$, where $p : \mathbb{R} \to S^1$ is the standard universal covering. Since $\mathbb{R}$ is contractible, $f$ has be homotopic (via some $H$) to a constant map. Hence $p \circ H$ is a homotopy from $f$ to a constant map.

6. Let $r : S^1 \to S^1$ be a reflection of the circle (e.g. $(x, y) \to (-x, y)$ in the plane). The *Klein bottle* $K$ is the quotient space of $[0, 1] \times S^1$ under the following equivalence relation: $(0, z) \sim (1, r(z))$ for all $z \in S^1$, and $(t, z)$ is not equivalent to anything except itself, for $t = 0, 1$.

(a) Explain why $K$ is compact.

(b) Let $C_1 \subset K$ be (the image of) the circle $1 \times S^1$, and let $C_2 \subset K$ be a small embedded circle inside $(\frac{1}{2}, \frac{3}{2}) \times S^1$. There is a continuous map $g : K \to \mathbb{R}^3$ such that $g|_{K-C_1}$ and $g|_{K-C_2}$ are injective. Assuming $g$ exists as described, use Urysohn’s Lemma to construct a continuous map of $K$ into $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$, which is an imbedding.

**Solution.** (a) Since the quotient map is a continuous map, and $[0, 1] \times S^1$ is compact, it follows that $K$ has to be compact.

(b) Since $K$ is Hausdorff and $C_1, C_2 \approx S^1$, they have to be closed sets in $K$. As $K$ is also compact, it is a normal space. By Urysohn’s Lemma, we obtain a function $f : K \to [0, 1]$ such that $f(C_1) = \{0\}$ and $f(C_2) = \{1\}$. Composing $f$ with the inclusion $[0, 1] \hookrightarrow \mathbb{R}$, we obtain a function $f' : K \to \mathbb{R}$ such that $f'(C_1) = \{0\}$ and $f'(C_2) = \{1\}$.
Now define \( h : K \to \mathbb{R}^3 \times \mathbb{R} \) by \( h(x) = (g(x), f'(x)) \). This map is continuous, as it has continuous coordinate functions. Consider two distinct points \( x, y \in K \). If both \( x \) and \( y \) are in \( C_1 \) and \( C_2 \) then \( f'(x) \neq f'(y) \), and so \( h(x) \neq h(y) \). Without loss of generality, if we assume that \( C_1 \) does not contain \( x \) or \( y \), then \( x, y \in K \setminus C_1 \), and \( g \) is injective on this subset. Hence \( g(x) \neq g(y) \), and so \( h(x) \neq h(y) \).

Finally, \( h \) is a continuous and injective map from a compact space to a Hausdorff space. This implies that \( h \) has to be an imbedding.

7. Let \( h, k : (X, x_0) \to (Y, y_0) \) be continuous maps.

(a) If \( h \simeq k \) (via \( H \)) such that \( H(x_0, t) = y_0 \) for all \( t \), then show that \( h_* = k_* \).

(b) Using (a) show that the inclusion \( j : S^n \to R^{n+1} \setminus \{0\} \) induces an isomorphism of fundamental groups. [Hint: Use the natural retraction map \( r : R^{n+1} \setminus \{0\} \to S^n \)].

**Solution.** Please see Lemma 58.2 and Theorem 58.2 (Page 360) from Munkres.

8. **[Bonus]** Let \( f : (S^1, x_0) \to (S^1, x_1) \) be a continuous map. Then the induced homomorphism \( f_* : \pi_1(S^1, x_0)(\cong \mathbb{Z}) \to \pi_1(S^1, x_1)(\cong \mathbb{Z}) \) is completely determined by the integer \( d \) given by \( f_*([\alpha_0]) = d[\alpha_1] \), where \( [\alpha_i] \) is a generator \( \pi_1(S^1, x_i) \) (for \( i = 0, 1 \)) that represents \( 1 \in \mathbb{Z} \). This integer \( d \) is called the **degree** of \( f \) (denoted by \( \text{deg}(f) \)).

(a) Show that if \( f \simeq g \), then \( \text{deg}(f) = \text{deg}(g) \).

(b) Show that if \( f \) is a homeomorphism, then \( \text{deg}(f) = \pm 1 \). In particular, show that \( \text{deg}(a) = -1 \), when \( a \) is the antipodal map.

**Solution.** (a) Note that \( \text{deg}(f) \) is independent of basepoint, for if we choose a different basepoint \( y_0 \in X \) with \( f(y_0) = y_1 \), then there exists isomorphisms \( \hat{\alpha} : \pi_1(X, x_0) \to \pi_1(Y, y_0) \) and \( \hat{\beta} : \pi_1(X, x_1) \to \pi_1(Y, y_1) \) such that \( f_* \circ \hat{\alpha} = \hat{\beta} \circ f_* \). Since any continuous map \( f : S^1 \to S^1 \) has to be a loop, it follows from 7(a) that \( f_* = g_* \), and consequently \( \text{deg}(f) = \text{deg}(g) \).

(b) To prove (b), we show first that \( \text{deg}(f \circ g) = \text{deg}(f) \cdot \text{deg}(g) \). But this follows directly from the fact that \( (f \circ g)_* = f_* \circ g_* \). Also, by definition
\[ \deg(id) = 1, \] as it induces the identity homomorphism. Therefore, if \( f \) is a homeomorphism, then \( \deg(f) \cdot \deg(f^{-1}) = 1 \). As \( \deg(f) \) is an integer, we have that \( \deg(f) = \pm 1 \). Finally, since \( a \) is a homeomorphism that is non-homotopic to \( f \), we conclude that \( \deg(a) = -1 \).