EXPANDER FAMILIES OF GRAPHS

A THESIS

submitted in partial fulfillment of the requirements

for the award of the dual degree of

Bachelor of Science-Master of Science

in

MATHEMATICS

by

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BHOPAL - 462066

April 2020

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भारतीय विज्ञान शिक्षा एवं अनुसंधान संस्थान भोपाल

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CERTIFICATE

This is to certify that **Vishal Gupta**, BS-MS (Dual Degree) student in Department of Mathematics, has completed bonafide work on the thesis entitled **Expander Families of Graphs** under my supervision and guidance.

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ACKNOWLEDGEMENTS

I heartily thank Dr. Kashyap Rajeevsarathy for offering me this project on my request. I thank him for his patience, cooperation, and encouragement during the presentations which have helped me a lot in honing my presentation skills. It was his constant support and guidance that made it possible for me to come this far in a totally new subject (to me).

I thank my PEC members Dr. Siddhartha Sarkar, Dr. Jyoti Prakash Saha, and my seniors Neeraj and Apeksha for attending my presentations and for always being available to solve my queries. I greatly benifited from their valuable remarks and suggestions regarding my thesis. I also thank everyone in the mathematics department for being so supportive, especially I thank Dr. Rahul Garg and Dr. Kumar Balasubramanian for guiding me whenever I felt lost. I am also very grateful to Dr. Ajit Bhand for all his help over the years I spent in IISER. I thank him for his generosity, approachability, and also for all the TT matches we have played together in the evenings.

I thank my friends Arundhati, Ekta, Garima, Radhika, Ranveer, Shreyas, Sreshta, and HUSTLERS for all those last night discussions and after exam celebrations. I am grateful to all of them for being the delta I needed for every epsilon to maintain a uniform continuity in this 5 years of journey at IISER. I thank them for keeping me sane and adding meaning to all of this through their presence.

I thank my family for raising me with love, and for teaching me morals of life. I am incredibly grateful to them for their support and encouragement that has made me what I am today.

Vishal Gupta

ABSTRACT

A family $(G_n)_{n\in\mathbb{N}}$ of finite connected d-regular graphs is said to be *expander* if the sequence $\{h(G_n)\}_{n\in\mathbb{N}}$, where $h(G_n)$ denotes the *isoperimetric constant* of the graph (G_n), is uniformly bounded away from zero. In this project, we study the various aspects of spectral graph theory from the viewpoint of expander graphs. We begin by discussing some of the basic facts of expander graph theory [8], and proof of the Alon-Boppana theorem. We go on to understand the relation between three graph invariants, namely the isoperimetric constant, the second largest eigenvalue, and the diameter, and which leads to the fact that sequences of finite abelian groups cannot yield expander families of Cayley graphs. Further, we study the notion of graph coverings and prove the classic result that sequences of solvable groups with bounded derived length do not yield expander families. We then use basic results from representation theory of finite groups to understand the spectrum of Cayley graphs and computed an explicit formula to calculate the spectrum of Cayley graphs on abelian groups. Finally, we study random lifts of graphs and the proof of the existence of infinite families of regular bipartite Ramanujan graphs of every degree bigger than 2, which uses the method of interlacing polynomials [9].

Contents

Chapter 1

Introduction

The origins of graph theory can be traced back to Euler's solution to the Konigsberg bridge problem from the 18th century, which asked if there was a way to walk on all the seven bridges of the Konigsberg city, exactly once, in a single trip, with the condition that the trip ended at the same place it began. Since then, the subject has evolved tremendously and has now become an important branch of pure mathematics, as well as applied mathematics, owing to its wide range of applications in science, engineering and technology. One such application is in the analysis and design of efficient communication networks.

A typical communication network can be modeled by a graph, where the vertices represent the entities that wish to communicate and edges represent the connections between these entities. Now, given a fixed set of vertices, the goal is to design a network that is better in terms of reliability, speed, and the cost-effectiveness. It is known that this is tantamount to determining whether the network has a large isoperimetric constant. There are no efficient algorithms available to explicitly compute the isoperimetric constant of an arbitrary graph, as this is known to be an NP-hard problem. However, some bounds have been derived for the isoperimetric constant (or, the Cheeger constant) of regular graphs in terms of the other graph invariants such as the diameter, and the second largest eigenvalue of the adjacency matrix associated to a graph.

In this chapter, we discuss some basic definitions, standard notations, and some results from graph theory that we will be using extensively in subsequent chapters. This chapter is based on [6, 8, 10].

1.1 Preliminaries

Definition 1.1.1. An *undirected graph* G is a pair (V, E), where V is a nonempty set, called the set of *vertices*, E is called the set of *edges*. Formally, if $V = \{v_1, v_2, ..., v_n\}$, then $E \subseteq \{\{v_i, v_j\} : v_i, v_j \in V\}$.

When there is more than one graph in discussion, to avoid any confusion we denote the vertex set of a graph G by $V(G)$, and edge set by $E(G)$.

Example 1.1.2. Let $G = (V, E)$ be a graph with $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 1\}, \{3, 4\}, \{3, 5\}\}$. Figure 1.1 gives the pictorial representation of G.

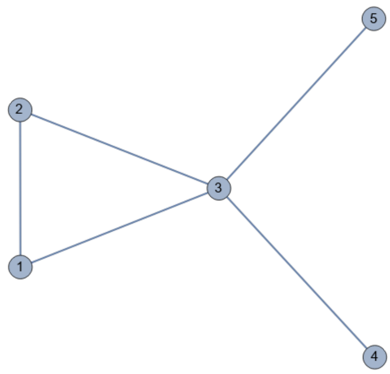


Figure 1.1: An undirected graph on 5 vertices.

A graph is *finite* if both its vertex set and edge set are finite. If $E = \phi$, then the graph is called a *null graph*, and if $E = \{\{v_i, v_j\} : v_i, v_j \in V, i \neq j\}$, then the graph is called a *complete graph*, denoted by K_n.

(a) K_4

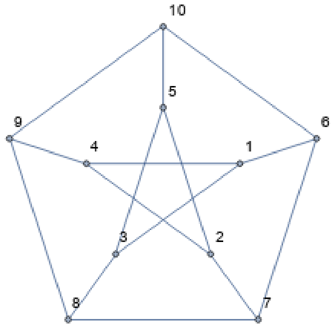
(b) K_6

Figure 1.2: The graphs K_4 and K_6.

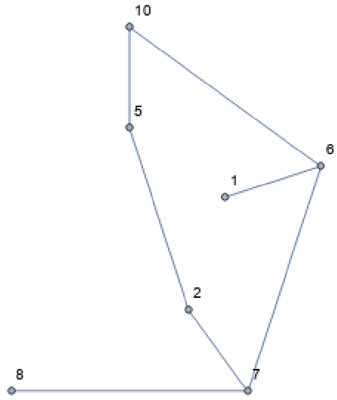
Remark 1.1.3. Let $G = (V, E)$ be a graph, and let $u, v \in V$. Then u, v are called *adjacent to each other* if $\{u, v\} \in E$. That is, if u, v are the endpoints of the same edge of the graph.

Definition 1.1.4. A graph H is said to be a *subgraph* of another graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then the graph H is called a *spanning subgraph* of G. A subgraph H is called an *induced subgraph* of G if any edge $e \in E(H)$ if and only if $e \in E(G)$.

Example 1.1.5. Consider the graph G as shown in Figure 1.3a. Observe that the graph H shown in Figure 1.3b is an induced subgraph of G. We note that H is not a spanning subgraph as $V(H) \subset V(G)$.



(a) G



(b) H

Figure 1.3: A graph G and its induced subgraph H.

Remark 1.1.6. The graph shown in Figure 1.3a is known as the *Petersen Graph*, named after the mathematician Julius Petersen.

Definition 1.1.7. An edge with equal endpoints is called a *loop*. Edges with the same pair of endpoints are called *multiple edges*.

Definition 1.1.8. A graph having no loops or multiple edges is called a *simple graph*.

Example 1.1.9. Consider the graph shown in Figure 1.4. It is not a simple graph as it has multiple edges and a loop. The edge in red color is a loop, and the green edges are multiple edges.

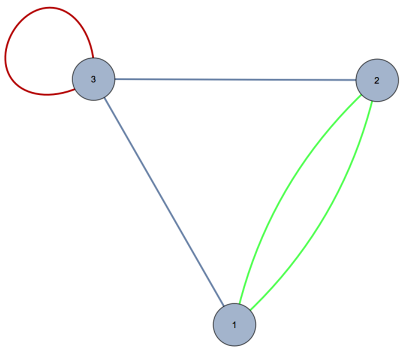


Figure 1.4: A multigraph on 3 vertices.

Remark 1.1.10. Throughout this thesis, we will assume graphs to be finite, undirected, and simple, unless stated otherwise.

Definition 1.1.11. The *order* of a graph $G = (V, E)$, denoted by $|G|$, is the cardinality of its vertex set V.

Definition 1.1.12. The *degree* of a vertex $v \in V$, denoted by $\deg(v)$, is the number of vertices in V that are adjacent to it.

Definition 1.1.13. A graph $G = (V, E)$ is *d-regular* if $\deg(v) = d$, for all $v \in V$.

Example 1.1.14. The graph depicted in Figure 1.5 is a 19-regular graph on 20 vertices K_{20}.

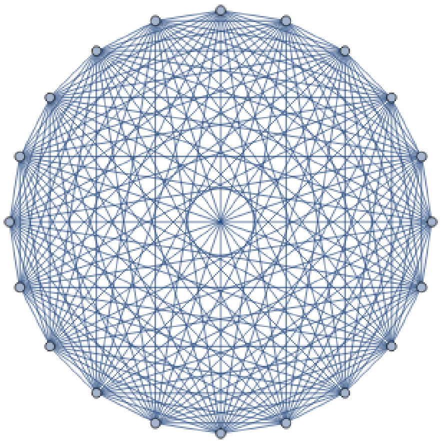


Figure 1.5: The complete graph K_{20}.

Lemma 1.1.15. *Let $G = (V, E)$ be a graph. Then*

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$

Proof. For any vertex $v \in V$, we know that

$$\deg(v) = |\{e \in E \mid e = \{u, v\}; u \in V\}|.$$

Since, each edge $e \in E$ has two endpoints, it contributes $+2$ to the sum. Thus,

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$

□

Definition 1.1.16. Let $G = (V, E)$ be a graph. A subset S of V is called *an independent set* if no two vertices of S are adjacent to each other.

Definition 1.1.17. A graph $G = (V, E)$ is called *bipartite* if its vertex set V can be partitioned into two independent sets. Similarly, G is called *r-partite* if V can be partitioned into r independent sets.

Example 1.1.18. Figure 1.6 below illustrates the graph $K_{3,8}$, which is the complete bipartite graph with independent sets of order 3 (red vertices) and 8 (blue vertices).

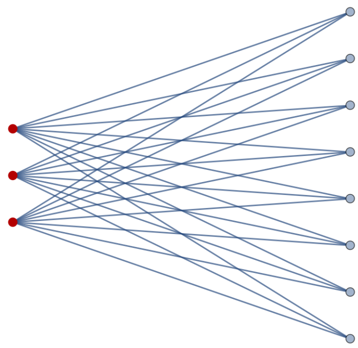


Figure 1.6: The graph $K_{3,8}$.

Definition 1.1.19. Let $G = (V, E)$ be a graph.

(i) A *walk* in G is a sequence $v_0, e_1, v_1, ..., e_k, v_k$ of vertices v_i and edges e_i such that for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. The number of edges in the sequence is called the *length* of the walk.

(ii) A walk is called a *closed walk* if $v_0 = v_k$.

(iii) A *path* is a walk with no repeated vertex. A closed path is called a *cycle*

(iv) A *trail* is a walk with no repeated edges. A closed trail is called a *circuit*.

Example 1.1.20. Consider the Petersen graph in Figure 1.3a. The sequence

$$1, \{1, 6\}, 6, \{6, 10\}, 10, \{6, 10\}, 6$$

is a walk in the graph starting from vertex 1 to vertex 6. But, note that it is not a path as 6 is the repeated vertex.

Definition 1.1.21. The *distance* between any two vertices u, v, denoted by $\mathrm{dist}(u, v)$, is the length of a shortest path joining u and v in the graph, where $\mathrm{dist}(u, v) = 0$ if and only if $u = v$.

Remark 1.1.22. We note that the defined notion of distance between two vertices from Definition 1.1.21 above defines a metric on G called the *path metric*.

Definition 1.1.23. The *diameter* of a graph G, denoted by $\mathrm{diam}(G)$, is defined as

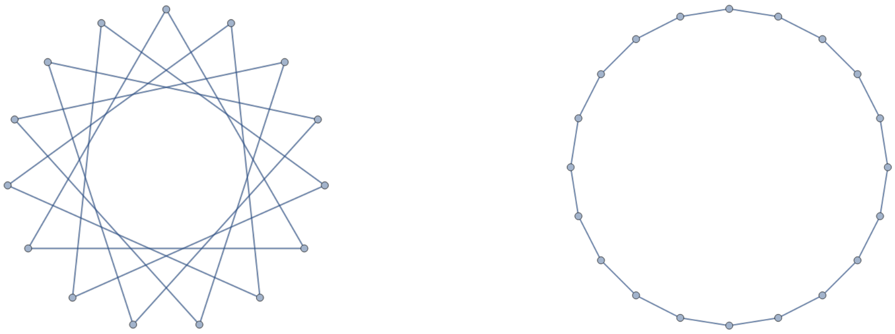
$$\mathrm{diam}(G) = \max\{\mathrm{dist}(u, v) : u, v \in V(G)\}.$$

Definition 1.1.24. A graph is called *connected* if for any two vertices of the graph there is a walk in the graph joining them. Otherwise, the graph is called *disconnected*.

Definition 1.1.25. Let G, H be graphs. A map $\phi : V(G) \to V(H)$ is said to be a *graph homomorphism* if $\{\phi(v), \phi(u)\} \in E(H)$, whenever $\{v, u\} \in E(G)$.

Definition 1.1.26. Two graphs G and H are *isomorphic* if there exists a bijective map $\phi : V(G) \to V(H)$ such that $\{\phi(v), \phi(u)\} \in E(H)$ if and only if $\{v, u\} \in E(G)$.

Example 1.1.27. Consider the graphs G, H as shown in Figure 1.7. Both G and H are connected and 2-regular graphs of order 20. Hence, they are isomorphic.



(a) G (b) H

Figure 1.7: Isomorphic graphs of order 20.

1.2 Adjacency Matrix

Definition 1.2.1. Let $G = (V, E)$ be a graph with $|V| = n$. The *adjacency matrix* of G, $A(G) = (a_{ij})_{n\times n}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } \{i, j\} \in E, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Remark 1.2.2. Note that $A(G)$ is a real and symmetric matrix. Hence, by Spectral Theorem, we know that all of its eigenvalues are real.

Example 1.2.3. Consider the complete graph on 4 vertices K_4. Then

$$A(K_4) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Definition 1.2.4. The *characteristic polynomial* of a graph G is defined to be the characteristic polynomial of $A(G)$. The *spectrum* of G is defined to be the multiset of eigenvalues of A. We write the spectrum of G as

$$\mathrm{Spec}(G) = \begin{pmatrix} \lambda_0 & \lambda_1 & \cdots & \lambda_k \\ n_0 & n_1 & \cdots & n_k \end{pmatrix},$$

where for each $0 \le i \le k$, λ_i are distinct eigenvalues of A, and $\sum_{i=0}^{k} n_i = n$.

Example 1.2.5. Consider the complete graph on n vertices K_n. Note that $A(K_n) = J - I_n$, where J is an $n \times n$ matrix with all of its entries equal to 1, and I_n denotes the identity matrix of order n. Since, J is real and symmetric, by Spectral Theorem we know that it is diagonalisable and has n real eigenvalues. But, note that $\mathrm{rank}(J) = 1$, so it has only one non-zero eigenvalue. Consider the vector $v = (1, 1, \cdots, 1) \in \mathbb{R}^n$. Clearly, $Jv = nv$, which implies the n is an eigenvalue of J. So,

$$\mathrm{Spec}(J) = \begin{pmatrix} n & 0 \\ 1 & n-1 \end{pmatrix}.$$

We also note that if (λ, v) is an eigenpair of J, then $Av = (J - I_n)v = Jv - I_nv = \lambda v - v = (\lambda - 1)v$. Thus, we get

$$\operatorname{Spec}(A) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

Remark 1.2.6. Relabelling the vertices of a graph G yields an adjacency matrix B, which is similar of $A(G)$, and hence has the same spectrum as G. Thus, the spectrum of a graph does not depend on its labelling. More generally, we have the following theorem, which we state without proof.

Theorem 1.2.7. *Let X and Y be two graphs of order n, and let $A(X)$ and $A(Y)$ be their corresponding adjacency matrices, respectively. Then $X \cong Y$ if and only if there exists a permutation matrix $P \in M_n(\mathbb{R})$ such that*

$$PA(X)P^T = A(Y).$$

Theorem 1.2.8. *Let G be a d-regular graph of order n with the spectrum $\lambda_{n-1} \leq \cdots \leq \lambda_1 \leq \lambda_0$. Then*

(i) $\lambda_0 = d$,

(ii) $|\lambda_i| \leq d$ *for all* $0 \leq i \leq n-1$, *and*

(iii) $|\lambda_1| < d$ *if and only if X is connected.*

Proof. Since G is d-regular, sum of the entries of any row of its adjacency matrix A is equal to d. Let $v = (1, 1, \cdots, 1)^t \in \mathbb{R}^n$. Then

$$A \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} = d \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}.$$

Thus, d is an eigenvalue of A. Now, let $v = (v_1, v_2, \cdots, v_n)^t \in \mathbb{R}^n$ be an eigenvector corresponding to some eigenvalue λ of A, and let $1 \leq i \leq n$ be such that

$$|v_i| = \max\{|v_j| \mid 1 \leq j \leq n\}.$$

Then

$$|\lambda||v_i| = |\lambda v_i| = |\sum_{j=1}^{n} a_{i,j}v_j| \leq \sum_{j=1}^{n} |a_{i,j}v_j| = \sum_{j=1}^{n} a_{i,j}|v_j| \leq \sum_{j=1}^{n} a_{i,j}|v_i| = d|v_i|.$$

Since $|v_i|$ is non-zero, we get $|\lambda| \leq d$.

Suppose that G is connected. We want to show that $|\lambda_1| < d$. Let $v = (v_1, v_2, \cdots, v_n)^t \in \mathbb{R}^n$ be an eigenvector corresponding to d. Again, let $1 \leq i \leq n$ be such that

$$|v_i| = \max\{|v_j| \mid 1 \leq j \leq n\}.$$

Replacing v by $-v$ if necessary, assume that v_i is positive. Then we have

$$dv_i = \sum_{j=1}^{n} a_{i,j} v_j \implies v_i = \sum_{j=1}^{n} \frac{a_{i,j}}{d} v_j.$$

This implies v_i is a convex linear combination of $v_j, 1 \leq j \leq n$. Since for each j, $|v_j| \leq v_i$, we get $v_j = v_i$ for all j such that $a_{i,j} = 1$. Since G is connected, any two distinct vertices are connected by a walk, which eventually gives $v_j = v_i$ for all j. Hence, v is a scalar multiple of $(1, 1, \cdots, 1)^t$, which implies eigenvalue d has multiplicity one.

To prove the converse, suppose that G is disconnected. Let $v \in V(G)$, and let V_1 be the set of all vertices $w \in V(G)$ such that there exists a walk in G connecting v and w. We note that, if $w \in V$ is adjacent to a vertex in V_1, then $w \in V_1$. Since, $V \backslash V_1 \neq \phi$, G splits into two d-regular graphs with vertex sets V_1 and $V_2 = V \backslash V_1$, respectively. Hence, d is an eigenvalue of both these graphs, which implies $\lambda_1(G) = d$. This completes the proof. $\square$

Definition 1.2.9. For a d-regular graph, the eigenvalue $\lambda_0 = d$ is called its trivial eigenvalue.

Theorem 1.2.10. *Let G be a d-regular bipartite graph of order n. Then the spectrum of G is symmetric about 0, that is, if $\lambda \in \operatorname{Spec}(G)$. then $-\lambda \in \operatorname{Spec}(G)$.*

Proof. Let $A(G)$ be the adjacency matrix of G. Let $V(G) = X \sqcup Y$, where X and Y are independent subsets such with $|X| = r < n/2$. We add $n - 2r$ isolated vertices to the independent set X to make the size of both the partite sets equal. We note that doing this adds rows and columns of zeros to $A(G)$ transforming it to a matrix A'.

We observe that we can permute the rows and columns of A' to obtain a matrix of the form

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix},$$

where B is an $(n-r) \times (n-r)$ square matrix. If λ is an eigenvalue with eigenvector $(x, y)^t$, then

$$A' \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} By \\ B^T x \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$$
$$\implies By = \lambda x \text{ and } B^T x = \lambda y.$$

Let $v' = (x, -y)^T$. Then

$$A' \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -By \\ B^T x \end{pmatrix} = -\lambda \begin{pmatrix} x \\ -y \end{pmatrix}.$$

Thus, v' is an eigenvector with eigenvalue $-\lambda$. $\square$

1.3 Cayley Graphs

Cayley graphs are graphs that capture the abstract structures of groups. Given a group and a subset of that group, we can construct a Cayley Graph with respect to the subset. Using the properties of the group we can derive properties of its Cayley graph.

Definition 1.3.1. Let G be a group and $\Gamma \subseteq G$. We say that Γ is a *symmetric subset* of G if whenever γ is an element of Γ, then γ^{-1} is an element of Γ.

Example 1.3.2. Consider the group D_8. Let $\Gamma_1 = \{1, r, r^2\}$, and $\Gamma_2 = \{1, r, r^3\}$ be two subsets of D_8. Then Γ_1 is not symmetric because $r^{-1} \notin \Gamma_1$, whereas Γ_2 is symmetric, since $r \cdot r^3 = 1$.

Definition 1.3.3. Let G be a finite group and $\Gamma \subseteq G$ be symmetric. The *Cayley graph* on G with respect to Γ is defined by $\mathrm{Cay}(G, \Gamma) := (V, E)$, where

(i) $V = G$, and

(ii) for $x, y \in G$, $\{x, y\} \in E$ if and only if $y^{-1}x \in \Gamma$.

Example 1.3.4. Let $G = \mathbb{Z}_n$, and let $\Gamma = \{1, -1\}$. Then $\mathrm{Cay}(G, \Gamma) = C_n$.

Proposition 1.3.5. *Let G be a finite group and $\Gamma \subseteq G$ be symmetric. Then:*

(i) $\mathrm{Cay}(G, \Gamma)$ *is $|\Gamma|$-regular, and*

(ii) $\mathrm{Cay}(G, \Gamma)$ *is connected if and only if Γ generates G.*

Proof. (i) For a vertex g of the graph $\mathrm{Cay}(G, \Gamma)$, let E_g denote the set of edges incident to g. By definition, it is clear that

$$E_g = \{\{g, g\gamma\} \mid \gamma \in \Gamma\} \implies |E_g| = |\Gamma|.$$

(ii) Suppose $\mathrm{Cay}(G, \Gamma)$ be connected. Then for any two vertices $a, b \in G$, there is a path in $\mathrm{Cay}(G, \Gamma)$ joining them. That is, there exists $\gamma_1, \gamma_2, \cdots, \gamma_r \in \Gamma$ such that

$$a = b\gamma_1\gamma_2\cdots\gamma_r \implies b^{-1}a = \gamma_1\gamma_2\cdots\gamma_r.$$

Since the vertices chosen were arbitrary, it proves that Γ generates G. Conversely, suppose Γ generates G. Then for any $g \in G$, there exists $\gamma_1, \gamma_2, \cdots, \gamma_r \in \Gamma$ such that $g = \gamma_1\gamma_2\cdots\gamma_r$. This implies that all the vertices are connected to the identity element of G, which proves the assertion.

□

Example 1.3.6. Consider the symmetric group on three symbols S_3. Let $\Gamma = \{(1\ 2), (1\ 3), (2\ 3)\}$. As Γ generates S_3, therefore by Proposition 1.3.5, the Cayley graph $X = \mathrm{Cay}(S_3, \Gamma)$ is connected and 3-regular. From Figure 1.8, it is apparent that X is also a bipartite graph with independent sets $A = \{e, (1\ 2\ 3), (1\ 3\ 2)\}$ and $B = \{(1\ 2), (1\ 3), (2\ 3)\}$.

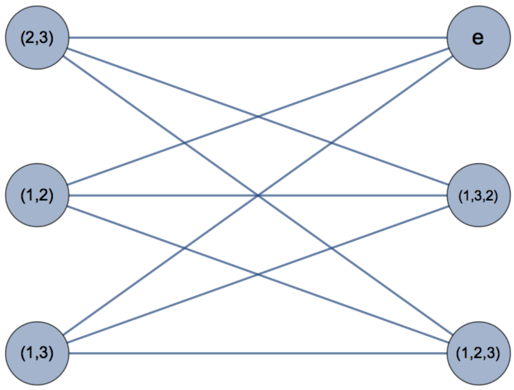


Figure 1.8: A Cayley graph on S_3.

Example 1.3.7. Consider the Petersen graph X as shown in the Figure 1.3a. We claim that X is not a Cayley graph. We know that D_{10} and $\mathbb{Z}_{10}$ are the only groups of order 10 up to isomorphism. So, if $X = \mathrm{Cay}(D_{10}, \Gamma)$, then Γ must be a symmetric generating subset of D_{10} with $|\Gamma| = 3$. Then at least one element of Γ, say x, must have order 2, which implies $x = sr^i$ where $0 \leq i \leq 4$. If there is an element of order 5, say $y = r^j, 1 \leq j \leq 4$, in Γ, then $(xy)^2 = (sr^i r^j)^2 = (sr^{i+j})^2 = 1$. This gives a cycle of length 4 in the graph, which is a contradiction, as X has no cycle of length 4. If all the elements of Γ have order 2, then there is no product of 5 elements of Γ equal to the identity, which is again a contradiction, as X has 5-cyclces. Thus, $X \neq \mathrm{Cay}(D_{10}, \Gamma)$.

Now, suppose that $X = \mathrm{Cay}(\mathbb{Z}_{10}, \Gamma)$. Note that for any two elements $x, y \in \Gamma$, $x^{-1}y^{-1}xy$ gives a cycle of length of 4, which is a contradiction.

Chapter 2

Expander Families

We say that a good communication network is one which is fast, reliable, and cost-effective. A natural question that arises in this context is given two communication network models (graphs), how does one tell which one is better? In other words, can one quantify the above mentioned properties? We answer this question in this chapter. In Sections 2.2-2.3, we discuss the adjacency and the Laplacian linear operators associated with a graph, and their relation to the graph spectrum. In Section 2.4, we prove the Rayleigh-Ritz Theorem, which provides a method to calculate the second-largest eigenvalue of a graph. Finally, we study the connection between $h(G)$ and $\lambda_1(G)$ of a graph G, which motivates the field of Spectral Graph Theory. This chapter is based on Chapter 1 of [8].

2.1 The Isoperimetric Constant

The speed of a network is roughly measured by the minimum number of edges that one needs to traverse to get from one vertex to another. So, smaller the diameter of the network, the faster it is. The reliability of a network is measured by the minimum number of edge cuts needed to disconnect the network. In other words, the edge-connectivity of a network is a measure of its reliability.

The isoperimetric constant of a network (graph) gives information about both the speed and the reliability of a communications network. Roughly speaking, the isoperimetric constant of a graph measures how quickly information can flow through the graph.

Definition 2.1.1. Let $G = (V, E)$ be a graph of order n, and let $S \subset V$. Then the *boundary* δS of S is defined by

$$\delta S := \{\{u, v\} \in E \mid u \in S \textit{ and } v \in V \backslash S\}.$$

Note that $\delta S = \delta(V \backslash S)$.

Definition 2.1.2. The *isoperimetric constant* $h(G)$ of a graph $G = (V, E)$ is defined by

$$h(G) = \min \left\{ \frac{|\delta S|}{|S|} \ : \ S \subset V \textit{ and } |S| \leq \frac{1}{2}|V| \right\}.$$

Remark 2.1.3. Note that as in Definition 2.1.1, it is redundant to consider subsets of cardinality greater than $|V|/2$, as $\delta(S) = \delta(V/S)$.

Example 2.1.4. Consider the graph in Figure 2.1. When $S = \{A\}$, we have. $|\delta S| = 2$, and $\frac{|\delta S|}{|S|} = \frac{2}{1} = 2$, and when $S = \{A, B\}$, we have $|\delta S| = 2$, and so $\frac{|\delta S|}{|S|} = \frac{2}{2} = 1$. Thus, we get that the isoperimetric constant $h(C_4) \leq 1$. Finally, when $S = \{A, C\}$, we have $|\delta S| = 4$, and so $\frac{|\delta S|}{|S|} = \frac{4}{2} = 2$. Since the graph is symmetric, we have considered all the possibilities. Therefore, $h(C_4) = 1$.

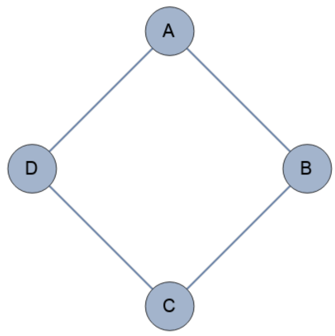


Figure 2.1: C_4

Remark 2.1.5. The isoperimetric constant goes by many other names. It is also called the Cheeger constant, the expansion constant, the edge expansion constant, or the conductance.

Remark 2.1.6. The isoperimetric constant of a graph captures the worst case scenario, in the sense that if $h(G) = \frac{|\delta S|}{|S|}$, then it means that every other subset of $V(G)$ has a larger boundary, relative to its size, when compared with S.

Proposition 2.1.7. *Let $G = (V, E)$ be a d-regular graph, then $h(G) \in [0, d]$.*

Proof. Firstly, it is clear by the definition that $h(G)$ is always non-negative. Further, when G is disconnected, S can be taken to be one of the disconnected components, for which $|\delta S| = 0$. Thus, we have $h(G) \geq 0$.

Finally, when G is connected, taking $S = \{v\}$, for some $v \in V$, we have

$$\frac{|\delta S|}{|S|} = \frac{d}{1} = d \implies h(G) \leq d.$$

□

Proposition 2.1.8. *Let G be a d-regular graph with $h(G) = d$. Then G is either K_2 or K_3.*

Proof. Let G be a d-regular graph (other than K_2 and K_3) with $h(G) = d$. Suppose $S = \{v_1, \cdots, v_r\} \subset V(G)$ such that

$$h(G) = \frac{|\delta S|}{|S|} \implies d \cdot |S| = |\delta S|.$$

This implies the set S is an independent set. For $v \in G$, let E_v denotes the set of edges of G that are incident to v. Consider the set $\hat{S} = \{v, v_1, v_2, \cdots, v_{r-1}\}$, where $v \in E_{v_1}$. Clearly,

$$|\delta \hat{S}| \leq (d-1)r \implies h(G) \leq (d-1),$$

which contradicts our assumption. $\square$

Example 2.1.9. Let $G = K_n$, and let $S \subset V = V(K_n)$. Then

$$\frac{|\delta S|}{|S|} = \frac{(|V| - |S|)|S|}{|S|} = |V| - |S| = n - |S|.$$

So, we have

$$h(K_n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even, and} \\ \frac{(n+1)}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

This tells us that $h(K_n)$ grows as n grows. While, K_n has the best edge-connectivity among graphs of order n, it has the least cost-effectiveness.

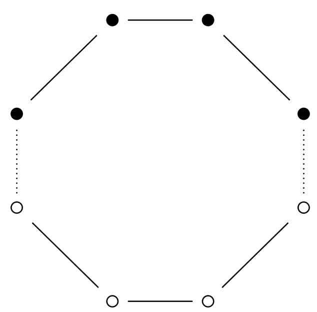
This leads us to the definition of exander family. Roughly speaking, expanders are highly connected, yet sparse graphs.

Definition 2.1.10. A family $(G_n)_{n \in \mathbb{N}}$ of non-empty connected d-regular graphs is an *expander family*, if:

(i) $|V(G_n)| \to \infty$ as $n \to \infty$, and

(ii) $\liminf_{n \to \infty} \{h(G_n)\} > 0$, that is, the sequence $\{h(G_n)\}$ is uniformly bounded away from 0.

Example 2.1.11. Consider $(C_n)_{n \in \mathbb{N}}$, an infinite family of 2-regular graphs. Take S to be the set of bottom half (white) vertices as shown in the figure below. Then



$$\frac{|\delta F|}{|F|} = \frac{2}{n/2} = \frac{4}{n}, \text{ and so we have}$$

$$h(C_n) \le \frac{4}{n} \implies h(C_n) \to 0 \text{ as } n \to \infty.$$

Thus, $(C_n)_{n\in\mathbb{N}}$ is not an expander family.

2.2 Adjacency Operator

Definition 2.2.1. Let S be a finite set. We define the complex vector space $L^2(S)$ by

$$L^2(S) = \{f : S \to \mathbb{C}\}.$$

Given $f, g \in L^2(S)$, $x \in S$, and $\alpha \in \mathbb{C}$, the sum in $L^2(S)$ is given by

$$(f+g)(x) = f(x) + g(x), \text{ and}$$

scalar multiplication is given by

$$(\alpha f)(x) = \alpha f(x).$$

The standard inner product and norm in $L^2(S)$ are given by

$$\langle f, g\rangle_2 = \sum_{x\in S} f(x)\overline{g(x)} \text{ and } \|f\|_2 = \sqrt{\langle f, f\rangle_2}, \text{ respectively.}$$

Remark 2.2.2. If $S = \{x_1, x_2, \cdots, x_n\}$. Then $L^2(S)$ is a complex n-dimensional vector space with standard basis $B = \{\delta_{x_1}, \delta_{x_2}, \cdots, \delta_{x_n}\}$ of Kronecker delta functions.

Let G be a finite graph of order n. Let $v_1, v_2, \ldots, v_n$ be an ordering of vertices in V, and let $A(G)$ be the adjacency matrix associated with respect to this ordering. Given an $f \in L^2(V)$, we may think of f as a vector $(f(v_1), f(v_2), \cdots, f(v_n))^t \in \mathbb{C}^n$.

$$Af = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n} \end{bmatrix} \begin{pmatrix} f(v_1) \\ f(v_2) \\ . \\ . \\ . \\ f(v_n) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{j=n} A_{1,j} f(v_j) \\ \sum_{j=1}^{j=n} A_{2,j} f(v_j) \\ . \\ . \\ . \\ \sum_{j=1}^{j=n} A_{n,j} f(v_j) \end{pmatrix}$$

Therefore, we may think of A as a linear transformation from $L^2(V)$ to $L^2(V)$ given by the formula

$$(Af)(v) = \sum_{w\in V} A_{v,w} f(w). \tag{2.2.1}$$

Definition 2.2.3. The linear operator $A : L^2(V) \to L^2(V)$ defined by Equation 2.2.1 is called the *adjacency operator* of G.

Remark 2.2.4. Considering A as an adjacency operator, the adjacency matrix is the matrix associated to the adjacency operator, with respect to the standard basis of $L^2(V)$.

Example 2.2.5. Consider the cycle graph C_4. Fix a cyclical ordering of the vertices as v_1, v_2, v_3, v_4. Define the function $f : V(C_4) \to \mathbb{C}$ by

$$f(v) = \begin{cases} -1, & \text{if } v = v_1 \text{ or } v = v_3, \text{ and} \\ 1, & \text{if } v = v_2 \text{ or } v = v_4. \end{cases}$$

We may think of the f as vector $(f(v_1), f(v_2), f(v_3), f(v_4))^t = (-1, 1, -1, 1)^t$. If A is the adjacency matrix associated with this ordering, then

$$Af = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \\ -2 \end{pmatrix} = -2f.$$

Thus, f is an eigenfunction of A associated with the eigenvalue -2.

2.3 The Laplacian

In this section, we will discuss another linear operator associated to a graph, called the Laplacian. The Laplacian on a graph is the discrete analogue of the Laplacian $\Delta = \text{div}(\text{grad}(f))$ from multivariable calculus.

Let $G = (V, E)$ be a graph with oriented edges. That is, for each edge $e \in E$, label one endpoint as e^+ and other as e^-. We call e^- the *origin*, and e^+ the *extremity* of the edge e.

$$\bullet_{e^-} \xrightarrow{e} \bullet_{e^+}$$

Definition 2.3.1. For a function $f \in L^2(V, \mathbb{R})$, the analog of the gradient operator d in the graph theory measures the change of f along the edge of the graph. Formally, $d : L^2(V) \to L^2(E)$ defined as $df(e) = f(e^+) - f(e^-)$.

Definition 2.3.2. We define $d^* : L^2(E) \to L^2(V)$, the finite analog of the divergence operator, by

$$d^*f(v) := \sum_{e \in E; v = e^+} f(e) - \sum_{e \in E; v = e^-} f(e).$$

That is, if we think of the function f as a flow on the edges of the graph G, then $(d^*f)(v)$ measures the total inward flow at the vertex v.
From the above definitions, it appears that both d and d^* depend on the orientation given to the edges. However, we will see in the following Lemma that the Laplacian is independent of the orientation of the edges.

Lemma 2.3.3. *Let $G = (V, E)$ be a k-regular graph and let A be the associated adjacency operator. Then $\triangle = kI - A$.*

Proof. Let $f \in L^2(V, \mathbb{R})$, and $v \in V$. Then

$$\begin{aligned}
(\triangle f)(v) &= (d^*(df))(v) \\
&= \sum_{e \in E; v = e^+} (df)(e) - \sum_{e \in E; v = e^-} (df)(e) \\
&= \left(\sum_{e \in E; v = e^+} f(v) - \sum_{e \in E; v = e^+; u = e^-} f(u) \right) - \left(\sum_{e \in E; v = e^-; u = e^+} f(u) - \sum_{e \in E; v = e^-} f(v) \right) \\
&= kf(x) - \sum_{u \in V} A_{v,u} f(u) \\
&= ((kI - A)f)(v).
\end{aligned}$$

□

Remark 2.3.4. As A is a linear operator, by Lemma 2.3.3, we get that Laplacian is also a linear operator.

Theorem 2.3.5. *Let $G = (V, E)$ be a k-regular graph of order n with an orientation on the edges in E. Then*

(i) If $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$ is the spectrum of the associated adjacency operator A, then the eigenvalues of the associated Laplacian operator is given by

$$0 = k - \lambda_0 \leq k - \lambda_1 \leq \cdots \leq k - \lambda_{n-1}.$$

(ii) Let $f \in L^2(V)$, and $g \in L^2(E)$. Then $\langle df, g \rangle_2 = \langle f, d^*g \rangle_2$, and

$$\langle \triangle f, f \rangle_2 = \sum_{e \in E} |f(e^+) - f(e^-)|^2.$$

Proof. If f is an eigenfunction of A corresponding to eigenvalue λ, then

$$\triangle f = (kI - A)f = kIf - Af = (k - \lambda)f,$$

which proves (i). For showing (ii), we note that

$$\begin{aligned}
\langle df, g \rangle_2 &= \sum_{e \in E} (df)(e)\overline{g(e)} = \sum_{e \in E} \left(f(e^+) - f(e^-) \right) \overline{g(e)} \\
&= \sum_{e \in E} f(e^+)\overline{g(e)} - \sum_{e \in E} f(e^-)\overline{g(e)} \\
&= \sum_{v \in V} f(v) \sum_{e \in E; v = e^+} \overline{g(e)} - \sum_{v \in V} f(v) \sum_{e \in E; v = e^-} \overline{g(e)} \\
&= \sum_{v \in V} f(v)\overline{(d^*g)(v)} \\
&= \langle f, d^*g \rangle_2.
\end{aligned}$$

Thus,

$$\langle \triangle f, f\rangle_2 = \langle d^* df, f\rangle_2 = \overline{\langle f, d^* df\rangle_2} = \langle df, df\rangle_2,$$

and

$$\langle df, df\rangle_2 = \sum_{e\in E} \big(f(e^+) - f(e^-)\big)\, \overline{(f(e^+) - f(e^-))} = \sum_{e\in E} |f(e^+) - f(e^-)|^2.$$

□

2.4 The Rayleigh-Ritz Theorem

In this section, we will discuss the relationship between $h(G)$ and $\lambda_1(G)$ of a graph G. The main result of this section states that a sequence (G_n) of d-regular graphs forms an expander family if and only if the sequence $(d - \lambda_1(G_n))$ is uniformly bounded away from zero.

Definition 2.4.1. Let X be a finite set and f_0 be the function that takes the constant value 1 on X. Define

$$L^2(X, \mathbb{R}) = \{f : X \to \mathbb{R}\}, \text{ and}$$

$$L_0^2(X, \mathbb{R}) = \{f \in L^2(X, \mathbb{R}) \mid \langle f, f_0\rangle_2 = 0\} = \{f \in L^2(X, \mathbb{R}) \mid \sum_{x\in X} f(x) = 0\}.$$

Theorem 2.4.2. *(Rayleigh-Ritz). Let G be a d-regular graph with vertex set V and adjacency matrix A. Then*

$$\lambda_1 = \max\left\{ \frac{\langle Af, f\rangle_2}{\|f\|_2^2} \;:\; f \in {L_0}^2(V, \mathbb{R}) \right\}.$$

Equivalently,

$$d - \lambda_1 = min\left\{ \frac{\langle \triangle f, f\rangle_2}{\|f\|_2^2} \;:\; f \in {L_0}^2(V, \mathbb{R}) \right\}.$$

Proof. Let $|V| = n$, and let $f_0 = \frac{f_0}{\|f_0\|_2}$. Since A is a $n \times n$ real symmetric matrix, we know that there exists an orthonormal basis $\{f_0', f_1, \cdots, f_{n-1}\}$ for $L^2(X, \mathbb{R})$, where each f_i is a real-valued eigenfunction of A associated with the eigenvalue λ_i. Let $f \in L_0^2(X, \mathbb{R})$ with $\|f\|_2 = 1$. Then $f = c_0 f_0' + c_1 f_1 + \cdots + c_{n-1} f_{n-1}$ for some scalars $c_i \in \mathbb{R}$. Note that

$$\langle f, f_0\rangle_2 = 0 = c_0\langle f_0', f_0\rangle_2 + \cdots + \langle f_{n-1}, f_0\rangle_2 = c_0.$$

This implies $c_0 = 0$ and $f = c_1 f_1 + \cdots + c_{n-1} f_{n-1}$. Now,

$$\begin{aligned}
\langle Af, f\rangle_2 &= \langle A \sum_{i=1}^{n-1} c_i f_i, \sum_{i=1}^{n-1} c_i f_i\rangle_2 \\
&= \langle \sum_{i=1}^{n-1} c_i \lambda_i f_i, \sum_{i=1}^{n-1} c_i f_i\rangle_2 \\
&= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} c_i c_j \lambda_i \langle f_i, f_j\rangle_2 = \sum_{i=1}^{n-1} c_i^2 \lambda_i \\
&\leq \lambda_1 \sum_{i=1}^{n-1} c_i^2 = \lambda_1 \|f\|^2 = \lambda_1.
\end{aligned}$$

Therefore,

$$\lambda_1 \geq \max\left\{ \frac{\langle Af, f\rangle_2}{\|f\|_2^2} \ : \ f \in {L_0}^2(V, \mathbb{R}) \right\}.$$

But, for f_1 we have

$$\langle Af_1, f_1\rangle_2 = \langle \lambda_1 f_1, f_1\rangle_2 = \lambda_1.$$

Thus,

$$\lambda_1 = \max\left\{ \frac{\langle Af, f\rangle_2}{\|f\|_2^2} \ : \ f \in {L_0}^2(V, \mathbb{R}) \right\}.$$

The equivalent statement follows from Lemma 2.3.3. □

Proposition 2.4.3. *Let $G = (V, E)$ be a d-regular graph. Then*

$$\frac{d - \lambda_1}{2} \leq h(G).$$

Proof. Let us fix an orientation of the edges. Let $F \subset V$ such that $h(G) = \frac{|\delta F|}{|F|}$. Let $a = |V \backslash F|$, and let $b = |F|$. Define $g : V \to \mathbb{R}$ as

$$g(x) = \begin{cases} a, & \text{if } x \in F, \text{ and} \\ -b, & \text{if } x \notin F. \end{cases}$$

Since,

$$\sum_{v \in V} g(v) = \sum_{v \in F} a - \sum_{v \notin F} (-b) = ab - ba = 0,$$

we have $g \in L_0^2(V, \mathbb{R})$. We know that

$$\langle \triangle g, g\rangle_2 = \sum_{e \in E} |g(e^+) - g(e^-)|^2 = \sum_{e \in \delta F} (b + a)^2 = |\delta F|(b + a)^2.$$

Also,

$$\langle g, g\rangle_2 = \sum_{v \in F} a^2 + \sum_{v \notin F} (-b)^2 = a^2 b + b^2 a = ab(a + b).$$

Since, $b = |F| \leq \frac{1}{2}|V|$, we have $b \leq a$. Thus,

$$\frac{\langle \triangle g, g \rangle_2}{\langle g, g \rangle_2} = \frac{|\delta F|(b+a)}{ba} = \left(1 + \frac{b}{a}\right) h(G) \leq 2h(G).$$

Finally, by the Rayleigh-Ritz Theorem we have

$$d - \lambda_1(G) \leq \frac{\langle \triangle g, g \rangle_2}{\langle g, g \rangle_2} \leq 2h(G),$$

from which the assertion follows.

□

Definition 2.4.4. If G is a connected d-regular graph, then $d - \lambda_1(G)$ is called the *spectral gap* of G.

Corollary 2.4.5. *Let (G_n) be a sequence of d-regular graphs with $|G_n| \to \infty$ as $n \to \infty$. Then (G_n) is a family of expanders if and only if the sequence $(d - \lambda_1(G_n))$ is uniformly bounded away from zero.*

Example 2.4.6. Consider the sequence of cycle graphs (C_n). We know that $\lambda_1(C_n) = 2\cos\frac{2\pi}{n}$. Note that the spectral gap $2 - 2\cos\frac{2\pi}{n} \to 0$ as $n \to \infty$. This implies that $h(C_n) \to 0$ as $n \to \infty$, which reaffirms the fact that (C_n) is not expander.

As calculating isoperimetric constant for graphs directly is quite difficult, this relationship between the second largest eigenvalue and the isoperimetric constant provides more traction. So, from now on, in our study for expander families, we will focus almost exclusively on graph spectra with particular attention on the second largest eigenvalue. In the following chapters, we will see many techniques to find out the spectrum of a graph. For example, in Chapter 6, we will use representations of finite groups to explicitly calculate the spectrum of Cayley graphs on those groups.

Chapter 3

The Alon-Boppana Theorem

Let G be a d-regular graph. In the previous chapter, we saw that $h(G)$ is large when $d - \lambda_1(G)$ is large. So, to construct a fast and reliable communication network, we need to find graphs with small $\lambda_1(G)$ relative to d. A natural question that arises in this direction is how small can $\lambda_1(G)$ be? We answer this question in this chapter, which is based on Chapter 3 of [8].

The following proposition gives a lower bound for $\lambda_1(G)$ in terms of d and $\operatorname{diam}(G)$.

Proposition 3.0.1. *Let $G = (V, E)$ be a connected d-regular graph. If* $\operatorname{diam}(G) \geq 4$*, then*

$$\lambda_1(G) > 2\sqrt{d-1} - \frac{2\sqrt{d-1} - 1}{\lfloor \frac{1}{2}\operatorname{diam}(G) - 1 \rfloor}.$$

Proof. Let $b = \lfloor \frac{1}{2}\operatorname{diam}(G) - 1 \rfloor$, and let $q = d - 1$. Let $v_1, v_2 \in V$ such that

$$\operatorname{dist}(v_1, v_2) \geq 2b + 2 = \begin{cases} \operatorname{diam}(G), & \text{if } \operatorname{diam}(G) \text{ is even, and} \\ \operatorname{diam}(G) - 1, & \text{otherwise.} \end{cases}$$

Define

$$\begin{aligned} A_i &= \{v \in V \mid \operatorname{dist}(v, v_1) = i\}, \\ B_i &= \{v \in V \mid \operatorname{dist}(v, v_2) = i\}, 0 \leq i \leq b. \end{aligned}$$

Let $x \in A_i \cap B_j$, for $0 \leq i, j, \leq b$. Then we have

$$\begin{aligned} \operatorname{dist}(v_1, v_2) &\leq \operatorname{dist}(v_1, x) + \operatorname{dist}(x, v_2) \\ \implies \operatorname{dist}(v_1, v_2) &\leq 2b < 2b + 2, \end{aligned}$$

which is not possible. Hence, we have that $A_i \cap B_j = \phi$.

Let $A = \bigcup_{i=0}^{b} A_i$, and $B = \bigcup_{j=0}^{b} B_j$. Suppose $x \in A$ and $y \in B$ are adjacent. Then

$$\begin{aligned} \operatorname{dist}(v_1, v_2) &\leq \operatorname{dist}(v_1, x) + \operatorname{dist}(y, v_2) + \operatorname{dist}(x, y) \\ &\leq 2b + 1 < 2b + 2, \end{aligned}$$

which is impossible. This implies no vertex in A is adjacent to any vertex in B.

Now, let $\alpha \in \mathbb{R}$. We define $f \in L_0^2(V, \mathbb{R})$ as

$$f(x) = \begin{cases} \alpha, & \text{if } x \in A_0, \\ \alpha q^{-\frac{(i-1)}{2}}, & \text{if } x \in A_i, i \geq 1, \\ 1, & \text{if } x \in B_0, \\ q^{-\frac{(i-1)}{2}}, & \text{if } x \in B_i, i \geq 1, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Since $f_0(x) = 1$ for all $x \in V$, we have

$$\begin{aligned} \langle f, f_0 \rangle_2 &= \sum_{x \in V} f(x) \overline{f_0(x)} \\ &= \sum_{x \in V} f(x) \\ &= \alpha \left(1 + \sum_{i=1}^{b} |A_i| q^{-\frac{(i-1)}{2}} \right) + 1 + \sum_{i=1}^{b} |B_i| q^{-\frac{(i-1)}{2}} \\ &= \alpha C_0 + C_1, \end{aligned}$$

where $C_0 = \left(1 + \sum_{i=1}^{b} |A_i| q^{-\frac{(i-1)}{2}}\right)$ and $C_1 = 1 + \sum_{i=1}^{b} |B_i| q^{-\frac{(i-1)}{2}}$. Let $\alpha = \frac{-C_0}{C_1}$. Then $\langle f, f_0 \rangle_2 = 0$. Moreover,

$$\begin{aligned} \langle f, f \rangle_2 &= \sum_{x \in V} f(x) \overline{f(x)} \\ &= \sum_{i=0}^{b} \sum_{x \in A_i} f(x) \overline{f(x)} + \sum_{i=0}^{b} \sum_{x \in B_i} f(x) \overline{f(x)} \\ &= \alpha^2 \left(1 + \sum_{i=1}^{b} |A_i| q^{-(i-1)} \right) + 1 + \sum_{i=1}^{b} |B_i| q^{-(i-1)} \\ &= S_A + S_B, \end{aligned}$$

where $S_A = \alpha^2 \left(1 + \sum_{i=1}^{b} |A_i| q^{-(i-1)}\right)$ and $S_B = 1 + \sum_{i=1}^{b} |B_i| q^{-(i-1)}$.

We now orient the edges of G such that for each edge $e \in E$, one endpoint is labelled as e^+ and the other is labelled as e^-. Since the Laplacian of G is independent of orientation, we have

$$\langle \triangle f, f \rangle_2 = C_A + C_B,$$

where

$$C_A = \sum_{e \in E;\, e^+ \text{ or } e^- \in A} \left(f(e^+) - f(e^-) \right)^2$$

and

$$C_B = \sum_{e \in E;\, e^+ \text{ or } e^- \in B} \left(f(e^+) - f(e^-)\right)^2.$$

Since, when both the end points are outside of A and B, we have $f(e^+) = f(e^-) = 0 \implies (f(e^+) - f(e^-))^2 = 0$. Since no vertex in A is adjacent to any vertex in B, we have

$$C_A = \sum_{i=0}^{b-1} \sum_{x \in A_i} \sum_{y \in A_{i+1}} A_{x,y} \left(f(x) - f(y)\right)^2 + \sum_{x \in A_b} \sum_{y \notin A} A_{x,y} \left(f(x) - 0\right)^2.$$

For each $x \in A_i$, there are at most q elements $y \in A_{i+1}$ that are adjacent to x. Therefore,

$$C_A \leq \sum_{i=1}^{b-1} q|A_i| \left(q^{-\frac{(i-1)}{2}} - q^{\frac{-i}{2}}\right)^2 \alpha^2 + q|A_b|q^{-(b-1)}\alpha^2.$$

Note that $\left(q^{-\frac{(i-1)}{2}} - q^{\frac{-i}{2}}\right)^2 = (\sqrt{q}-1)^2 q^{-i}$ and $q = (\sqrt{q}-1)^2 + 2\sqrt{q} - 1$. So, we have

$$\begin{aligned} C_A &\leq \sum_{i=1}^{b-1} q|A_i|(\sqrt{q}-1)^2 q^{-i}\alpha^2 + \left((\sqrt{q}-1)^2 + 2\sqrt{q} - 1\right)|A_b|q^{-(b-1)}\alpha^2 \\ &= (\sqrt{q}-1)^2\alpha^2 \left(\sum_{i=1}^{b} |A_i|q^{-(i-1)}\right) + \alpha^2(2\sqrt{q}-1)|A_b|q^{-(b-1)}. \end{aligned}$$

As $S_A - \alpha^2 = \alpha^2 \left(\sum_{i=1}^{b} |A_i|q^{-(i-1)}\right)$, we have

$$C_A \leq (\sqrt{q}-1)^2(S_A - \alpha^2) + \alpha^2 \frac{(2\sqrt{q}-1)}{b} b|A_b|q^{-(b-1)}.$$

If $x \in A_i$ where $1 \leq i \leq b-1$, then there is at least one vertex from A_{i-1} that is adjacent to x, and at most q vertices from A_{i+1} that are adjacent to x. Hence, $|Ai+1| \leq q|A_i|$ for $1 \leq i \leq b-1$. Similarly, $|Bi+1| \leq q|B_i|$, for $1 \leq i \leq b-1$. So

$$|A_1| \geq q^{-1}|A_2| \geq q^{-2}|A_3| \geq \cdots \geq q^{-(b-1)}|A_b|.$$

In particular,

$$\alpha^2 b|A_b|q^{-(b-1)} = \alpha^2 \sum_{i=1}^{b} |A_b|q^{-(b-1)} \leq \alpha^2 \sum_{i=1}^{b} |A_i|q^{-(i-1)} = S_A - \alpha^2.$$

Since G is connected and $\mathrm{diam}(G) \geq 4$, we have that $d \geq 2$ and $(2\sqrt{q}-1)/b > 0$. Also, $0 < (\sqrt{q}-1)^2 = q + 1 - 2\sqrt{q}$. Thus, we have

$$\begin{aligned} C_A &\leq \left((\sqrt{q}-1)^2 + \frac{(2\sqrt{q}-1)}{b}\right)(S_A - \alpha^2) \\ &= \left((q+1+2\sqrt{q} + \frac{(2\sqrt{q}-1)}{b}\right)(S_A - \alpha^2) \\ &< \left((q+1+2\sqrt{q} + \frac{(2\sqrt{q}-1)}{b}\right)S_A. \end{aligned}$$

Similarly,

$$C_B < \left((q + 1 + 2\sqrt{q} + \frac{(2\sqrt{q} - 1)}{b}\right) S_B.$$

Therefore,

$$\langle \triangle f, f \rangle_2 = C_A + C_B < \left((q + 1 + 2\sqrt{q} + \frac{(2\sqrt{q} - 1)}{b}\right) (S_A + S_B).$$

Applying the Rayleigh-Ritz theorem, we have

$$\begin{aligned} d - \lambda_1(G) &= \min_{g \in L_0^2(V, \mathbb{R})} \langle \triangle g, g \rangle_2 \\ &\leq \frac{\langle \triangle f, f \rangle_2}{\langle f, f \rangle_2} \\ &= \frac{C_A + C_B}{S_A + S_B} \\ &< q + 1 + 2\sqrt{q} + \frac{(2\sqrt{q} - 1)}{b} \\ &= d - 2\sqrt{d-1} + \frac{2\sqrt{d-1} - 1}{b}, \end{aligned}$$

from which the assertion follows. □

Remark 3.0.2. From Proposition 3.0.1, we get an upper bound on the spectral gap of a d-regular graph

$$\frac{d - \lambda_1(G)}{2} < \frac{d}{2} - \sqrt{d-1} + \frac{2\sqrt{d-1} - 1}{2\lfloor \frac{1}{2}\mathrm{diam}(G) - 1 \rfloor}.$$

The following proposition provides the key ingredient to prove the main result in this chapter.

Proposition 3.0.3. *If (G_n) is a sequence of connected d-regular graphs with $|G_n| \to \infty$ as $n \to \infty$, then*

$$\liminf_{n \to \infty} \lambda_1(G_n) \geq 2\sqrt{d-1}.$$

That is, for all $\epsilon > 0$, there exists an $N > 0$ such that $\lambda_1(G_n) > 2\sqrt{d-1} - \epsilon$ for all $n > N$.

Proof. Let G be a finite d-regular graph, and let $v \in V(G)$. Note that the number of walks of length 1 starting from v in G is d. Similarly, the number of non-backtracking (trek) walks of length 2 starting from v is d^2. In general, the number of non-backtracking walks of length k starting from v is d^k, where each walk contains at most $k+1$ vertices.

Note that, we can cover the entire graph by taking all walks of length $\mathrm{diam}(G)$ from the fixed vertex v. There are $d^{\mathrm{diam}(G)}$ such walks, each containing at most $\mathrm{diam}(G) + 1$ distinct vertices. Hence,

$$|G| < (\mathrm{diam}(G) + 1) d^{\mathrm{diam}(G)}. \tag{3.0.1}$$

Now, let (G_n) be a sequence of connected, d-regular graphs such that $|G_n| \to \infty$ as $n \to \infty$. Then, by Equation 3.0.1, we get $\text{diam}(G_n) \to \infty$ as $n \to \infty$. Also, by Proposition 3.0.1, we have

$$\lambda_1(G_n) > 2\sqrt{d-1} - \frac{2\sqrt{d-1}-1}{\lfloor \frac{1}{2}\text{diam}(G_n) - 1 \rfloor}.$$

Since $|\lambda_1(G_n)| < d$, and $2\sqrt{d-1} - \frac{2\sqrt{d-1}-1}{\lfloor \frac{1}{2}\text{diam}(G_n)-1 \rfloor}$ are bounded sequences, we can take lim inf on both sides. Therefore, we have

$$\begin{aligned}
\liminf_{n\to\infty} \lambda_1(G_n) &\geq \liminf_{n\to\infty} \left(2\sqrt{d-1} - \frac{2\sqrt{d-1}-1}{\lfloor \frac{1}{2}\text{diam}(G_n) - 1 \rfloor} \right) \\
&= 2\sqrt{d-1} + \liminf_{n\to\infty} \left(-\frac{2\sqrt{d-1}-1}{\lfloor \frac{1}{2}\text{diam}(G_n) - 1 \rfloor} \right) \\
&= 2\sqrt{d-1} + \lim_{n\to\infty} \left(-\frac{2\sqrt{d-1}-1}{\lfloor \frac{1}{2}\text{diam}(G_n) - 1 \rfloor} \right) \\
&= 2\sqrt{d-1} - \lim_{n\to\infty} \left(\frac{2\sqrt{d-1}-1}{\lfloor \frac{1}{2}\text{diam}(G_n) - 1 \rfloor} \right) \\
&= 2\sqrt{d-1}.
\end{aligned}$$

□

The proposition tells us that the spectral gap of a d-regular graph is bounded above by $d - \sqrt{d-1}$.

Recall, for a d-regular graph, d is called its trivial eigenvalue, and for a d-regular bipartite graph, both d and $-d$ are called its trivial eigenvalues.

Definition 3.0.4. Let G be a d-regular graph. We define

$$\lambda(G) = \max\{|\lambda| \mid \lambda \text{ is a non trivial eigenvalue of G}\}.$$

By definition, $\lambda(G) \geq \lambda_1(G)$, and so the main theorem of this chapter will now follow from Proposition 3.0.3.

Theorem 3.0.5. *(Alon-Boppana) Let (G_n) be a sequence of connected d-regular graphs with $|G_n| \to \infty$ as $n \to \infty$, then*

$$\liminf_{n\to\infty} \lambda(G_n) \geq 2\sqrt{d-1}.$$

Since $d - \lambda(G) \leq d - \lambda_1(G)$, the Alon-Boppana theorem asserts that the strongest upper bound for $\lambda(G)$ is $2\sqrt{d-1}$. So, a natural question that arises in this context is whether there exist graphs with $\lambda \leq 2\sqrt{d-1}$? This motivates the following definition.

Definition 3.0.6. We say that a d-regular graph G is *Ramanujan* if $\lambda(G) \leq 2\sqrt{d-1}$.

Remark 3.0.7. Note that a d-regular Ramanujan graph must be connected, for otherwise $\lambda(G) = d$.

Example 3.0.8. Consider the Petersen graph P. It can be shown that

$$\mathrm{Spec}(P) = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 5 & 4 \end{pmatrix}.$$

So, $\lambda = 2 < 2\sqrt{d-1} = 2\sqrt{3-1} = 2.828$. Thus, the Petersen graph is Ramanujan.

Example 3.0.9. The cycle graph C_n is a Ramanujan graph for $n \geq 3$. It is known that

$$\lambda_1(C_n) = 2\cos\frac{2\pi}{n} \leq 2\sqrt{2-1} = 2.$$

Example 3.0.10. The complete graph K_n is a Ramanujan graph for $n \geq 3$. Its spectrum is given by

$$\mathrm{Spec}(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

Since, K_n is $n-1$ regular, $(n-1)$ is the trivial eigenvalue of K_n. Therefore, $\lambda(K_n) = 1 < 2\sqrt{n-2}$ for $n \geq 3$.

Remark 3.0.11. Since for all $d \geq 3$, $d \geq 2\sqrt{d-1}$, we have

$$h(G) \geq \frac{d-\lambda_1(G)}{2} \geq \frac{d-\lambda(G)}{2} \geq \frac{d-2\sqrt{d-1}}{2} > 0.$$

This tells us that for $d \geq 3$, any sequence of d-regular Ramanujan graphs is an expander family. But, for $d < 3$ this is not true, since (C_n) forms a family of 2-regular cycle that is not an expander family.

The proof of the following proposition is analogous to the Rayleigh-Ritz theorem.

Proposition 3.0.12. *Let $G = (V, E)$ be a non-bipartite, d-regular graph. Let A be the adjacency operator for G. Then*

$$\lambda(G) = \max\left\{ \frac{|\langle Af, f\rangle|_2}{\|f\|_2^2} \ : \ f \in {L_0}^2(V, \mathbb{R}) \right\}.$$

Remark 3.0.13. If the graph G is d-regular and bipartite, then

$$\lambda(G) \leq \max\left\{ \frac{|\langle Af, f\rangle|_2}{\|f\|_2^2} \ : \ f \in {L_0}^2(V, \mathbb{R}) \right\} = d,$$

since $\lambda_{n-1} = -d$. The equality holds if and only if G is disconnected.

Chapter 4

Diameter of Cayley Graphs and Expander Families

In this chapter, we discuss another graph invariant, diameter, and its connection with the isoperimetric constant for regular graphs. In Section 4.1, we prove that an expander family has logarithmic diameter. As of now, there are no necessary and sufficient conditions known that a sequence of finite groups must satisfy in order to yield an expander family. So, answering this question completely is not easy. However, we can answer it partially by identifying some of those families of finite groups that never yield expander families. One such family is the family of abelian groups. In Section 4.3, we show that a sequence Cayley graphs on abelian groups can never be an expander family. This chapter is based on Chapter 4 of [8].

4.1 Diameter of an Expander Family

The natural path metric on a graph yields the following discrete analogs of the notion of a closed ball and a sphere in the graph-theoretic setting.

Definition 4.1.1. Let $G = (V, E)$ be a graph and let $w \in V$. Let r be a non-negative integer. Define *closed ball of radius r centred at w*, denoted as $B_r[w]$, by

$$B_r[w] = \{v \in V \mid d(v, w) \leq r\},$$

and *sphere of radius r centred at w*, denoted as $S_r[w]$, by

$$S_r[w] = \{v \in V \mid d(v, w) = r\}.$$

Remark 4.1.2. We note that if the $|B_r[w]|$ grows quickly as a function of r, then it means that we can reach more vertices in fewer steps from w, and since the choice of w was arbitrary, this implies that the graph has small diameter. On the other hand, if $|B_r[w]|$ grows slowly as a function of r, then the graph has large diameter.

Remark 4.1.3. We claim that for a finite graph G, and a vertex w of G, if $|B_r[w]| \leq r^2$, then $\operatorname{diam}(G) \geq |G|^{1/2}$. As this holds trivially when G is disconnected, we consider the case when G is connected. Denoting diam(G) by k, we have $|G| = |B_k[w]|$, and

$$|B_k[w]| \leq k^2 \Rightarrow |G|^{1/2} \leq k.$$

Lemma 4.1.4. *Let $G = (V, E)$ be a finite d-regular graph with $d \geq 3$ and* $\operatorname{diam}(X) \geq 3$. *Then* $\operatorname{diam}(X) \geq \log_d |G|$.

Proof. We know that $|S_0[v]| = 1$ and $|S_1[v]| \leq d$. Moreover, for $j \geq 2$, and a vertex $w \in S_j[v]$, at least one edge incident to w is also incident to a vertex in $S_{j-1}[v]$. Consequently, at most $(d-1)$ edges that are incident to w are incident to vertices in $S_{j+1}[v]$. So, we have

$$|S_{j+1}[v]| = (d-1)|S_j[v]|.$$

Similarly, $|S_j[v]| \leq (d-1)|S_{j-1}[v]|$ and so on. Thus, we have $|S_j[v]| \leq d(d-1)^{j-1}$, and so we have

$$\begin{aligned} B_r[v] &= S_0[v] + \cdots + S_r[v] \\ &\leq 1 + \sum_{j=0}^{r-1} d(d-1)^j \\ &\leq 1 + d\frac{(d-1)^r - 1}{d-2} \\ &= \frac{d(d-1)^r - 2}{d-2}. \end{aligned} \tag{4.1.1}$$

It now suffices to show that $|B_r[v]| \leq d^r$, as the assertion would them follow from Remark 4.1.3. Since $(d-1)^3 = d^2(d-2) - (d^2 - 3d + 1)$ and $d^2 - 3d + 1 \geq 0$, for all $d \geq 3$, we have

$$\begin{aligned} (d-1)^r &\leq d^2(d-2)(d-1)^{r-3} \\ &\leq d^2(d-2)(d)^{r-3} \\ &= d^{r-1}(d-2). \end{aligned}$$

$$\text{So, } d(d-1)^r - 2 \leq d(d-1)^r \leq d^r(d-2). \tag{4.1.2}$$

Therefore, from Equation 4.1.1 and 4.1.2 , we get $|B_r[v]| \leq d^r$.

□

Lemma 4.1.5. *Let G be a connected graph and let $a > 1$. Suppose that for any vertex $v \in V(G)$, we have that $|B_r[v]| \geq a^r$ whenever $|B_{r-1}[v]| \leq \frac{1}{2}|G|$. Then*

$$\operatorname{diam}(G) \leq \left(\frac{2}{\log a}\right) \log |G|.$$

Proof. Let $u, v \in V(G)$. Let r_1 be the smallest integer such that $|B_{r_1}[u]| > \frac{1}{2}|G|$, and let r_2 be the smallest integer such that $|B_{r_2}[v]| > \frac{1}{2}|G|$. Since, $r_1 - 1 < r_1$ and $r_2 - 1 < r_2$, we have $|B_{r_1}[u]| \geq a^{r_1}$ and $|B_{r_2}[v]| \geq a^{r_2}$. Also, we get $|B_{r_1}[u]| + |B_{r_2}[v]| > |G|$, which implies that the two closed balls $B_{r_1}[u], B_{r_2}[v]$ are not disjoint. That is, there exists a vertex $w \in B_{r_1}[u] \bigcap B_{r_2}[v]$. Since, $\text{dist}(u, w) \le r_1$ and $\text{dist}(v, w) \le r_2$, by the triangle inequality we have

$$\begin{aligned}\text{dist}(u, v) &\le r_1 + r_2 \\ &\le \frac{\log |B_{r_1}[u]|}{\log a} + \frac{\log |B_{r_2}[v]}{\log a} \\ &\le \frac{2}{\log a} \log |G|.\end{aligned}$$

Since this is independent of the choice of vertex, it follows that $\text{diam}(G) \le \left(\frac{2}{\log a}\right) \log |G|$. ◻

The following proposition establishes the existence and uniqueness of the constant a from Lemma 4.1.5.

Proposition 4.1.6. *Let G be a connected finite d-regular graph. Let $a = 1 + \frac{h(G)}{d}$. Then*

$$\text{diam}(G) \le \left(\frac{2}{\log a}\right) \log |G|. \tag{4.1.3}$$

Proof. Let $v \in V(G)$. Suppose that $|B_{r-1}[v]| \le \frac{1}{2}|G|$ for some non-negative integer r. Then by definition of $h(G)$, we have

$$h(G)|B_{r-1}[v]| \;\le\; |\delta B_{r-1}[v]|.$$

We know that any edge in $\delta B_{r-1}[v]|$ must be incident to a vertex in $S_r[v]$. Moreover, two edges in $\delta B_{r-1}[v]|$ can be incident at the same vertex in $S_r[v]$. So, we have that $|\delta B_{r-1}[v]| \ge |S_r[v]|$ As G is $d-$regular, we have

$$|S_r[v]| \ge \frac{|\delta B_{r-1}[v]}{d} \ge \frac{h(G)|B_{r-1}[v]|}{d}.$$

Note, $B_r[v]$ is disjoint union of $B_{r-1}[v]$ and $S_r[v]$. So,

$$|B_r[v]| = |B_{r-1}[v]| + S_r[v] \;\ge\; |B_{r-1}[v]| + \frac{h(G)|B_{r-1}[v]|}{d} = a|B_{r-1}[v]|.$$

Therefore, by induction, we have that, $|B_r[v]| \;\ge\; a^r$, whenever $|B_{r-1}[v]| \le \frac{1}{2}|G|$. The assertion now follows directly from Lemma 4.1.5. ◻

Proposition 4.1.6 tells us that if the isoperimetric constant of a sequence (G_n) of d-regular graphs is bounded away from zero, then the diameters of the graphs grow at most logarithmically as a function of $|G_n|$.

Definition 4.1.7. Let $(G_n)_{n\in\mathbb{N}}$ be a family of graphs. We say (G_n) has *logarithmic diameter* if $\text{diam}(G_n) = O(\log |G_n|)$, that is, there exists a $N \in \mathbb{N}$ and $C > 0$ such that $\text{diam}(G_n) \le C \log |G_n|$, for all $n > N$.

Corollary 4.1.8. *If (G_n) is a family of d-regular expanders, then (G_n) has logarithmic diameter.*

Proof. Since (G_n) is an expander family, there exists an $\epsilon > 0$ such that $h(G_n) \geq \epsilon$ for all n. Let $C_n = 1 + \frac{h(G_n)}{d}$ and let $C = 1 + \frac{\epsilon}{d}$. Since, $h(G_n) \geq \epsilon$, we have

$$\frac{2}{\log C_n} \leq \frac{2}{\log C}.$$

$$\text{Therefore, } \operatorname{diam}(G_n) \leq \frac{2}{\log C_n} \log |G_n| \leq \left(\frac{2}{\log C}\right) \log |G_n|,$$

from which our assertion follows.

□

Remark 4.1.9. Corollary 4.1.8 asserts that if a sequence of graphs does not have logarithmic diameter, then it cannot be an expander family. But, note that the converse of this is not true.

4.2 Diameter of Cayley Graphs

Definition 4.2.1. Let (G_n) be a sequence of finite groups. We say that (G_n) has *logarithmic diameter* if for some positive integer d, there exists a sequence (Γ_n), where for each n we have that Γ_n is symmetric subset of G_n with $|\Gamma_n| = d$, so that the family of Cayley graphs $(\operatorname{Cay}(G_n, \Gamma_n))$ has logarithmic diameter.

Definition 4.2.2. A *word of length n* in a set Γ is an element of the Cartesian product $\Gamma \times \Gamma \times \cdots \times \Gamma = \Gamma^n$.

Definition 4.2.3. Let G be a group , and let $\Gamma \subset G$.

(i) Let $w = (w_1, ..., w_n)$ is a word in Γ, then w *evaluates to g* (or, g can be expressed as w) if $g = w_1 w_2 \cdots w_n$.

(ii) Let $g \in G$ such that g can be expressed as a word in Γ. Then the *word norm* of g is the minimal length of any word in τ which evaluates to g.

Example 4.2.4. Let $G = \mathbb{Z}_{10}$ and let $\Gamma = \{-1, 1, 2, 5\}$. Then $w_1 = (2, 1)$ is a word of length 2, and $w_2 = (1-, 5, -1)$ is a word of length 3. Notice that both words w_1 and w_2 evaluate to 3, but the word norm of 3 in Γ is 2, as 2 is the minimum length of any word in Γ that evaluates to 3.

Remark 4.2.5. The standard convention is that the word of length 0 evaluates to the identity element of the group.

Proposition 4.2.6. *Let G be a finite group and Γ be a symmetric subset of G. Let $X = \operatorname{Cay}(G, \Gamma)$.*

(i) X is connected if and only if every element of G can be expressed as a word in Γ.

(ii) If $a, b \in G$ and there is a walk in X from a to b, then distance from a to b is the word norm of $a^{-1}b$ in Γ.

(iii) The diameter of X equals the maximum of the word norms in Γ of the elements of G.

Proof. (i) It follows from Proposition 1.3.5.

(ii) Let $(g_0, g_1, \cdots, g_n)$ be a walk of length n from a to b. That is, $g_0 = a$ and $g_n = b$. Since, g_{i-1}, g_i are adjacent, there exists a $\gamma_i \in \Gamma$ such that $g_{i-1}\gamma_i = g_i$. So, $(\gamma_1, \cdots, \gamma_n)$ is a word of length n that evaluates to $a^{-1}b$. Also, given a word in Γ of length n in Γ which evaluates to $a^{-1}b$, we have a walk of length n in G from a to b. Thus, by Definition 1.1.21, the assertion follows.

(iii) It follows from (ii) and Definition 1.1.23.

$\square$

Example 4.2.7. Consider the dihedral group $D_8 = \langle r, s \mid r^4 = s^2 = 1\rangle$. Let $\Gamma = \{s, r, r^3\}$. Since the s, r are the generators of the group, Γ generates the group. Hence, the Cayley graph $X = \mathrm{Cay}(D_8, \Gamma)$ is connected. As the word norm of any element of D_8 in $\Gamma \leq 3$, we get $\mathrm{diam}(X) = 3$.

4.3 Abelian groups never yield expander families

In this section, we prove that no sequence of Cayley graphs on abelian groups has logarithmic diameter. Then, by Corollary 4.1.8, we conclude that no sequence of abelian groups yields an expander family.

Lemma 4.3.1. *If $a, b \in \mathbb{N}$ with $b \leq a$, then*

$$\binom{a}{b} \leq (a - b + 1)^b.$$

Proof. First, observe that $0 \leq q \leq p$, then $\frac{p+1}{q+1} \leq \frac{p}{q}$. Since $p \geq q \geq 0$, we have $p - q \geq 0$

$$\begin{aligned}
&\Rightarrow p - q + pq - pq \geq 0 \\
&\Rightarrow p(1 + q) - q(1 + p) \geq 0 \\
&\Rightarrow p(1 + q) \geq q(1 + p) \\
&\Rightarrow \frac{p}{q} \geq \frac{(1 + p)}{(1 + q)}.
\end{aligned}$$

Therefore,

$$\frac{a}{b} \leq \frac{a-1}{b-1} \leq \cdots \leq \frac{a-b+1}{1}.$$

Thus,

$$\binom{a}{b} = \left(\frac{a}{b}\right)\left(\frac{a-1}{b-1}\right)\cdots\left(\frac{a-b+1}{1}\right) \leq (a - b + 1)^b.$$

$\square$

We state the following elementary lemma from combinatorics without proof.

Lemma 4.3.2. *The number of solutions to the equation $a_1 + \cdots + a_n = k$, where the a_i are non-negative integers, is $\binom{n+k-1}{k}$.*

Proposition 4.3.3. *No sequence of finite abelian groups has logarithmic diameter.*

Proof. Let G be a finite abelian group and let $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_d\}$ be a symmetric subset of G. Let $X = \mathrm{Cay}(G, \Gamma)$ with $\mathrm{diam}(X) = k$. If Γ does not generate G, then the graph X is disconnected, which implies $\mathrm{diam}(X) = \infty$.

Now, suppose that Γ generates G. Since X is connected, for any two vertices a, b of X, there is a path joining them. So, we have

$$a = b\gamma_1^{a_1} \cdots \gamma_d^{a_d}.$$

In particular, let $b = e$. Then

$$a = e^{a_0}\gamma_1^{a_1} \cdots \gamma_d^{a_d},$$

where e is the identity of G and $\Sigma_{i=0}^{i=d}\, a_i = k$, each a_i being a non-negative integer. By Lemma 4.3.2, the number of solutions of $\Sigma_{i=0}^{i=d}\, a_i = k$ is $\binom{k+d}{k}$. Therefore, by Lemma 4.3.1, we get

$$|X| \leq \binom{k+d}{k} = \binom{k+d}{d} \leq (k+1)^d,$$

$$\implies \mathrm{diam}(X) \geq |X|^{1/d} - 1.$$

So, for any sequence (X_n) of d-regular Cayley graphs on abelian groups, we have $\mathrm{diam}(X_n) \geq |X_n|^{1/d} - 1$. Since, root functions grow faster than logarithmic functions, we get $\mathrm{diam}(X_n) \neq O(\log |X_n|)$. □

Proposition 4.3.3 together with Corollary 4.1.8 yields the following.

Corollary 4.3.4. *No sequence of abelian groups yields an expander family.*

Example 4.3.5. Since, $\mathrm{Cay}(\mathbb{Z}_n, \Gamma = \{1, -1\}) = C_n$. Corollary 4.3.4 above yields yet another proof of the fact that (C_n) is not an expander family.

Chapter 5

Graph Coverings And Coset Graphs

In this chapter, we continue our discussion on the sequences of groups that do not yield expander families. In Section 5.1, we discuss the covering of a graph and show that if K is a subgroup or quotient of a group G, then the isoperimetric constant of a Cayley graph on G is bounded by the isoperimetric constant of a certain related Cayley graph on K. Using this, we prove that if a sequence (G_n) of finite groups admits a nonexpanding sequence of quotients or bounded-index subgroups, then (G_n) does not yield an expander family. By applying this result, in Section 5.5, we show that Cayley graphs on a sequence of solvable groups with bounded derived length do not form an expander family, which is the main result of this chapter. This chapter is based on Chapters 2 and 4 of [8], and Section 2 of [9].

5.1 Graph Covering

Let $G = (V, E)$ be a graph, not necessarily simple. For a vertex $v \in V$, we denote E_v to be the set of edges incident to v. Recall that if $\phi : G \to H$ is a graph homomorphism, then ϕ maps E_v to $E_{\phi(v)}$.

Definition 5.1.1. Let $\phi : \widehat{G} \to G$ be a graph homomorphism.

(i) For a vertex $v \in V(\widehat{G})$, we say ϕ is *bijective at* v if ϕ maps E_v to $E_{\phi(v)}$ bijectively.

(ii) We say ϕ is *locally bijective* if it is bijective at each vertex of $\widehat{G}$.

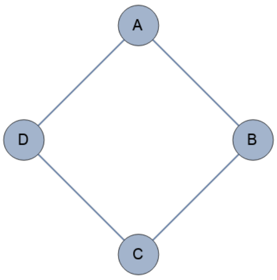
(iii) We say ϕ is a *covering* from $\widehat{G}$ to G if ϕ is surjective and locally bijective. If $\phi : \widehat{G} \to G$ is a covering, then we say $\widehat{G}$ covers G.

Remark 5.1.2. Suppose that $\widehat{G}$ covers G. Then $\widehat{G}$ is d-regular if and only if G is d-regular.

Example 5.1.3. Let $\hat{G}$ and G be the graphs shown in Figure 5.1. Let $\phi : \widehat{G} \to G$ be a graph homomorphism such that

$$\phi(A) = \phi(C) = 1 \text{ and } \phi(B) = \phi(D) = 2.$$

Then ϕ is a covering.



(a) $\hat{G}$

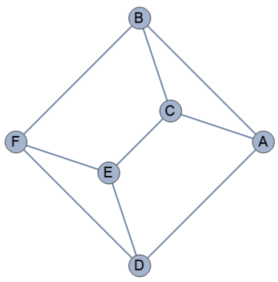
(b) G

Figure 5.1: A cover $\hat{G}$ of the 2-cycle graph G.

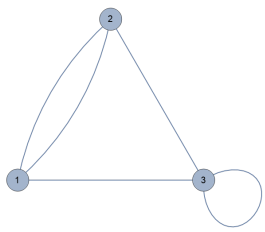
Example 5.1.4. Let $\hat{G}, G$ be the graphs depicted in Figure 5.2. Let $\phi : \widehat{G} \to G$ be a graph homomorphism such that

$$\begin{aligned} &\phi(A) = \phi(D) = 3, \\ &\phi(C) = \phi(F) = 2, \text{ and} \\ &\phi(E) = \phi(B) = 1. \end{aligned}$$

Then ϕ is a covering.



(a) $\hat{G}$



(b) G

Figure 5.2: A 2-cover $\hat{G}$ of graph G.

Lemma 5.1.5. *If $\widehat{G}$ covers G and $\widehat{G}$ is connected, then G is connected.*

Proof. Let $\phi : \widehat{G} \to G$ be a covering, and let $a, b \in V(G)$. If they are adjacent in G, then there is nothing to do. Suppose that they are not adjacent. As ϕ is surjective, there exists $c, d \in V(\widehat{G})$ such that $\phi^{-1}(a) = c$ and $\phi^{-1}(b) = d$. Since, $\widehat{G}$ is connected, there is a path $(c = v_1, e_1, v_2, e_2, \cdots, e_n, v_{n+1} = d)$. As ϕ is a graph homomorphism, it follows that the sequence of edges $\phi(e_1), \cdots, \phi(e_n)$ define a path in G joining a and b. $\square$

Definition 5.1.6. Let $\phi : \widehat{G} \to G$ be a covering, and let $w \in V(G)$. Then the *fibre* $\phi^{-1}(w)$ of ϕ at w is the set of all vertices v of $\widehat{G}$ such that $\phi(v) = w$.

Lemma 5.1.7. *Suppose that $\phi : \widehat{G} \to G$ is a covering. If G is connected, then*

$$|\phi^{-1}(w_1)| = |\phi^{-1}(w_2)|$$

for all vertices w_1, w_2 of G.

Proof. Let w_1, w_2 be two distinct vertices of G. Clearly $\phi^{-1}(w_1)$ and $\phi^{-1}(w_2)$ are disjoint. We break our argument into two cases.

<u>Case 1:</u> w_1 and w_2 are adjacent vertices.
Let $v_1 \in \phi^{-1}(w_1)$, $v_2 \in \phi^{-1}(w_2)$, and let $E_{v_1} = \{x_1, \cdots, x_r\}$. If $v_2 \notin E_{v_1}$, then $|E_{w_1}| > |E_{v_1}|$, which is not possible as ϕ is locally bijective. So, v_1 and v_2 are adjacent, and if there are m edges edges between w_1 and w_2, then there are m edges edges between v_1 and v_2 as well. Hence, the number of edges between a vertex in $\phi^{-1}(w_1)$ and a vertex in $\phi^{-1}(w_2)$ is $m \cdot |\phi^{-1}(w_1)|$. Reversing the roles of w_1 and w_2, we get that there are $m \cdot |\phi^{-1}(w_2)|$ edges joining a vertex $\phi^{-1}(w_2)$ to a vertex in $\phi^{-1}(w_1)$. Since $m > 0$, we have $|\phi^{-1}(w_1)| = |\phi^{-1}(w_2)|$.

<u>Case 2:</u> w_1 and w_2 are not adjacent vertices.
As G is connected, there exists a path $(w_1 = v_1, e_1, v_2, e_2, \cdots, e_n, v_{n+1} = w_2)$. From the arguments above, it follows that $|\phi^{-1}(v_i)| = |\phi^{-1}(v_j)|$, for $1 \leq i \neq j \leq n+1$. Thus, we have $|\phi^{-1}(w_1)| = |\phi^{-1}(w_2)|$, and this completes the proof.

□

Definition 5.1.8. Let $\phi : \widehat{G} \to G$ be a covering. We say $\widehat{G}$ is an *n-lift* of G if $|\phi^{-1}(v)| = n$ for all $v \in G$.

Lemma 5.1.9. *If $\widehat{G}$ and G are finite graphs such that $\widehat{G}$ covers G, then $h(\widehat{G}) \leq h(G)$.*

Proof. Let $\phi : \widehat{G} \to G$ be a covering. If G is not connected, then by Lemma 5.1.5 $\widehat{G}$ is not connected, and so we have

$$h(\widehat{G}) = 0 = h(G).$$

Now suppose that G is connected. Let $S \subset G$ such that $h(G) = \frac{|\partial S|}{|S|}$, and let

$$\phi^{-1}(S) = \{v \in V(\widehat{G}) \mid \phi(v) \in S\}.$$

Let $w \in V(G)$, and let $a = |\phi^{-1}(w)|$. Then by Lemma 5.1.7, we know that $|\widehat{G}| = a \cdot |G|$ and $|\phi^{-1}(S)| = a \cdot |S|$. Since $|S| \leq \frac{1}{2}|G|$, we get $|\phi^{-1}(S)| \leq \frac{1}{2}|\widehat{G}|$. Every edge in ∂S has exactly a preimages in $\widehat{G}$. So $|\partial \phi^{-1}(S)| = a|\partial S|$. Therefore, we have

$$h(\widehat{G}) \leq \frac{|\partial \phi^{-1}(S)|}{|\phi^{-1}(S)|} = \frac{a|\partial S|}{a|S|} = h(G),$$

which completes the proof. □

5.2 2-Lifts

Let $G = (V, E)$ be a simple graph. By Definition 5.1.8, a *2-lift* of G is a graph $\widehat{G} = (\widehat{V}, \widehat{E})$ that has two vertices $\{v_0, v_1\}$ for each $v \in V$, in other words, the fibre of each vertex $v \in V$ has two elements.

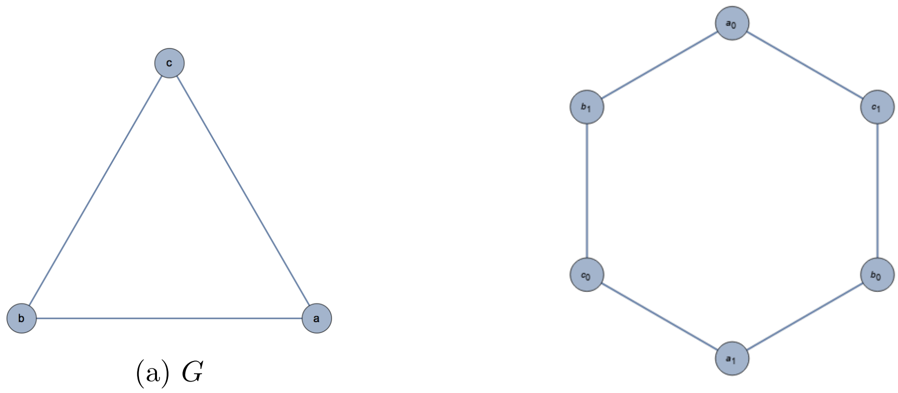
Remark 5.2.1. By the definition of graph covering, every edge in G corresponds to two edges in $\widehat{G}$. Once the vertex set $\widehat{V}$ is fixed, for an edge $\{u, v\} \in E$, the edge set $\widehat{E}$ can either contain the pair of edges

$$\{\{u_0, v_0\}, \{u_1, v_1\}\}, \text{ or } \{\{u_0, v_1\}, \{u_1, v_0\}\}.$$

If only edge pairs of first type appear, then the 2-lift is just two disjoint copies of the original graph G. Note that if the original graph G is a tree, then no matter what type of edge pairs appear, its 2-lift is two disjoint copies of G.

Remark 5.2.2. If the original graph is bipartite, then its 2-lift is also bipartite as the preimage of a independent set is also independent.

Example 5.2.3. Consider the graphs G, H depicted in Figure 5.3. H is a 2-lift of G having edges only of the second type, and therefore H is a double cover of G.



(a) G

(b) H

Figure 5.3: A 2-lift of the 3-cycle graph.

Definition 5.2.4. We define *sign* function $s : E \to \{\pm 1\}$ given by

$$s(\{u, v\}) = \begin{cases} 1, & \text{if corresponding edges are of type (1) in the 2-lift, and} \\ -1, & \text{if corresponding edges are of type (2) in the 2-lift.} \end{cases}$$

Using the sign function, we define the *signed adjacency matrix* $A_s(G)$ to be same as the adjacency matrix $A(G)$, except the entries corresponding to an edge $\{u, v\}$ are $s(\{u, v\})$. We refer to the eigenvalues of $A(G)$ as old eigenvalues, and eigenvalues of $A_s(G)$ as *new* eigenvalues.

Lemma 5.2.5. *Let $G = (V, E)$ be a simple graph and let $\widehat{G}$ be a 2-lift of G. Then every eigenvalue of $A(G)$ and $A_s(G)$ are eigenvalues of $A(\widehat{G})$. Furthermore, the multiplicity of each eigenvalue of $A(\widehat{G})$ is the sum of its multiplicities in $A(G)$ and $A_s(G)$.*

Proof. Let A_1 be the adjacency matrix of the graph $G_1 = (V, s^{-1}(1))$, and let A_2 be the adjacency matrix of the graph $G_2 = (V, s^{-1}(-1))$. Then note that

$$A(G) = A_1 + A_2, \text{ and } A_s(G) = A_1 - A_2.$$

Also, observe that

$$A(\widehat{G}) = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix},$$

where the second block diagonal matrix is also A_1 in $A(\widehat{G})$ because a covering is locally bijective.

Now, if (λ, u) is an eigenpair of $A(G)$. Then

$$A(\widehat{G})(u,u)^T = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} u \\ u \end{bmatrix} = \begin{bmatrix} A_1(u) + A_2(u) \\ A_2(u) + A_1(u) \end{bmatrix} = \begin{bmatrix} A(G)(u) \\ A(G)(u) \end{bmatrix} = \lambda(u,u)^T.$$

Similarly, if (μ, v) is an eigenpair of $A_s(G)$. Then

$$A(\widehat{G})(v,-v)^T = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} v \\ -v \end{bmatrix} = \begin{bmatrix} A_1(v) + A_2(-v) \\ A_2(v) + A_1(-v) \end{bmatrix} = \begin{bmatrix} A_s(G)(v) \\ -A_s(G)(v) \end{bmatrix} = \mu(v,-v)^T.$$

Since the dot product of (u, u) and $(v, -v)$ is zero, these eigenvectors are linearly independent and form a basis. □

5.3 Coset Graphs

Definition 5.3.1. Let G be a finite group, and let $\Gamma \subset G$ be symmetric. Given a subgroup H of G, we construct a *coset graph* $\mathrm{Cos}(H\backslash G, \Gamma) := (V, E)$, where V is the set $H\backslash G$ of right cosets of H in G, and two vertices Hx, and Hy are adjacent if there is $\gamma \in \Gamma$ such that $Hx = Hy\gamma$.

Example 5.3.2. Consider the dihedral group D_6. Let $H = \{1, s\}$, and let $\Gamma = \{s, r, r^2\}$. Then $H\backslash D_6 = \{H, Hr, Hr^2\}$. The coset graph $\mathrm{Cos}(H\backslash G, \Gamma)$ is shown in Figure 5.4 below.

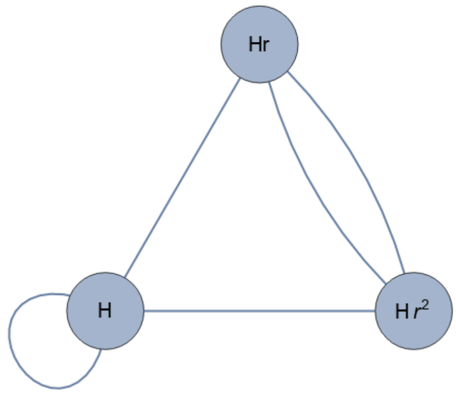


Figure 5.4: A Cayley coset graph of D_6.

Remark 5.3.3. Note that, as is case with $\mathrm{Cay}(G, \Gamma), \mathrm{Cos}(H\backslash G, \Gamma)$ is also $|\Gamma|$-regular.

If $H \lhd G$, then we have $\mathrm{Cos}(H\backslash G, \Gamma) = \mathrm{Cos}(G/H, \Gamma) = \mathrm{Cay}(G/H, \overline{\Gamma})$, where $\overline{\Gamma}$ is the image of Γ under the canonical homomorphism $\phi : G \to G/H$.

Lemma 5.3.4. *Let G be a finite group, let $H < G$, and let $\Gamma \subset G$ be symmetric. Then* $\mathrm{Cay}(G, \Gamma)$ *covers* $\mathrm{Cos}(H\backslash G, \Gamma)$.

Proof. Define a map $\phi : G \to H\backslash G$ such that for all $g \in G$, $\phi(g) = Hg$. Whenever $\{g, g\gamma\}$ is an edge in $\mathrm{Cay}(G, \Gamma)$, $\{Hg, Hg\gamma\}$ is an edge in $\mathrm{Cos}(H\backslash G, \Gamma)$. Therefore, ϕ is a graph homomorphism. Moreover, it is a surjective graph homomorphism as for any $Hg \in H\backslash G$ there is a g in G. As both the graphs $\mathrm{Cay}(G, \Gamma)$ and $\mathrm{Cos}(H\backslash G, \Gamma)$ are regular with degree $= |\Gamma|$, we get that ϕ is locally bijective. This implies $\mathrm{Cay}(G, \Gamma)$ covers $\mathrm{Cos}(H\backslash G, \Gamma)$. □

The following is a direct consequence of Lemma 5.1.9.

Corollary 5.3.5. *Let G, H, Γ be as in Lemma 5.3.4. Then*

$$h(\mathrm{Cay}(G, \Gamma)) \leq h(\mathrm{Cos}(H\backslash G, \Gamma)).$$

Definition 5.3.6. Let (G_n) and (Q_n) be sequences of finite groups. We say that (G_n) *admits* (Q_n) *as a sequence of quotients* if for each n there exists $H_n \lhd G_n$ such that $G_n/H_n \cong Q_n$.

Definition 5.3.7. Let (G_n) be a sequence of finite groups. We say that (G_n) *yields an expander family* if for some positive integer d, there exists a sequence (Γ_n), such that for each n, Γ_n is a symmetric subset of G_n with cardinality d so that the sequence of Cayley graphs $(\mathrm{Cay}(G_n, \Gamma_n))$ is an expander family.

Proposition 5.3.8. *(Quotients Non-expansion Principle) Let (G_n) be a sequence of finite groups. Suppose that (G_n) admits (Q_n) as a sequence of quotients. If (Q_n) does not yield an expander family, then (G_n) does not yield an expander family.*

Proof. We will prove the contrapositive of the statement. Suppose (G_n) yields an expander family, that is, for a positive integer d, there exists a sequence of symmetric subsets (Γ_n) such that $(\mathrm{Cay}(G_n, \Gamma_n))$ is an expander family. Let (H_n) be the sequence of normal subgroups of G_n such that $G_n/H_n \cong Q_n$. Then by Lemma 5.3.4 and Corollary 5.3.5, it follows that (Q_n) yields an expander family. □

5.4 Subgroups and Schreier Generators

Definition 5.4.1. Let G be a finite group, and let $H < G$. Let $T \subset G$ such that it contains exactly one element from each right coset of H in G. Then T is called a *set of transversals* for H in G.

Note that the subset T can not be a multiset, since the cosets are disjoint and partition the group. For G, H, and T as in Definition 5.4.1, we denote by $\bar{x}$ the unique element of T such that $Hx = H\bar{x}$. If $t, \gamma \in G$, we introduce the notation

$$\widehat{(t, \gamma)} = t\gamma(\overline{t\gamma})^{-1}.$$

Definition 5.4.2. Let G, H, and T be as in Definition 5.4.1, and let $\Gamma \subset G$. We call

$$\hat{\Gamma} := \{\widehat{(t, \gamma)} \mid (t, \gamma) \in T \times \Gamma\}$$

the set of *Schreier generators* for H in G with respect to Γ.

Example 5.4.3. Consider the dihedral group D_8. Let $H = \{1, r, r^2\}$, and let $\Gamma = \{s, sr\}$. The right cosets of H in G are H and Hs. Let $T = \{1, s\}$. Then

$$\begin{aligned}
\widehat{(s, sr)} &= ssr(\overline{ssr})^{-1} = r \cdot 1 = r, \\
\widehat{(1, s)} &= s(\overline{s})^{-1} = s \cdot s = 1, \\
\widehat{(s, s)} &= ss(\overline{ss})^{-1} = 1, \text{ and} \\
\widehat{(1, sr)} &= sr(\overline{sr})^{-1} = sr \cdot s^{-1} = srs = r^{-1} = r^2,
\end{aligned}$$

Thus, the complete multiset of Schreier generators is

$$\hat{\Gamma} = \{\widehat{(1, s)}, \widehat{(1, sr)}, \widehat{(s, s)}, \widehat{(s, sr)}\}.$$

Lemma 5.4.4. *Let G, H, and T be as in Definition 5.4.1.*

(i) For all $x \in G$, there exists a unique $h \in H$ such that $x = h\bar{x}$.

(ii) For all $h \in H, a \in G$, we have $\overline{ha} = \bar{a}$.

(iii) For all $a, b \in G$, we have $\overline{\bar{a}b} = \overline{ab}$.

(iv) For all $t \in T$, we have $\bar{t} = t$.

Proof. (i) $H\bar{x} = Hx$ if and only if $H = Hx\bar{x}^{-1}$, which implies there exist some $h \in H$ such that $h = x\bar{x}^{-1}$, and so we have $x = h\bar{x}$. This is unique because if $x = h_1\bar{x}$ and $x = h_2\bar{x}$, then $h_1 = h_2$.

(ii) This follows from the fact $Hha = Ha$.

(iii) We know $Ha = H\bar{a} \Rightarrow (Ha)b = (H\bar{a})b$. Also, as $(H\bar{a})b = H\overline{\bar{a}b}$ we have $H\overline{\bar{a}b} = Hab = H\overline{ab}$, and the assertion follows.

(iv) This follows from Definition 5.4.1.

$\square$

Lemma 5.4.5. $\hat{\Gamma} \subset H$. *Moreover, if* $\Gamma \subset G$ *is symmetric, then* $\hat{\Gamma}$ *is symmetric in* H.

Proof. For $t \in T$ and $\gamma \in \Gamma$, let $x = t\gamma$. Then by Lemma 5.4.4, we know that there exists a unique $h \in H$ such that $x = h\bar{x}$, which implies $x\bar{x}^{-1} \in H$, or $t\gamma(\overline{t\gamma})^{-1} \in H$. Therefore, $\hat{\Gamma} \subset H$.

Now, assume $\Gamma \subset G$ is symmetric. Define a map $\phi : T \times \Gamma \to T \times \Gamma$ by $\phi(t, \gamma) = (\overline{t\gamma}, \gamma^{-1})$. Once again, from Lemma 5.4.4, it follows that

$$(\phi \circ \phi)(t, \gamma) = \phi(\overline{t\gamma}, \gamma^{-1}) = (\overline{\overline{t\gamma}\gamma^{-1}}, \gamma) = (t, \gamma),$$

which shows that ϕ is bijective.
We know that

$$\widehat{(t, \gamma)}^{-1} = \overline{t\gamma}\gamma^{-1}t^{-1} = \overline{t\gamma}\gamma^{-1}(\overline{\overline{t\gamma}\gamma^{-1}})^{-1} = \widehat{\phi(t, \gamma)}.$$

Finally, from the bijectivity of ϕ, it follows that $\widehat{(t, \gamma)}^{-1} \in \hat{\Gamma}$, which concludes the proof. $\square$

Lemma 5.4.5 tells us that if $\mathrm{Cay}(G, \Gamma)$ is an undirected graph, then $\mathrm{Cay}(H, \hat{\Gamma})$ is also an undirected graph.

Lemma 5.4.6. *There is a one-to-one correspondence between the set of directed edges in* $\mathrm{Cay}(G, \Gamma)$ *and the set of directed edges in* $\mathrm{Cay}(H, \hat{\Gamma})$.

Proof. Let $E_1 = E(\mathrm{Cay}(G, \Gamma)), E_2 = E(\mathrm{Cay}(H, \hat{\Gamma}))$, and let T be a set of transversals for H in G. Then any element $g \in G$ can be written as ht, for some $h \in H$ and $t \in T$. Define a map $\phi : E_1 \to E_2$ given by

$$\phi(ht, \gamma) = (h, \widehat{(t, \gamma)}).$$

We want to show that ϕ is a bijection. Since G is a finite group, we have

$$|E_2| = |H|[G : H]|\Gamma| = |H|\frac{|G|}{|H|}|\Gamma| = |G| \cdot |\Gamma| = |E_1|.$$

Since for any edge $(h, \widehat{(t, \gamma)}) \in E_2$, there is an edge $(ht, \gamma) \in E_1$, the map ϕ is sujective, and the assertion follows. $\square$

Lemma 5.4.7. *Let G, H, and T be as in Definition 5.4.1 Let $\Gamma \subset G$ be symmetric. Then*

$$h(\mathrm{Cay}(G,\Gamma)) \;\leq\; \frac{h(\mathrm{Cay}(H,\hat{\Gamma}))}{[G:H]}.$$

Proof. Let $S \subset H$ such that $|S| \leq \frac{1}{2}|H|$ and $h(\mathrm{Cay}(H,\hat{\Gamma})) = \frac{|\delta S|}{|S|}$. Let $\hat{S} = \{ht \mid h \in S, t \in T\}$. Then we have

$$|\hat{S}| = |S| \cdot |T| \leq \frac{1}{2}|H| \cdot |T| = \frac{1}{2}|G|.$$

Note that the map $ht \mapsto h$ is the same as the map $g \mapsto g((\bar{g}))^{-1}$. So if $h \in H, t \in T$, and $\gamma \in \Gamma$, then

$$ht\gamma \in \hat{S} \iff ht\gamma(\overline{ht\gamma})^{-1} \in S \iff h\widehat{(t,\gamma)} \in S.$$

By Lemma 5.4.6, we have

$$\begin{aligned}
|\delta\hat{S}| &= |\{(g,\gamma) \mid g \in \hat{S}, \gamma \in \Gamma, g\gamma \notin \hat{S}\}| \\
&= |\{(ht,\gamma) \mid h \in S, t \in T, \gamma \in \Gamma, ht\gamma \notin \hat{S}\}| \\
&= |\{(h,\widehat{(t,\gamma)}) \mid h \in S, t \in T, \gamma \in \Gamma, ht\gamma \notin \hat{S}\}| \\
&= |\{(h,\widehat{(t,\gamma)}) \mid h \in S, t \in T, \gamma \in \Gamma, h\widehat{(t,\gamma)} \notin S\}| \\
&= |\delta S|.
\end{aligned}$$

Therefore,

$$h(\mathrm{Cay}(G,\Gamma)) \leq \frac{|\delta\hat{S}|}{|\hat{S}|} = \frac{|\delta S|}{|S| \cdot [G:H]} = \frac{h(\mathrm{Cay}(H,\hat{\Gamma}))}{[G:H]}.$$

$\square$

Remark 5.4.8. If Γ generates G, then $\mathrm{Cay}(G,\Gamma)$ is connected, which implies that $h(\mathrm{Cay}(G,\Gamma)) > 0$, and so by Lemma 5.4.7, it follows that $h(\mathrm{Cay}(H,\hat{\Gamma})) > 0$. Hence, we have that $\mathrm{Cay}(H,\hat{\Gamma})$ is connected, which implies that $\hat{\Gamma}$ generates H. Therefore, we conclude if Γ generates G, then $\hat{\Gamma}$ generates H. This is known as the *Schreier subgroup lemma* in Group Theory.

Definition 5.4.9. Let (G_n) and (H_n) be sequences of finite groups. We say that (G_n) admits (H_n) as a bounded-index sequence of subgroups if $H_n < G_n$ for all n and the sequence $([G_n : H_n])$ is bounded.

Proposition 5.4.10. *(Subgroups Non-expansion Principle) Let (G_n) be a sequence of finite groups. Suppose that (G_n) admits (H_n) as a bounded-index sequence of subgroups. If (H_n) does not yield an expander family, then (G_n) does not yield an expander family.*

Proof. We will prove the contrapositive of the statement. Suppose (G_n) yields an expander family, that is, for a positive integer d, for each n, there exists a sequence of symmetric subsets (Γ_n) with $|\Gamma_n| = d$ such that $(\mathrm{Cay}(G_n, \Gamma_n))$ is an expander family. Let $\epsilon > 0$ such that $h((\mathrm{Cay}(G_n, \Gamma_n))) \geq \epsilon$, for all n. Let M be a positive integer such that $[G_n : H_n] \leq M$, for all n. Let T_n be a set of transversals for H_n in G_n for each n, and let

$$S_n = \hat{\Gamma} \cup \{(M - [G : H])d \cdot e_n\},$$

where e_n denotes the identity. So, $|S_n| = M \cdot d$ for all n. Then

$$h(\mathrm{Cay}(H_n, S_n)) = h(\mathrm{Cay}(H_n, \hat{\Gamma_n})) \geq h((\mathrm{Cay}(G_n, \Gamma_n))) \geq \epsilon,$$

for each n. Therefore, (H_n, S_n) is an expander family as desired. □

5.5 Solvable Groups with bounded Derived Length

In this section, we will prove that Cayley graphs yielded by a sequence of solvable groups with bounded derived length do not form an expander family.

Definition 5.5.1. Let G be a group. An element of the form $a^{-1}b^{-1}ab$ for some $a, b \in G$ is called a *commutator*. We denote by $[G, G]$ to be the subgroup of G generated by the set of commutators in G.

Definition 5.5.2. Let G be a group. We recursively define a sequence of subgroups of G, as follows:

$$G^{(0)} = G, \text{ and } G^{(k+1)} = [G^{(k)}, G^{(k)}], \text{ for } k \geq 0. \tag{5.5.1}$$

The group $G^{(k)}$ is called the k^{th} *derived subgroup* of G.

Definition 5.5.3. Let G be a group.

(i) We say that G is *solvable with derived length* 0 if G is the trivial group.

(ii) We say that G is *solvable with derived length* $k+1$ if $G^{(k)} \neq 1$, but $G^{(k+1)} = 1$.

Remark 5.5.4. A group G is abelian iff G is solvable with derived length 1.

Theorem 5.5.5. *Let (G_n) be a sequence of finite non-trivial groups such that $|G_n| \to \infty$. Let k be a positive integer. Suppose that for all n, we have that G_n is solvable with derived length $\leq k$. Then (G_n) does not yield an expander family.*

Proof. We will prove this using induction on k. By Corollary 4.3.4, we know that a sequence of abelian groups (G_n) does not yield an expander family. Therefore, the theorem holds for $k = 1$.

Assuming that the theorem holds true when the derived length is k, we want to show that it holds for $k + 1$. We divide the argument into two cases.

Case 1: The sequence (G'_n) has bounded index in (G_n).
For all n, let l_n be the derived length of G_n. Note that (G'_n) is solvable with derived length $l_n - 1 \leq k$. By the inductive hypothesis, (G'_n) does not yield an expander family. Therefore, by the Subgroups Non-expansion Principle, we have that (G_n) does not yield an expander family.

Case 2: The sequence $(|G_n/G'_n|)$ is unbounded.
Since G_n/G'_n is abelian, it follows from Corollary 4.3.4 that (G_n/G'_n) does not yield an expander family. Therefore, by the Quotients Non-expansion Principle, we have that (G_n) does not yield an expander family, which completes the proof. $\square$

Chapter 6

Representation Theory and Eigenvalues of Cayley Graphs

In this chapter, we use the results and techniques from the representation theory of finite groups to determine the eigenvalues of Cayley graphs. In Section 6.1, we recall some basic notions from representation theory, and in Section 6.2, we express the adjacency matrix A of a Cayley graph in terms of regular representation of the involved group. Since every representation of finite group is completely reducible (Maschke's Thereom), we show that we can further decompose A into the direct sum of inequivalent irreducible representations of the group. In Section 6.3, we give a formula to explicitly calculate the spectrum of Cayley graphs on abelian groups. This chapter is based on Chapters 6 and 7 of [8].

6.1 Representations of finite groups

Definition 6.1.1. Let G be a group and V be a finite-dimensional vector space. A *representation of* G in V is a homomorphism $\pi\colon G \to \mathrm{GL}(V)$, where $\mathrm{GL}(V) = \{T\colon V \to V \mid T \text{ is an isomorphism}\}$. The dimension of vector space V is called the *degree* of π.

We denote a representation of G as in Definition 6.1.1 by the pair (π, V).

Example 6.1.2. Let G be a finite group, and let $\pi : G \to \mathrm{GL}(\mathbb{C}) = \mathbb{C}^\times$ be defined by $\pi(g) = 1$, for all $g \in G$. Then $(\pi, \mathbb{C})$ is a one-dimensional representation of G called the *trivial representation.*

Example 6.1.3. The homomorphism

$$\mathbb{Z}_n \to \mathbb{Z}^\times : [m] \overset{\phi}{\mapsto} e^{i\,2\pi m/n}$$

defines a representation $(\phi, \mathbb{C})$ of $\mathbb{Z}_n$ of degree one.

Definition 6.1.4. Let (π, V) and (π', V') be two representations of a group G. Then an *intertwining operator* is a linear map $\phi\colon V \to V'$ such that for all $g \in G$,

$$\phi \circ \pi(g) = \pi'(g) \circ \phi. \tag{6.1.1}$$

The space of intertwining operators from V to V' is denoted by $\mathrm{Hom}_G(V, V')$ or $Hom_G(\pi, \pi')$.

Definition 6.1.5. Two representations (π, V) and (π', V') of a group G are said to be *equivalent* if there exists an isomorphism $\phi\colon V \to V'$ such that it satisfies Equation 6.1.1.

Definition 6.1.6. Let (π, V) be a representation of G, and let W be a subspace of V. We say that W is *G-invariant* if for all $g \in G, w \in W, \pi(g)w \in W$.

Definition 6.1.7. Let (π, V) be a representation of G. We say that π is an *irreducible representation of G* if it has no non-trivial G-invariant subspace.

We will now state some results from the representation theory of finite groups, that will be used later in this chapter.

Theorem 6.1.8. *(Maschke's Thereom). Every representation of a finite group is completely reducible.*

Lemma 6.1.9. *(Schur's lemma) Let (π, V) and (π', V') be two irreducible representations of a group G, and let $\phi \in \mathrm{Hom}_G(V, V')$. Then $\phi = 0$ or ϕ is an isomorphism. Moreover,*

(i) If $\pi = \pi'$ and $\phi \in \mathrm{Hom}_G(\pi, \pi')$, then $\phi = \lambda \cdot 1$, for some $\lambda \in \mathbb{C}$.

(ii) Let (π, V) and (π', V') be two irreducible representations of a group G, and let ϕ_1, ϕ_2 be two non-zero intertwining maps. Then $\phi_1 = \lambda\phi_2$, for some $\lambda \in \mathbb{C}$.

Corollary 6.1.10. *Let (π, V) and (π', V') be two irreducible representations of a group G.*

(i) If $\pi \nsim \pi'$, then $\mathrm{Hom}_G(\pi, \pi') = 0$.

(ii) If (π, V) is an irreducible representation of G, then center of G acts by scalars.

(iii) If G is an abelian group, (π, V) is an irreducible representation of G, then π is 1-dimensional.

Definition 6.1.11. Let $\phi : G \to \mathrm{GL}(V)$ be a representation. The *character* $\chi_\phi : G \to \mathbb{C}$ of ϕ is a function defined by setting $\chi_\phi(g) = \mathrm{tr}(\phi(g))$, where tr denotes the trace.

The character of an irreducible representation is called an *irreducible character.*

Remark 6.1.12. If $\phi : G \to \mathbb{C}^*$ is a degree 1 representation, then $\chi_\phi = \phi$.

Theorem 6.1.13. *Let G be a finite group. Let ϕ, ρ be two irreducible representations of G. Then*

$$\langle \chi_\phi, \chi_\rho \rangle = \begin{cases} 1, & \phi \sim \rho, \text{ and} \\ 0, & \phi \nsim \rho. \end{cases}$$

Thus, the irreducible characters of G form an orthonormal set.

Definition 6.1.14. Let G be a finite group. The *right regular representation* of G is the homomorphism $R : G \to \mathrm{GL}(L^2(G))$ such that $R(g)f(h) = f(hg)$ for all $f \in L^2(G)$ and $g, h \in G$.

Remark 6.1.15. To see that R in Definition 6.1.14 is indeed a representation, we need to show that R is a group homomorphism. Let $g, h, i \in G$, and let $f \in L^2(G)$. Then

$$R(gh)f(i) = f(igh)$$

and

$$(R(g)R(h)f)(i) = (R(g)(R(h)f))(i) = R(h)f(ig) = f(igh).$$

Therefore, $R(gh) = R(g)R(h)$.

Proposition 6.1.16. *Let G be a finite group of order n, and let $\Gamma \subseteq G$ be symmetric. Let $\phi : G \to GL(L^2(G))$ be a map defined as $\phi(g) = A(Cay(G, \{g\})) := A_g$. Then ϕ is the right regular representation of G, and we have that*

$$A(\mathrm{Cay}(G, \Gamma)) = \sum_{g \in \Gamma} A_g.$$

Proof. First, to prove that ϕ is a group homomorphism, we need to show that $A_{g_1}A_{g_2} = A_{g_1g_2}$. We know that $(A_g)_{ij} = 1$ if and only if the vertex g_i is connected to g_j, or when $g_ig = g_j$. Now, we note that

$$(A_{g_1g_2})_{ij} = 1 \Leftrightarrow g_j = g_i(g_1g_2).$$

Further,

$$(A_{g_1}A_{g_2})_{ij} = 1 \Leftrightarrow \sum_{k=1}^{k=n} (A_{g_1})_{ik}(A_{g_2})_{kj} = 1 \Leftrightarrow g_k = g_ig_1 \text{ and } g_j = g_kg_2 \Leftrightarrow g_i(g_1g_2) = g_j.$$

Hence, ϕ is a group homomorphism. Now, we note that for all $g, h \in G$ we have

$$\phi(g)(\delta_h) = A_g(\delta_h) = \delta_{hg} = R(g)(\delta_h),$$

which proves the first part.

Now, let $X = \mathrm{Cay}(G, \Gamma)$. We know that $(A(X))_{ij} = 1$ if and only if there exists $g, g^{-1} \in \Gamma$ such that $g_ig = g_j$ and $g_i = g_jg^{-1}$. Therefore, we have $A(\mathrm{Cay}(G, \Gamma)) = \sum_{g \in \Gamma} A_g$. □

Theorem 6.1.17. *Let G be a finite group. Then there are only finitely many irreducible representations of G, up to equivalence. Suppose that $V_1, V_2, \cdots, V_n$ form a complete list of inequivalent irreducible representations of G. Let $d_i = dim(V_i)$. Then $L^2(G)$ is orthogonally equivalent to*

$$d_1V_1 \oplus \cdots \oplus d_nV_n.$$

Moreover, $|G| = \sum d_i^2$.

6.2 Decomposing the Adjacency Operator into irreducible representations

Proposition 6.2.1. *Let G be a finite group, $\Gamma \subset G$ be symmetric, and let A the adjacency operator of* $\mathrm{Cay}(G, \Gamma)$. *If $\pi_1, \cdots, \pi_k$ is a complete set of inequivalent matrix irreducible representations of G, then*

$$A \cong d_1 M_{\pi_1} \oplus d_2 M_{\pi_2} \oplus \cdots \oplus d_k M_{\pi_k},$$

where d_i is the dimension of π_i and $M_\pi = \sum_{\gamma \in \Gamma} \pi(\gamma)$.

Proof. Let R be the right regular representation of G, and let $f \in L^2(G)$. Then

$$\begin{aligned}(Af)(g) &= \sum_{h \in G} A_{g,h} f(h), \\ &= \sum_{\gamma \in \Gamma} f(g\gamma), \\ &= \sum_{\gamma \in \Gamma} (R(\gamma)f)(g).\end{aligned}$$

Therefore, $A = \sum_{\gamma \in \Gamma} R(\gamma)$. By Maschke's theorem and Theorem 6.1.17, we have

$$R \cong d_1 \pi_1 \oplus \cdots \oplus d_k \pi_k, \text{ and so we have}$$
$$\sum_{\gamma \in \Gamma} R(\gamma) \cong \sum_{\gamma \in \Gamma} d_1 \pi_1(\gamma) \oplus \cdots \oplus d_k \pi_k(\gamma).$$

Therefore,

$$A \cong \sum_{\gamma \in \Gamma} d_1 \pi_1(\gamma) \oplus \cdots \oplus d_k \pi_k(\gamma),$$

from which the assertion follows. $\square$

Remark 6.2.2. The matrices A and $d_1 M_{\pi_1} \oplus d_2 M_{\pi_2} \oplus \cdots \oplus d_k M_{\pi_k}$ in Proposition 6.2.1 have the same eigenvalues. Since $\det(A \oplus B) = \det(A)\det(B)$, if all M_{π_i} have degree ≤ 4, we can find out all the eigenvalues.

Example 6.2.3. Consider the symmetric group S_3, and let $\zeta = e^{\frac{2\pi i}{3}}$. Its complete set of irreducible representations up to equivalence is given by $\{\pi_0, \pi_1, \pi_2\}$, where π_0 is defined to be the trivial representation, $\pi_1(\sigma) := (\mathrm{sgn}(\sigma))$ for all $\sigma \in S_3$, and π_2 is defined as

$$\pi_2(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pi_2((1,3)) = \begin{pmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{pmatrix}, \pi_2((1,2)) = \begin{pmatrix} 0 & \zeta^2 \\ \zeta & 0 \end{pmatrix},$$

$$\pi_2((2,3)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pi_2((1,2,3)) = \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta \end{pmatrix}, \pi_2((1,3,2)) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}.$$

Let $X = \mathrm{Cay}(S_3, \Gamma)$, where $\Gamma = \{(1\ 2), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$, and let A be the adjacency operator of X. Then we have

$$\sum_{\gamma\in\Gamma} \pi_0(\gamma) = 4, \sum_{\gamma\in\Gamma} \pi_0(\gamma) = 0, \text{ and} \sum_{\gamma\in\Gamma} \pi_0(\gamma) = \begin{pmatrix} -1 & -\zeta \\ -\zeta^2 & -1 \end{pmatrix}.$$

Therefore, we get

$$A \cong \begin{pmatrix} 4 & & & & & \\ & 0 & & & & \\ & & -1 & -\zeta & & \\ & & -\zeta^2 & -1 & & \\ & & & & -1 & -\zeta \\ & & & & -\zeta^2 & -1 \end{pmatrix}.$$

The characteristic polynomial of the matrix $\begin{pmatrix} -1 & -\zeta \\ -\zeta^2 & -1 \end{pmatrix}$ is $x(x+2)$. Hence, its eigenvalues are 0 and -2. Thus, we get

$$\mathrm{Spec}(A) = \begin{pmatrix} -2 & 0 & 4 \\ 2 & 3 & 1 \end{pmatrix}.$$

Corollary 6.2.4. *Let G be a finite group, and let $\Gamma, \Gamma' \subset G$ be symmetric such that $\Gamma \subset \Gamma'$. Let $X = \mathrm{Cay}(G, \Gamma)$ and $X' = \mathrm{Cay}(G, \Gamma')$. Then $|\Gamma| - \lambda_1(X) \leq |\Gamma'| - \lambda_1(X')$.*

Proof. Let A and A' be the adjacency operators for X and X', respectively. Let $f \in L^2(G)$ such that $||f|| = 1$. Since, R is unitary with respect to the standard inner product on $L^2(G)$, by Cauchy-Schwarz inequality, we get

$$|\langle R(g)f, f\rangle| \leq ||R(g)f||||f|| = ||f||^2 = 1 \quad \forall g \in \Gamma'. \tag{6.2.1}$$

Also,

$$\begin{aligned} \langle R(g)f, f\rangle + \langle R(g^{-1})f, f\rangle &= \langle f, R(g^{-1})f\rangle + \langle f, R(g)f\rangle, \\ &= \overline{\langle R(g^{-1})f, f\rangle} + \overline{\langle R(g)f, f\rangle}, \\ &= \overline{\langle R(g^{-1})f, f\rangle + \langle R(g)f, f\rangle}, \\ &= \overline{\langle R(g)f, f\rangle + \langle R(g^{-1})f, f\rangle}. \end{aligned}$$

$$\Rightarrow \langle R(g)f, f\rangle + \langle R(g^{-1})f, f\rangle \in \mathbb{R}. \tag{6.2.2}$$

Now,

$$\begin{aligned} |\Gamma| - \langle Af, f\rangle &= |\Gamma| - \sum_{\gamma\in\Gamma} \langle R(g)f, f\rangle \\ &= \sum_{\gamma\in\Gamma} (1 - \langle R(g)f, f\rangle) \\ &\leq \sum_{\gamma\in\Gamma'} (1 - \langle R(g)f, f\rangle). \end{aligned}$$

Since, Γ' is symmetric, by Equations 6.2.1 and 6.2.2, we get

$$2 - \langle R(g)f, f\rangle - \langle R(g^{-1})f, f\rangle \geq 0.$$

If γ has order 2, then

$$\langle R(\gamma)f, f\rangle = \langle f, R(\gamma^{-1})f\rangle = \langle f, R(\gamma)f\rangle = \overline{\langle R(\gamma)f, f\rangle}.$$

$$\Rightarrow \langle R(g)f, f\rangle \in \mathbb{R}.$$

So, by Equation 6.2.2, we get $1 - |\langle R(g)f, f\rangle| = 1 - \langle R(g)f, f\rangle \geq 0$. Thus, the last inequality holds true. Thus, for all $f \in L^2(G)$ such that $||f|| = 1$, we have

$$|\Gamma| - \langle Af, f\rangle \leq |\Gamma'| - \langle A'f, f\rangle.$$

Let $f_0 \in L^2(G)$ such that $||f_0|| = 1$ and $\langle A'f_0, f_0\rangle = \lambda_1(X')$. Then

$$|\Gamma| - \langle Af_0, f_0\rangle \leq |\Gamma'| - \lambda_1(X'), \text{ and as}$$

$$|\Gamma| - \lambda_1(X) \leq |\Gamma| - \langle Af_0, f_0\rangle,$$

the assertion follows. □

Theorem 6.2.5. *Let G be a finite group and $\Gamma \subset G$ be symmetric such that $g\Gamma g^{-1} = \{g\gamma g^{-1} | \gamma \in \Gamma\} = \Gamma$, for all $g \in G$. Let $X = \mathrm{Cay}(G, \Gamma)$ and let A be the adjacency operator of X. Let $\rho_1, \cdots, \rho_r$ be a complete set of inequivalent irreducible representations of G; let χ_i be the character of ρ_i; and let d_i be the degree of ρ_i. Then the eigenvalues of A are given by*

$$\mu_i = \frac{1}{d_i} \sum_{\gamma \in \Gamma} \chi_i(\gamma),$$

for $1 \leq i \leq r$, where each eigenvalue μ_i occurs with multiplicity d_i^2.

Proof. Since Γ is closed under conjugation, for all $g \in G$, we have

$$\begin{aligned} AR(g) &= \sum_{\gamma \in \Gamma} R(\gamma)R(g) = \sum_{\gamma \in \Gamma} R(\gamma g) \\ &= \sum_{\gamma \in \Gamma} R((g\gamma g^{-1})g) = \sum_{\gamma \in \Gamma} R(g\gamma) = R(g)A. \end{aligned}$$

By Theorem 6.2.1, we can decompose

$$L^2(G) = d_1 V_1 \oplus \cdots \oplus d_n V_n.$$

Fix $i \in \{1, 2, \ldots, r\}$, and let β be a basis of V_i. Then we have

$$[A]_\beta [R(g)]_\beta = [R(g)]_\beta [A]_\beta, \text{ for all } g \in G.$$

Since, restriction of $R(g)$ to any V_i is ρ_i (i.e., $R(g)v_i = \rho_i(g)v_i$ for all $v_i \in V_i$), by Schur's lemma, we get $[A]_\beta = \mu_i I_{d_i}$, where $\mu_i \in \mathbb{C}$. This gives

$$\begin{aligned} d_i\mu_i &= \operatorname{tr}(A|_{V_i}) \\ &= \operatorname{tr}\left(\sum_{\gamma\in\Gamma}\rho_i(\gamma)\right) = \sum_{\gamma\in\Gamma}\chi_i(\gamma). \end{aligned}$$

Thus, we have $\mu_i = \frac{1}{d_i}\sum_{\gamma\in\Gamma}\chi_i(\gamma)$, for each i, which would imply that

$$A \cong d_1(\mu_1 I_{d_1}) \oplus \cdots \oplus d_r(\mu_r I_{d_r}).$$

Therefore, the eigenvalues of A are $\mu_1, \cdots, \mu_r$, where μ_i has multiplicity d_i^2. $\square$

Remark 6.2.6. Theorem 6.2.5 does require not imply that the μ_i have to be necessarily distinct, as there can be some $\mu_i = \mu_j$. In that case, the multiplicity of the eigenvalue μ_i will be $d_i^2 + d_j^2$.

Example 6.2.7. Again, consider the group S_3. We denote by $K_{(1\ 2)}$, the conjugacy class of the permutation (1 2). Recall the complete set of irreducible representations. $\{\pi_0, \pi_1, \pi_2\}$ given in Example 2.3. Let χ_0, χ_1, and χ_2 be the characters of π_0, π_1, and π_2, respectively. Let $X = \mathrm{Cay}(S_3, K_{(1\ 2)})$, and let A be the adjacency operator of X. Then by Theorem 2.5, the eigenvalues of A are

$$\begin{aligned} \mu_0 &= \frac{1}{1}\sum_{\gamma\in\Gamma}\chi_0(\gamma) = 3, \\ \mu_1 &= \frac{1}{1}\sum_{\gamma\in\Gamma}\chi_1(\gamma) = -3, \text{ and} \\ \mu_2 &= \frac{1}{2}\sum_{\gamma\in\Gamma}\chi_2(\gamma) = 0. \end{aligned}$$

Therefore,

$$\operatorname{Spec}(A) = \begin{pmatrix} -3 & 0 & 3 \\ 1 & 4 & 1 \end{pmatrix}.$$

6.3 Eigenvalues of Cayley graphs on abelian groups

In this section, we describe the set of irreducible representations of any finite abelian group. Using this, we give a formula to explicitly calculate the spectrum of any Cayley graph of an abelian group.

Proposition 6.3.1. *The complete set of inequivalent irreducible matrix representations of $\mathbb{Z}_n$ are given by*

$$\Phi = \{\Phi_k \mid \text{ for } k = 0, 1, 2, \cdots, n-1\},$$

where

$$\Phi_k(a) = e^{\frac{2\pi i a k}{n}}, \text{ for all } a \in \mathbb{Z}_n.$$

Proof. First, Φ_k is a representation, since for all $a, b \in \mathbb{Z}_n$ and for all $0 \leq k \leq n-1$,

$$\Phi_k(a+b) = e^{\frac{2\pi i(a+b)k}{n}} = e^{\frac{2\pi iak}{n}} e^{\frac{2\pi ibk}{n}} = \Phi_k(a)\Phi_k(b).$$

Since each Φ_k is a degree 1 representation, it is irreducible. Further,

$$\begin{aligned}
\langle \Phi_k(g), \Phi_j(g) \rangle &= \sum_{g \in \mathbb{Z}_n} e^{\frac{2\pi igk}{n}} e^{\frac{-2\pi igj}{n}} \\
&= e^{\pi ik(n-1)} e^{-\pi ij(n-1)} \\
&= \begin{cases} 1, & \text{if } k = j, \text{ and} \\ 0, & \text{if } k \neq j. \end{cases}
\end{aligned}$$

Therefore, representations are inequivalent. Since $|\mathbb{Z}_n| = \sum_{i=0}^{n-1} 1^2$, this is the complete set of inequivalent irreducible representations of $\mathbb{Z}_n$. □

The following corollary, which gives the complete set of irreducible representations for finite abelian groups, is a direct consequence of Proposition 6.3.1.

Corollary 6.3.2. *Let $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ be a finite abelian group. Then the set of irreducible representations of G is*

$$\{\Phi_k \mid k = (k_1, \cdots, k_r) \in \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}\},$$

where

$$\Phi_k(a_1, \cdots, a_r) = \left(e^{\frac{2\pi ia_1k_1}{n_1}} e^{\frac{2\pi ia_2k_2}{n_2}} \cdots e^{\frac{2\pi ia_rk_r}{n_r}} \right).$$

A direct application of Proposition 6.3.1 and Corollary 6.3.2 is the following.

Corollary 6.3.3. *Let $\Gamma \subset \mathbb{Z}_n$ be symmetric, and let $\zeta = e^{\frac{2\pi i}{n}}$. Then the eigenvalues of* $\mathrm{Cay}(\mathbb{Z}_n, \Gamma)$ *are given by*

$$\lambda_k = \sum_{\gamma \in \Gamma} \zeta^{k\gamma},$$

where $0 \leq k \leq n-1$.

Example 6.3.4. Let $X_n = \mathrm{Cay}(\mathbb{Z}_{2n}, \Gamma_n)$ where $\Gamma_n = \{1, -1, n\}$. By Corollary 6.3.3, the eigenvalues of X_n are given by

$$\lambda_k = \sum_{\gamma \in \Gamma_n} e^{\frac{2\pi ik\gamma}{2n}} = e^{\frac{2\pi ik}{2n}} + e^{\frac{-2\pi ik}{2n}} + e^{\pi ik} = 2\cos(\frac{\pi k}{n}) + (-1)^k,$$

where $0 \leq k \leq 2n-1$. Therefore, we have

$$\lambda_1(X_n) = 2\cos(\frac{2\pi}{n}) + 1 \to 3, \text{ as } n \to \infty,$$

which implies that the spectral gap $(3 - \lambda_1(X_n)) \to 0$. Therefore, X_n is not an expander family.

Chapter 7

Families I: Bipartitte Ramanujan Graphs of All Degrees

In this chapter, we prove the existence of infinite families of regular bipartite Ramanujan graphs of every degree greater than 2, and the existence of infinite families of *irregular Ramanujan* graphs. To do this, we will prove a variant of a conjecture of Bilu and Linial [1] about the existence of certain special 2-lifts of every graph using the method of interlacing polynomials. The key idea is to start with a d-regular complete bipartite graph G, and using this variant, cosntruct a 2-lift of G which is Ramanujan. Since a 2-lift of a bipartite graph is also bipartite, we inductively form the appropriate 2-lifts to obtain an infinite sequence of d-regular bipartite Ramanujan graphs. This chapter is based on [1, 2, 3, 4, 5, 7, 9].

7.1 Matching Polynomial

Definition 7.1.1. A matching M in a graph G is a subset of its edge set $E(G)$ such that no two edges in M share a common vertex. A matching which covers all the vertices of a graph is called a *perfect matching* of that graph.

The number of matchings with r edges of a graph G is denoted by $\rho(G, r)$. We set $\rho(G, 0) = 1$ (empty set).

Remark 7.1.2. By definition, it is apparent that a graph of odd order cannot have a perfect matching.

Example 7.1.3. In the graph $\overline{C_6}$ depicted in Figure 7.1, $M_1 = \{\{1, 2\}, \{5, 6\}\}$ is a matching in $\overline{C_6}$, whereas $M_2 = \{\{1, 2\}, \{1, 5\}\}$ is not. However, $M_3 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$ is a perfect matching in $\overline{C_6}$.

Definition 7.1.4. The *matching polynomial* of a graph $G = (V, E)$, denoted by $\mu_G(x)$, is defined as

$$\mu_G(x) := \sum_{r\geq 0}(-1)^r \rho(G, r)x^{n-2r},$$

where n is the order of G.

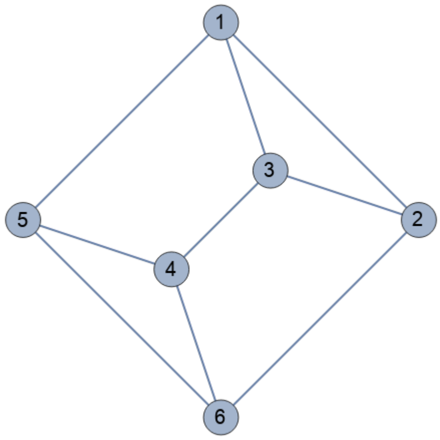


Figure 7.1: The $\overline{C_6}$ (prism) graph.

Example 7.1.5. Let G be an empty graph with n vertices, then $\mu_G(x) = x^n$, as the only non-zero coefficient is $\rho(G, 0) = 1$.

Example 7.1.6. Let P_n be a path on n vertices. Then its matching polynomial is

$$\mu_{P_n}(x) = \sum_{r \geq 0} (-1)^r \binom{n-r}{r} x^{n-2r}.$$

To prove this, we show that there is a one-to-one correspondence between the set of matchings in P_n having r edges and the set of paths on $(n-r)$ vertices with i distinguished vertices. Given a matching M with $|M| = r$, contract each edge that belongs to M onto its left endpoint in P_n to obtain a path on $(n-r)$ vertices. Conversely, given a path on $(n-r)$ vertices, select any r vertices and we can reconstruct a matching having r edges in P_n. Thus, $\rho(P_n, r) = \binom{n-r}{r}$, and we have

$$\mu_{P_n}(x) = \sum_{r \geq 0} (-1)^r \binom{n-r}{r} x^{n-2r}.$$

In the following theorem, we establish some fundamental properties of the matching polynomial.

Theorem 7.1.7. *Let G and H be two graphs on different vertex sets.*

(i) $\mu_{G \cup H}(x) = \mu_G(x)\mu_H(x)$.

(ii) For $e = \{u, v\} \in E(G)$, we have $\mu_G(x) = \mu_{G \setminus e}(x) - \mu_{G \setminus uv}(x)$, where $G \setminus uv$ is the graph obtained by removing the vertices u and v from the graph G.

(iii) $\mu_G(x) = x\mu_{G \setminus u}(x) - \sum_{i \sim u} \mu_{G \setminus ui}(x)$.

Proof. (i) Every matching in $G \cup H$ is the union of a matching of G and a matching of H. Thus, we have

$$\rho(G \cup H, r) = \sum_{s=0}^{r} \rho(G, s)\rho(H, r-s).$$

So the coefficient of x^{n-2r} in $\mu_G(x)\mu_H(x)$ is given by

$$\sum_{s=0}^{r}(-1)^s\rho(G,s)(-1)^{r-s}\rho(H,r-s)=\sum_{s=0}^{r}(-1)^r\rho(G,s)\rho(H,r-s),$$

which completes the proof.

(ii) We note that the number of r-matchings which contain the edge e is $\rho(G\setminus uv,r-1)$, and the number of r-matchings which do not contain the edge e is $\rho(G\setminus e,r)$. So, for $r\geq 0$, we have

$$\rho(G,r)=\rho(G\setminus e,r)+\rho(G\setminus uv,r-1).$$

Hence, it follows that

$$\begin{aligned}
\mu_G(x) &= \sum_{r\geq 0}(-1)^r\rho(G\setminus e,r)x^{n-2r}+\sum_{r\geq 1}(-1)^r\rho(G\setminus uv,r-1)x^{n-2r},\\
&= \sum_{r\geq 0}(-1)^r\rho(G\setminus e,r)x^{n-2r}+(-1)\sum_{r\geq 1}(-1)^{r-1}\rho(G\setminus uv,r-1)x^{n-2(r-1+1)},\\
&= \sum_{r\geq 0}(-1)^r\rho(G\setminus e,r)x^{n-2r}+(-1)\sum_{r\geq 1}(-1)^{r-1}\rho(G\setminus uv,r-1)x^{n-2-2(r-1)},\\
&= \mu_{G\setminus e}(x)+(-1)\sum_{t\geq 0}(-1)^t\rho(G\setminus uv,r-1)x^{n-2-2t},\\
&= \mu_{G\setminus e}(x)+\mu_{G\setminus uv}(x),
\end{aligned}$$

as $|V(G\setminus uv)|=n-2$. Therefore, the assertion follows.

(iii) First, we note that

$$\rho(G,r)=\rho(G\setminus i,r)+\sum_{i\sim j}\rho(G\setminus ij,r-1). \tag{7.1.1}$$

Now, plugging it in the expression for $\rho(G,r)$ from Equation 7.1.1 above into the expression for the matching polynomial $\mu_G(x)=\sum_{r\geq 0}(-1)^r\rho(G,r)x^{n-2r}$, we get

$$\begin{aligned}
\mu_G(x) &= \sum_{r\geq 0}(-1)^r\left(\rho(G\setminus i,r)+\sum_{i\sim j}\rho(G\setminus ij,r-1)\right)x^{n-2r}\\
&= \sum_{r\geq 0}(-1)^r\rho(G\setminus i,r)x^{n-2r}+\sum_{r\geq 0}(-1)^r\sum_{i\sim j}\rho(G\setminus ij,r-1)x^{n-2r}\\
&= x\sum_{r\geq 0}(-1)^r\rho(G\setminus i,r)x^{n-1-2r}+(-1)\sum_{r\geq 0}(-1)^{r-1}\sum_{i\sim j}\rho(G\setminus ij,r-1)\\
&= x\mu_{G\setminus i}(x)-\sum_{i\sim j}\mu_{G\setminus ij}(x).
\end{aligned}$$

□

In the following theorem, we show that the matching polynomial of a tree is exactly the same as the characteristic polynomial of its adjacency matrix.

Theorem 7.1.8. *Let G be a tree, and let $A(G)$ be its adjacency matrix. Then $\mu_G(x) = det(xI - A(G))$.*

Proof. Let $|V(G)| = n$. Since there are no loops in the graph G, all the diagonal entries of $(xI - A(G)) = x$. Now, expanding the determinant using Leibniz formula, we get

$$\det(xI - A(G)) = \sum_{\sigma \in Sn} (-1)^{sgn(\sigma)} x^{|a:\sigma(a)=a|} \prod_{a:\sigma(a)\neq a} (-A(G)(a, \sigma(a)).$$

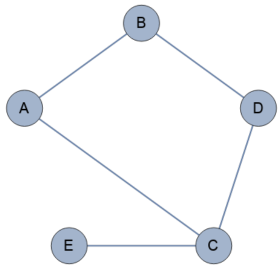
We claim that the only permutations that contribute to the sum are the involutions ($\sigma(\sigma(a)) = a$ for all $a \in V(G)$). This condition ensures that whenever $A(G)(a, \sigma(a))$ appears in the product, so does $A(G)(\sigma(a), a)$, thereby making the product positive. We prove this using contradiction. Let $\sigma = (a_1, \cdots, a_k)$ be a permutation that contributes to the sum. This means, $\sigma(a_i) = a_{i+1}$ for $1 \leq i \leq k-1$ and $\sigma(a_k) = a_1$. Since $A(G)$ is an adjacency matrix, for these terms to contribute, $A(G)(a_i, a_{i+1})$ must be equal to 1 for all $1 \leq i \leq k-1$ and $A(G)(a_k, a_1) = 1$, which implies there is a cycle of length k in the graph. But, G is a tree and hence cannot have cycle, which is a contradiction. Thus, the only permutations that contribute to the sum are the involutions.

We know that any involution σ has either fixed points or transpositions, and each transposition that contributes to the sum, corresponds to an edge in the graph. Thus, the number of permutations with k cycles of length 2 is equal to the number of matchings with k edges. Also, we know that the sign of a permutation with k transpositions is $(-1)^k$, so the coefficient of x^{n-2k} in the expansion above is $(-1)^k \rho(G, k)$, which complete the argument. □

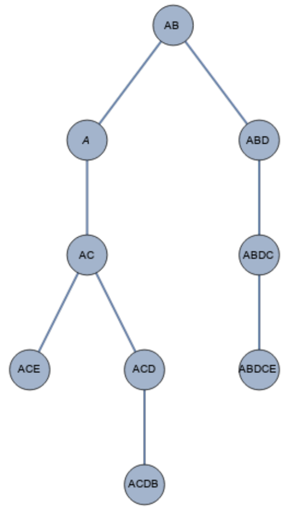
Since the matching polynomial of a tree is same as its characteristic polynomial, we get that it has real roots. Next, we prove that the matching polynomial of any graph has real roots by proving that it divides the matching polynomial of its path tree.

Definition 7.1.9. Given a graph G and a vertex a of G, the *path tree* $P(G, a)$ is a graph whose vertices correspond to paths in G that start at a and do not contain any vertex twice. Two vertices in $P(G, a)$ are adjacent if one path extends the other by one vertex.

Example 7.1.10. Depicted in Figure 7.2, is a graph G and its path tree $P(G, A)$ starting at the vertex A.



(a) G



(b) $P(G, A)$

Figure 7.2: Example of a path tree.

Remark 7.1.11. Note that when G itself is a tree, then for any vertex a of G, $P(G, a)$ is isomorphic to G.

Theorem 7.1.12. *Given a graph G, and a vertex $a \in V(G)$, we have*

$$\frac{\mu_{G\setminus a}(x)}{\mu_G(x)} = \frac{\mu_{P(G,a)\setminus a}(x)}{\mu_{P(G,a)}(x)}.$$

Proof. We will prove the theorem using induction on $|G|$. First, we note that $P(G, a) \setminus a = \bigcup_{b\sim a} P(G \setminus a, b)$. Moreover, if the graph G is a tree, then we know that $P(G, a) \cong G$, and so the equality holds for trees. As all graphs with number of vertices ≤ 2 are trees, the statement of the theorem holds. Let $|G| = n$, and let us assume the assertion holds true for all proper subgraphs of G. Then by Theorem 7.1.7, we have

$$\begin{aligned}
\frac{\mu_G(x)}{\mu_{G\setminus a}(x)} &= \frac{x\mu_{G\setminus a}(x) - \sum_{b\sim a} \mu_{G\setminus ab}(x)}{\mu_{G\setminus a}(x)} \\
&= x - \sum_{b\sim a} \frac{\mu_{G\setminus ab}(x)}{\mu_{G\setminus a}(x)} \\
&= x - \sum_{b\sim a} \frac{\mu_{P(G\setminus a,b)\setminus b}(x)}{\mu_{P(G\setminus a,b)}(x)} \quad \text{(by our inductive hypothesis).} \qquad (1)
\end{aligned}$$

Now, since

$$P(G \setminus a, a) \setminus b = \bigcup_{c\sim b, c\neq a} P(G \setminus ab, c), \text{ and } P(G, a) \setminus a = \bigcup_{c\sim a} P(G \setminus a, c),$$

we have

$$\mu_{P(G,a)\setminus a}(x) = \prod_{c\sim a} \mu_{P(G\setminus a,c)}(x).$$

Let ab be a vertex in $P(G,a)$ that corresponds to a path from a to b. Then

$$\begin{aligned}
(P(G,a)\setminus a)\setminus ab &= \left(\bigcup_{c\sim a, c\neq b} P(G\setminus a,c)\right) \cup \left(\bigcup_{c\sim b, c\neq a} P(G\setminus ab,c)\right),\\
&= \left(\bigcup_{c\sim a, c\neq b} P(G\setminus a,c)\right) \cup \left(P(G\setminus a,b)\setminus b\right).\\
\Rightarrow \mu_{P(G,a)\setminus a)\setminus ab}(x) &= \left(\prod_{c\sim a, c\neq b} \mu_{P(G\setminus a,c)}(x)\right) \mu_{P(G\setminus a,b)\setminus b}(x)\\
\Rightarrow \frac{\mu_{P(G,a)\setminus a)\setminus ab}(x)}{\mu_{P(G,a)\setminus a}(x)} &= \frac{\left(\prod_{c\sim a, c\neq b} \mu_{P(G\setminus a,c)}(x)\right) \mu_{P(G\setminus a,b)\setminus b}(x)}{\prod_{c\sim a} \mu_{P(G\setminus a,c)}(x)}\\
&= \frac{\mu_{P(G\setminus a,b)\setminus b}(x)}{\mu_{P(G\setminus a,b)}(x)}.
\end{aligned}$$

This expression when substituted in Equation (1) yields

$$\begin{aligned}
\frac{\mu_G(x)}{\mu_{G\setminus a}(x)} &= x - \sum_{b\sim a} \frac{\mu_{P(G,a)\setminus a)\setminus ab}(x)}{\mu_{P(G,a)\setminus a}(x)}\\
&= \frac{x\mu_{P(G,a)\setminus a}(x) - \sum_{b\sim a} \mu_{P(G,a)\setminus a)\setminus ab}(x)}{\mu_{P(G,a)\setminus a}(x)}\\
&= \frac{\mu_{P(G,a)}(x)}{\mu_{P(G,a)\setminus a}(x)}.
\end{aligned}$$

□

Theorem 7.1.13. *Let G be graph and let $a \in V(G)$. Then $\mu_G(x) \mid \mu_{P(G,a)}(x)$.*

Proof. Let $|G| = n$. We will induct on n to establish the theorem. When $n \leq 2$, G is a path three, and the assertion holds. Now, for $b \sim a$ let us assume $\mu_{G\setminus a}(x)$ divides $\mu_{P(G\setminus a)}(x)$ (this is true for any $b \in V(G\setminus a)$, but we are interested only in the case when $b \sim a$).
Since

$$P(G,a)\setminus a = \bigcup_{b\sim a} P(G\setminus a, b),$$

we have

$$\mu_{P(G,a)\setminus a}(x) = \prod_{b\sim a} \mu_{P(G\setminus a,b)}(x).$$

This implies that for any $b \sim a$, $\mu_{P(G\setminus a,b)}(x) \mid \mu_{P(G,a)\setminus a}(x)$, and so we have that

$$\mu_{G\setminus a}(x) \mid \mu_{P(G,a)\setminus a}(x).$$

In other words, $\left(\frac{\mu_{P(G,a)\setminus a}(x)}{\mu_{G\setminus a}(x)}\right)$ is a polynomial in x, which implies $\left(\frac{\mu_{P(G,a)}(x)}{\mu_G(x)}\right)$ is a polynomial in x. Thus, $\mu_G(x) \mid \mu_{P(G,a)}(x)$, as desired. □

The following is a consequence of Theorems 7.1.8, 7.1.12, and 7.1.13.

Theorem 7.1.14. *(Godsil) Let $P(G, u)$ be a path tree of G. Then the matching polynomial of G divides the characteristic polynomial of the adjacency matrix of $P(G, u)$. In particular, all the roots of the $\mu_G(x)$ are real and have absolute value at most $\rho(P(G, u))$.*

Theorem 7.1.15. *For every graph G of maximum degree d, all the roots of the $\mu_G(x)$ have absolute value at most $2\sqrt{d-1}$.*

Proof. Fix $a \in V(G)$ be the root vertex, let $|P(G, a)| = n$, and let $A(P(G, a)) = A_P$. It is apparent that the maximum degree of $P(G, a)$ is d. We define a height function $h : V(P(G, a)) \longrightarrow \mathbb{N} \cup \{0\}$ by $h(w) = \text{dist}(w, v)$, for all $w \in V(P(G, a))$. Let D be a diagonal matrix of order of n with

$$D_{ii} = (\sqrt{d-1})^{h(i)}.$$

Then

$$DA_PD^{-1} = \begin{bmatrix} \frac{D_{11}a_{11}}{D_{11}} & \frac{D_{11}a_{12}}{D_{22}} & \cdots & \frac{D_{11}a_{1n}}{D_{nn}} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \frac{D_{nn}a_{11}}{D_{11}} & \frac{D_{nn}a_{12}}{D_{22}} & \cdots & \frac{D_{nn}a_{nn}}{D_{nn}} \end{bmatrix},$$

where $A_p = (a_{ij})_{n \times n}$.

There are three types of vertices in a tree, namely the root vertex, leaf vertices, and intermediate vertices. Let j^{th} row corresponds to the root vertex a. Since the degree of $P(G, a)$ is at most, d, we have $D_{jj} = 1$ and $a_{jk} = 1$, for at most d values of k. Hence, we have

$$(DA_PD^{-1})_{jk} = \frac{1}{\sqrt{d-1}},$$

for at most d values of k. Therefore, for $d \geq 2$,

$$\sum_{k=1}^{n} (DA_PD^{-1})_{jk} \leq \frac{d}{\sqrt{d-1}} \leq 2\sqrt{d-1}.$$

Again, let i^{th} row correspond to one of the leaf vertices, and let $h(i) = k$. As one is the only nonzero entry of the i^{th} row of A_P, it follows that $(\sqrt{d-1})^{k-(k-1)} = \sqrt{d-1}$ is the only nonzero entry of the i^{th} row of DA_PD^{-1}.

The case of intermediate vertices is a combination of above two cases, i.e., the row corresponding to any intermediate vertex in the matrix DA_PD^{-1} has one entry equal to $\sqrt{d-1}$, and up to $(d-1)$ entries equal to $\frac{1}{\sqrt{d-1}}$. Thus, the maximum row sum of the matrix $DA_PD^{-1} = 2\sqrt{d-1}$.

Next, we claim that eigenvalues of a non-negative matrix are bounded above in absolute value by its maximum row sum. We know that the spectral radius $\rho(A_P)$ is the norm of the adjacency operator. That is,

$$\rho(A_P) = \sup\{\|A_P x\|_2 \mid \|x\|_2 = 1\}.$$

Let d_i denotes the row sum of the i^{th} row of the matrix A_P, and let $\lambda = \max\{d_i | 1 \leq i \leq n\}$. Let $x = \left(\frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}}\right)$. Then $A_P x = \left(\frac{d_1}{\sqrt{n}}, \cdots, \frac{d_n}{\sqrt{n}}\right)$, and so we have

$$\begin{aligned}
\|A_P x\|_2 &= \sqrt{\sum_{i=1}^{n} \frac{d_i^2}{n}} \\
&\leq \sqrt{\frac{\lambda^2 n}{n}}, \\
&= \lambda \\
\Rightarrow \rho(A_P) &\leq \lambda.
\end{aligned}$$

We have proved above that the maximum row sum of the matrix DA_PD^{-1} is $2\sqrt{d-1}$, which tells us that $\rho(DA_PD^{-1}) = \rho(A_P) \leq 2\sqrt{d-1}$. Thus, all the roots of $\mu_G(x)$ have absolute value at most $2\sqrt{d-1}$. □

Definition 7.1.16. Given a graph G and a vertex a of G, its *universal cover* T is a graph whose vertices correspond to non-backtracking walks in G that start at a. Two vertices in T are adjacent if one walk extends the other by one vertex.

Remark 7.1.17. Since in a graph there can be infinitely many non-backtracking walks starting from a fixed vertex, the universal cover T in an infinite tree with infinite symmetric adjacency matrix A_T.

Lemma 7.1.18. *Let G be a graph, and let T be its universal cover. Then the roots of $\mu_G(x)$ are bounded in absolute value by $\rho(T)$.*

Proof. Let $a \in V(G)$. Note that $P(G, a)$ is a finite induced subgraph of T, and A_P is a finite submatrix of A_T. By Theorem 7.1.14, we know that the roots of μ_G are bounded by $\rho(P(G, a))$. Therefore, we have

$$\begin{aligned}
\rho(P(G, a)) = \|A_P\|_2 &= \sup_{\|x\|_2 = 1} \|A_P x\|_2 \\
&\leq \sup_{\|y\|_2 = 1, supp(y) \subset P} \|A_T y\|_2 \\
&\leq \sup_{\|y\|_2 = 1} \|A_T y\|_2 = \rho(T).
\end{aligned}$$

□

Let $G = (V, E)$ be a finite graph with $|E| = m$. Let $E = \{e_1, e_2, \cdots, e_m\}$ be an arbitrary ordering. Let s be a signing of these edges, and let A_S be the corresponding adjacency matrix. We define $f_s(x) := \det(xI - A_s)$ to be the characteristic polynomial of A_s. This leads us to the following.

Theorem 7.1.19. *Let $\mathbb{E}$ denote the expected value of a random variable. Then*

$$\mathbb{E}_{s\in\{+1,-1\}^m}[f_s(x)] = \mu_G(x). \tag{7.1.2}$$

Proof. By the Leibniz formula, we have

$$\begin{aligned}\mathbb{E}(\det(xI - A_s)) &= \mathbb{E}\left[\sum_{\sigma\in Sn}(-1)^{sgn(\sigma)}x^{|a:\sigma(a)=a|}\prod_{a:\sigma(a)\neq a}(-A_s(a,\sigma(a))\right] \\ &= \sum_{\sigma\in Sn}(-1)^{sgn(\sigma)}x^{|a:\sigma(a)=a|}\mathbb{E}\left(\prod_{a:\sigma(a)\neq a}(-A_s(a,\sigma(a))\right).\end{aligned}$$

So, for a fixed $\sigma \in S_n$,

$$\mathbb{E}\left(\prod_{a:\sigma(a)\neq a}(-A_s(a,\sigma(a))\right) = \frac{\prod_{a:\sigma(a)\neq a}(-A_1(a,\sigma(a)) + \cdots + \prod_{a:\sigma(a)\neq a}(-A_{2^m}(a,\sigma(a))}{2^m}.$$

We now claim that the only way we can get a non-zero contribution in the expectation is when the following holds:

(i) $(a, \sigma(a)) \in E(G)$, and

(ii) $\sigma(\sigma(a)) = a$.

By the definition of A_s, it is apparent that the first condition is necessary. To see the necessity of the second condition, we first note that for any $\sigma \in S_n$ and $a \in V(G), \mathbb{E}(A_s(a,\sigma(a)) = 0$. Since both -1 and +1 appear 2^{m-1} times, the numerator sum equals to zero. Now, if $\sigma = (a_1 \cdots a_k)$, then $\mathbb{E}(A_s(a_1,a_2)\cdots A_s(a_k,a_1)) = \mathbb{E}(A_s(a_1,a_2))\cdots\mathbb{E}(A_s(a_k,a_1)) = 0$. Thus, the only permutations contributing non-trivially to the sum are the involutions. As we have already seen in Theorem 7.1.8, these correspond exactly to the matchings in the graph.

□

Remark 7.1.20. We can rewrite Equation 7.1.2 as $\sum_s f_s(x) = 2^m\mu_G(x)$. As $\mu_G(x)$ has real roots, $\sum_s f_s(x)$ has real roots.

7.2 Interlacing Families

Definition 7.2.1. We say that a polynomial $g(x) = \prod_{i=1}^{i=n-1}(x-\alpha_i)$ with real roots *interlaces a polynomial* $f(x) = \prod_{i=1}^{i=n}(x-\beta_i)$ if

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \cdots \leq \alpha_{n-1} \leq \beta_n.$$

Example 7.2.2. Note that a polynomial with real roots, and its derivative, interlace.

Definition 7.2.3. We say that polynomials $f_1, f_2, \cdots, f_k$ have a *common interlacing* there is a polynomial g that interlaces each of f_i.

Lemma 7.2.4. *Let $f_1, f_2, \cdots, f_k$ be polynomials of degree n with real roots and positive leading coefficients, and define*

$$f_0 := \sum_{i=1}^{k} f_i.$$

If $f_1, f_2, \cdots, f_k$ have a common interlacing, then there exists an i for which the largest root of f_i is at most the largest root of f_0.

Proof. Let g be a polynomial that interlaces all of the f_i, and let α_{n-1} be the largest root of g. First, note that the polynomial f_0 has real roots, since at each root of g, either $f_0 \leq 0$, or $f_0 \geq 0$. As each f_i has a positive leading coefficient, it is positive for sufficiently large x. As each f_i is non-positive at α_{n-1}, f_0 is also non-positive at α_{n-1} and will eventually becomes positive. This tells us that f_0 has a root β_n such that $\beta_n \geq \alpha_{n-1}$. As f_0 is the sum of the f_i, there must be some i for which $f_i(\beta_n) \geq 0$. Thus, the largest root of f_i is at least α_{n-1} and at most β_n. $\square$

Remark 7.2.5. Using similar arguments, we can show that the conclusion of Lemma 7.2.4 also holds for the k^{th} largest root.

Remark 7.2.6. Note that the assumption of common interlacing in Lemma 7.2.4 is crucial to conclude that the sum f_0 has real roots. For example, consider the polynomial $(x+1)(x+2)+(x-1)(x-2)$. It does not have real roots, while the polynomials $(x+1)(x+2)$ and $(x-1)(x-2)$ both have real roots. Even if the sum of two polynomials has real roots, the conclusion of Lemma 7.2.4 may fail to hold if we do not assume a common interlacing. For example, consider the polynomials $(x+5)(x-9)(x-10)$ and $(x+6)(x-1)(x-8)$. The largest root of the sum is 7.4 which is less than the largest roots of both the polynomials.

Definition 7.2.7. Let $S_1, S_2, \cdots, S_m$ be finite sets, and for every assignment $s_1, \cdots, s_m \in S_1 \times \cdots \times S_m$, let $f_{s_1,s_2,\cdots,s_m}(x)$ be a polynomial of degree n with real roots and positive leading coefficient. For $s_1, \cdots, s_k \in S_1 \times \cdots \times S_k$ with $k < m$, we define

$$f_{s_1,\cdots,s_k} := \sum_{s_{k+1}\in S_{k+1},\cdots,s_m\in S_m} f_{s_1,\cdots,s_k,s_{k+1},\cdots,s_m}$$

and

$$f_0 := \sum_{s_1\in S_1,\cdots,s_m\in S_m} f_{s_1,\cdots,s_m}.$$

We say that the polynomials $\{f_{s_1,\cdots,s_m}\}_{s_1,\cdots,s_m}$ form an *interlacing family* if for all $k < m$ and all $s_1, \cdots, s_k \in S_1 \times \cdots \times S_k$, the polynomials

$$\{f_{s_1,\cdots,s_k,t}\}_{t\in S_{k+1}}$$

have a common interlacing.

Theorem 7.2.8. *Let $S_1, \cdots, S_m$ be finite sets, and let $\{f_{s_1,\cdots,s_m}\}_{s_1,\cdots,s_m}$ be an interlacing family of polynomials. Then, there exists some $s_1, \cdots, s_m$ so that the largest root of the polynomial $f_{s_1,\cdots,s_m}$ is at most the largest root of the polynomial f_0.*

Proof. We note that $f_0 = \{f_t\}_{t \in S_1}$. By definition, we know that $\{f_t\}_{t\in S_1}$ has a common interlacing. By Lemma 7.2.4, we know that there exists an $i \in S_1$ such that f_i has its largest root to be no greater than the largest root of f_0. Similarly, note that $f_i = \{f_{i,t}\}_{t\in S_2}$, and there exists an $j \in S_2$ such that $f_{i,j}$ has its largest root bounded above by the largest root of f_i. Proceeding inductively, for any $s_1, \cdots, s_k$, we have that the polynomials $\{f_{s_1,\cdots,s_k,t}\}_{t\in S_{k+1}}$ have a common interlacing, and $f_{s_1,\cdots,s_k} = \{f_{s_1,\cdots,s_k,t}\}_{t\in S_{k+1}}$. By Lemma 7.2.4, there exists an $s_{k+1} \in S_{k+1}$ such that the largest root of the polynomial $f_{s_1,\cdots,s_k,s_{k+1}}$ is bounded above by the largest root of $f_{s_1,\cdots,s_k}$. □

We want to prove that the polynomials $\{f_s\}_{s\in\{\pm1\}^m}$ defined in Section 7.1 form an interlacing family. According to Definition 7.2.4, this requires establishing that the existence of certain common interlacings. We circumvent this by showing that for a set of polynomials $\{f_1, f_2, \cdots, f_k\}$ of degree n, having a common interlacing is equivalent to requiring that $\sum_{i=1}^k f_i$ has real roots.

Definition 7.2.9. Let $f_1(x), \cdots, f_k(x)$ be polynomials in one variable with real coefficients. We say they are *compatible* if for all $c_1, \cdots, c_k \geq 0$, all the roots of the polynomial $\sum_{i=1}^k c_i f_i(x)$ are real.

For a polynomial f, let $n_f(X)$ denote the number of real roots of $f(x)$ that lie in the interval $[x, \infty)$ (counted with their multiplicities). It is can be shown that for two compatible polynomials f, g with positive leading coefficients, $|n_f(x) - n_g(x)| \leq 1$ for all $x \in \mathbb{R}$.

Lemma 7.2.10. *Let $f_1, \cdots, f_k$ be univariate polynomials of degree n with positive leading coefficients. Then $f_1, \cdots, f_k$ have a common interlacing if and only if $\sum_{i=1}^k \lambda_i f_i$ has real roots for all non-negative $\lambda_1, \cdots, \lambda_k$.*

Proof. The proof of forward implication is straightforward. Conversely, suppose $\sum_{i=1}^k \lambda_i f_i$ has real roots for all non-negative $\lambda_1, \cdots, \lambda_k$. To prove $f_1, \cdots, f_k$ have a common interlacing, note that it is suffices to prove that for all s, t such that $1 \leq s < t \leq k$, the polynomials f_s, f_t have a common interlacing.

For $1 \leq i \leq k$, let $(r_1^i, \cdots, r_n^i)$ be the root sequence of f_i. We define intervals $I_1^i, \cdots, I_{n+1}^i$, as follows: $I_1^i = [r_1^i, \infty), I_{n+1}^i = (-\infty, r_n^i]$, and $I_j^i = [r_j^i, r_{j-1}^i]$, for $2 \leq j \leq n$. Next, for $1 \leq s < t \leq k$, to show that the polynomials f_s, f_t have a common interlacing, it suffices to prove that for $1 \leq j \leq n+1$, $I_j^s \bigcap I_j^t$ is nonempty. Suppose we assume on the contrary. Let j be the smallest possible index such that $I_j^s \cap I_j^t = \phi$. Since the leading coefficients of both f_s and f_t are positive, we have $j \geq 2$. Assuming $r_{j-1}^s \leq r_{j-1}^t$, we see that r_j^t exists and $r_{j-1}^s < r_j^t$. But then $n_{f_t}(r_j^t) = j$ and $n_{f_s}(r_j^t) \leq j - 2$, which is a contradiction, and so the assertion follows. □

7.3 Infinite Families of Regular Bipartite Ramanujan Graphs

In this section, we prove that the characteristic polynomials $\{f_s\}_{s\in\{\pm 1\}^m}$ form an interlacing family. Using this we prove that for every d-regular graph, there exists a signing such that all of the new eigenvalues are at most $2\sqrt{d-1}$, which is a variant of what was conjectured by Bilu and Linial [1]. The following theorem says that if the sign of each edge is chosen independently, each with its own probability, then the resulting polynomial has real roots.

Theorem 7.3.1. *The polynomial*

$$\sum_{s\in\{\pm 1\}^m}\left(\prod_{i:s_i=1} p_i\right)\left(\prod_{i:s_i=-1}(1-p_i)\right) f_s(x)$$

has real roots for all values of $p_1,\cdots,p_m \in [0,1]$.

Theorem 7.3.2. *The polynomials* $\{f_s\}_{s\in\{\pm 1\}^m}$ *form an interlacing family.*

Proof. By Lemma 7.2.10, it is sufficient to show that for $0 \le k \le m-1$, the polynomial

$$\lambda f_{s_1,\cdots,s_k,1}(x) + (1-\lambda) f_{s_1,\cdots,s_k,-1}(x)$$

has real roots, for each partial assignment $s_1,\ldots,s_k$ and every $\lambda \in [0,1]$. By Theorem 7.3.1, we know that

$$\sum_{s\in\{\pm 1\}^m}\left(\prod_{i:s_i=1} p_i\right)\left(\prod_{i:s_i=-1}(1-p_i)\right) f_s(x)$$

has real roots for all values of $p_1,\cdots,p_m \in [0,1]$. So, we want to choose the $p_i's$ such that we get the coefficients of $f_{-1}=0$ in the sum. Putting $p_2=\lambda, p_3,\cdots,p_m = 1/2$, and $p_1 = (1+1)/2 = 1$, we get the desired result. Similarly, for any partial assignment $s_1,\cdots,s_k$, this follows by putting $p_{k+1}=\lambda, p_{k+2},\cdots,p_m = 1/2$, and $p_i = (1+s_i)/2$ for $1 \le i \le k$. □

Theorem 7.3.3. *Let G be a graph with adjacency matrix A and universal cover T. Then there exists a signing s of A so that all of the eigenvalues of A_s are at most $\rho(T)$. In particular, if G is d-regular, there is a signing s so that the eigenvalues of A_s are at most $2\sqrt{d-1}$.*

Proof. By Theorems 7.3.2 and 7.2.8, there exists a signing $s \in \{\pm 1\}^m$ such that the largest root of the polynomial f_s is bounded above by the largest root of the characteristic polynomial f_0. Moreover, by Theorem 7.1.19, we have $f_0 = 2^m \mu_G(x)$. Thus, by Theorem 7.1.14 and Lemma 7.1.18, we get that the roots of f_s are no larger than $\rho(T)$. The second statement follows form Theorem 7.1.15. □

Lemma 7.3.4. *Every nontrivial eigenvalue of a complete (c,d)-biregular graph is zero.*

Proof. As adjacency matrix of such a graph has rank 2, there can be at most two non-zero eigenvalues, and those are $\pm\sqrt{cd}$, the trivial eigenvalues. $\square$

Theorem 7.3.5. *For every $d \geq 3$, there exists an infinite sequence of d-regular bipartite Ramanujan graphs.*

Proof. By Lemma 7.3.4, we know that a complete d-regular bipartite graph G is Ramanujan. By Theorem 7.3.3, there exists a 2-lift of G such that all the new eigenvalues are bounded above by $2\sqrt{d-1}$, and we know the spectrum of a bipartite graph is symmetric about 0, so all the non-trivial eigenvalues of this 2-lift lie in $[-2\sqrt{d-1}, 2\sqrt{d-1}]$. As a 2-lift of a d-regular bipartite graph is also d-regular and bipartite, we get that the obtained 2-lift of G is also a d-regular bipartite Ramanujan graph. Thus, for every d-regular bipartite Ramanujan graph, there is another d-regular bipartite Ramanujan graph with twice as many vertices. Proceeding in this manner, by considering successive 2-lifts, we obtain an infinite family, as desired. $\square$

Remark 7.3.6. By Remark 3.0.11, Theorem 7.3.5 also proves the existence of infinite expander families.

Remark 7.3.7. Since every connected 2-regular graph of order n is C_n, Theorem 7.3.5 does not hold for $d = 2$.

Definition 7.3.8. We say a *(c, d)-biregular graph is Ramanujan* if all of its non-trivial eigenvalues have absolute value at most $\sqrt{c-1} + \sqrt{d-1}$.

Theorem 7.3.9. *There exists an infinite sequence of (c, d)-biregular bipartite Ramnujan graphs for all $c, d \geq 3$.*

Proof. Let G be any (c, d)-biregular bipartite Ramanujan graph (for example, complete (c, d)-biregular graph). By Theorem 7.3.3, we know that there is a 2-lift of G such that all the new eigenvalues are bounded above by $\rho(T) = \sqrt{c-1} + \sqrt{d-1}$. As a 2-lift is also (c, d)-biregular bipartite, all the non-trivial eigenvalues of the 2-lift have absolute value at most $\sqrt{c-1} + \sqrt{d-1}$. Therefore, the resulting graph is a larger (c, d)-biregular bipartite Ramanujan graph. By applying Theorem 7.3.3 repeatedly, we can construct an infinite sequence of (c, d)-biregular bipartite Ramnujan graphs for all $c, d \geq 3$. $\square$

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