AN INTRODUCTION TO MAPPING CLASS GROUPS OF SURFACES

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VIDIT DAS

(14193)

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH BHOPAL
BHOPAL - 462066

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Indian Institute of Science Education and Research
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ABSTRACT

Let S be a closed orientable surface of genus $g \geq 1$. The mapping class group of S, denoted by $\mathrm{Mod}(S_g)$, is the group of isotopy classes of orientation-preserving self diffeomorphisms of S_g which are identity on the boundary and preserve the set of punctures. We start by introducing some basic properties of $\mathrm{Mod}(S)$ followed by some explicit computation of the group for some surfaces such as a closed disk, the sphere, etc.. Then we discuss some fundamental examples of infinite-order elements in $\mathrm{Mod}(S_g)$, known as *Dehn twists*. Further introducing the representation $\mathrm{Mod}(S_g) \longrightarrow \mathrm{Sp}(2g, \mathbb{Z})$ afforded by the natural action of $\mathrm{Mod}(S_g)$ on $\mathrm{H}_1(S_g, \mathbb{Z})$, we conclude the project by showing that the kernel of this representation namely the *Torelli group*, is torsion-free.

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1. INTRODUCTION

Let S be a compact, connected, orientable surface of genus $g \geq 1$, then the *mapping class group* of a surface S, denoted by $\mathrm{Mod}(S)$ is the group of all the path components of the orientation-preserving homeomorphisms which are identity on the boundary and preserve the set of punctures. The added conditions may change in different studies. The mapping class group is also commonly known as the homeotopy group [18]. These groups play a central role in the field of Geometric topology, particularly in the areas of Riemann surface theory, Moduli space theory, Teichm*ü*ller theory, and Theory of 3-manifolds. Fricke [11] called this group the "Automorphic Modular Group," as it is the generalisation of the classical modular group $\mathrm{SL}(2, \mathbb{Z})$.

The study of mapping class groups of surfaces started in the 1920s, with Dehn's [1–3] and Nielsen's [4–6] work. Dehn's work found a natural continuation in the works of Harvey [7, 8] in the late 1970s. The works of Nielsen were extended and brought to a complete form by Thurston in his theory of surface diffeomorphisms [9, 10].

The primary reference for this project was [11]. The main references for hyperbolic geometry, which was a prerequiste for this project, are [13, 15]. The notes by Massuyeau [24] was also a helpful reference.

2. PRELIMINARIES

In this chapter, we would be discussing some preliminary notions of algebraic topology which we will be using.

2.1 Surfaces

A *surface* S is defined to be a 2-manifold. We look at it as a 2-dimensional submanifold embedded in a 3-dimensional Euclidean space. There are instances where these cannot be embedded in 3-dimensions and we require higher dimensions.

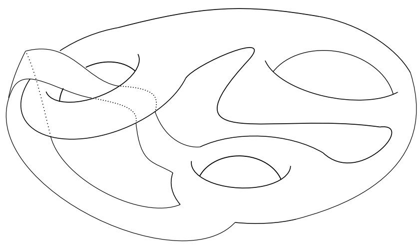

Fig. 2.1: A closed surface.

Given the above notion of a surface, we perceive a closed surface to be a complex mathematical object floating in space like some crumble cuboid with a lot of holes, punctures and cusps. The figure demonstrates a complicated closed surface.

We know from a result proved by Radò [26] that every compact surface admits a triangulation which led to one of the most fundamental results in surface theory.

Theorem 2.1 (Classification theorem of surfaces)**.** *Any closed, connected, orientable surface is homeomorphic to the connected sum of a 2-dimensional sphere with $g \geq 0$ tori. Any compact, connected, orientable surface is obtained from a closed surface by removing $b \geq 0$ open disks with disjoint closures. The set of* *homeomorphism types of compact surfaces is in bijective correspondence with the set* $\{(g, b) : g, b \geq 0\}$.

Here, the g stands for the *genus* of the surface and b stands for the number of *boundary components. Punctures* are basically points removed from the interior of the surface and are often denoted by marked points. We keep track of the position of these points and often switch notations between marked points and punctures. Thus, any surface can be characterised by the triple (g, b, n). Following this we denote any surface as $S_{g,b,n}$. Throughout this project, will denote the boundary of the surface S by ∂S, and the surface $S_{g,0,0}$ simply by S_g.

Thus, the surface in Figure 2.1 is equivalent to the surface in Figure 2.2 below.

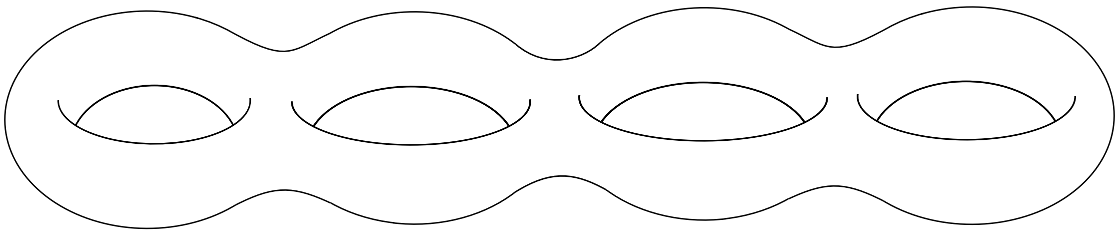

Fig. 2.2: The same closed surface in Figure 2.1.

Now, we will discuss about an invariant which plays an important role in the classification of surfaces. The *Euler characteristic of a surface* $S = S_{g,b,n}$ is given by

$$\chi(S) = 2 - 2g - (b + n),$$

which remains invariant under homeomorphisms. It turns out that the value of $\chi(S)$ determines the intrinsic geometry of S. If the value of $\chi(S)$ is positive, then the geometry on S is spherical, if it is zero, then it is Euclidean, and if it is negative, then the metric on S would be hyperbolic 2.2. We will focus on hyperbolic surfaces.

2.2 Curves on surfaces

Let S be a connected, orientable surface. We know that a continuous mapping

$$S^1 \xrightarrow{\gamma} S,$$

is called a *closed curve*. It is called *simple* when we do not have any self intersections in the loop besides the base point.

Given an oriented closed curve $\gamma \subset S$, we have a bijective correspondence:

$$\left\{\begin{array}{c}\textit{Non-trivial conjugacy}\\ \textit{classes in } \pi_1(S)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{Non-trivial}\\ \textit{free homotopy classes of}\\ \textit{oriented closed curves in } S\end{array}\right\}.$$

By a *free homotopy class* we mean homotopy of loops without considering the base point. Since our surface is connected, we will be able to conjugate the free homotopy loops to the fixed homotopy loops by the path joining the base point to a point in the free homotopy loop. In other words, a loop based at x_1 can be seen as a loop based at x_0 by conjugating it with the path connecting x_0 and x_1. Thus, if homotopic we get the correspondence. Moreover, we have another bijective correspondence:

$$\left\{\begin{array}{c}\textit{elements of the conjugacy}\\ \textit{class of } \gamma \in \pi_1(S)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{lifts of these curves}\\ \textit{free homotopy classes of}\\ \textit{to the universal cover of } S\end{array}\right\}.$$

These topological curves on the surface are relevant to geometers as well. In dimensions, 2 and 3 topology and geometry go hand in hand. For instance, there is a bijective correspondence between the conjugacy classes in $\pi_1(S)$ and the oriented geodesics in S. We implicitly use the Uniformisation theorem [27] to prove Theorem 2.3.

Theorem 2.2 (Uniformisation theorem). *Every simply connected Riemann surface is conformally equivalent to the unit disk, the complex plane, or the Riemann sphere.*

Theorem 2.3. *Let S be a hyperbolic surface. If γ be a closed curve in S which is not homotopic to a neighbourhood of a puncture, then γ is homotopic to a unique geodesic closed curve α.*

Proof (a sketch). Take a simple closed curve in S, say γ. Lift it to the universal cover which is $\mathbb{H}^2$ in this case. Choose a lift of γ and see that in $\mathbb{H}^2$ it would be a curve from boundary to boundary. Thus, it would be homotopic to a geodesic in $\mathbb{H}^2$. Thus, we push the homotopy down to S to obtain our desired result. □

While studying loops on a surface, it suffices to consider the simple closed curves as these are primitive in the space of all closed curves. We broadly classify simple closed curves on a surface S into two different classes.

Definition 2.4. We refer to a simple closed curve as *separating* if cutting the surface along the curve breaks the surface into connected components and if cutting the surface along the curve does not result in breaking the surface into connected components we call it *nonseparating*.

Definition 2.5. An *essential simple closed curve* is a simple closed curve which is not homotopic to a point or into a boundary component or into a neighbourhood of a puncture.

Let S_g be a closed surface of genus g. Then we have that $\mathrm{H}_1(S_g) \cong \mathbb{Z}^{2g}$ is generated by the latitudinal and longitudinal curves around the genera. In practice, we write every curve in terms of the g generators of $\mathbb{Z}^g$. An example would be $S_{2,0,0}$ the double torus. It has four curves generating its homology group namely,

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

which are represented by the curves in the Figure 2.3. Later on we will realise more of their importance. Also, notice that we cannot write the separating curve in terms of these.

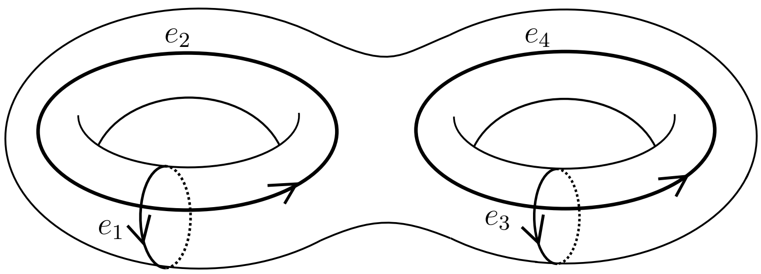

Fig. 2.3: A basis of $H_1(S_2)$.

2.3 Bigon Criterion

Definition 2.6. Let α and β be two closed curves in S. Let the free homotopy class of α be a and that of β be b. We define the minimum number of intersection points between the representatives of the classes a and b, i.e.,

$$i(a, b) = \min\{|\alpha \cap \beta| : \alpha \in a \text{ and } \beta \in b\}$$

to be the ***geometric intersection number*** of a and b.

Remark 2.7. Observe that geometric intersection number is always non-negative. Moreover, for any homotopy class a we have $i(a, a) = 0$.

Remark 2.8. The representative curves $\alpha \in a$ and $\beta \in b$ are said to be in *minimal position* if $i(a, b) = i(\alpha, \beta)$.

The word *bigon* stands for a polygon with two sides. The presence of bigons reduces intersections between curves. Figure 2.4 give an illustration of this phe-

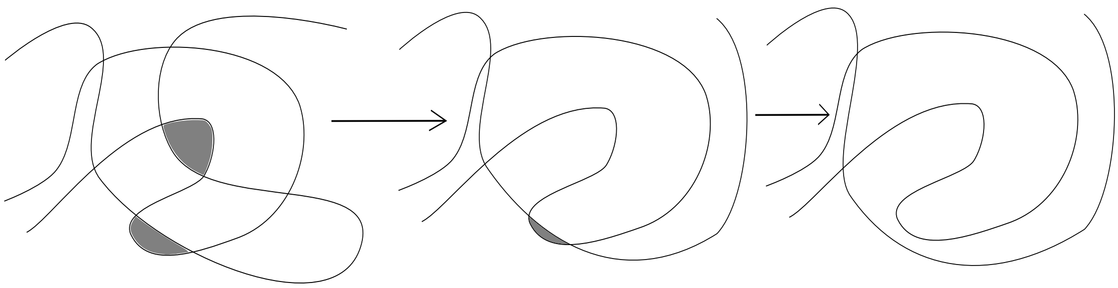

Fig. 2.4: Reduction of intersections by sliding across bigons.

nomenon. Formalizing this idea, we have:

Theorem 2.9 (Bigon criterion)**.** *Two transverse simple closed curves in a surface S are in minimal position if and only if they do not form a bigon.*

Proof:(a sketch). We find the formed bigons and reduce the intersections by contracting the embedded disks in between the arms of the bigons as shown in the Figure 2.4. □

Corollary 2.10. *Distinct, simple closed geodesics in a hyperbolic surface are in minimal position.*

2.4 Change of co-ordinates principle

A homeomorphism takes a separating curve to a separating curve and a nonseparating curve to a nonseparating one. This is because, if we cut along a curve, the number of connected components remain invariant under a homeomorphism. The change of co-ordinates principle tells us that there is a unique nonseparating curve up to homeomorphism. Therefore, on a surface S there is only one nonseparating simple closed curve up to homeomorphism. We will be using this principle often in arguments later on.

Theorem 2.11. *There is an orientation-preserving homeomorphism of a surface taking one simple closed curve to another if and only if the corresponding cut surfaces (which may be disconnected) are homeomorphic.*

To demonstrate the usage of the principle let us look at an example.

Example 2.12. Consider the surface S_2, and let us look at the curve γ on S_2 shown in the Figure 2.5. We are to find an essential simple closed curve in S_2 such that the geometric intersection number of γ and our desired curve is 1. Since it is separating, by change of coordinates principle we know that there is a homeomorphism of $S_{2,0,0}$ which takes γ to one of the known separating curves, and now by the Figure 2.5 it makes the problem easier. Evidently, there are none.

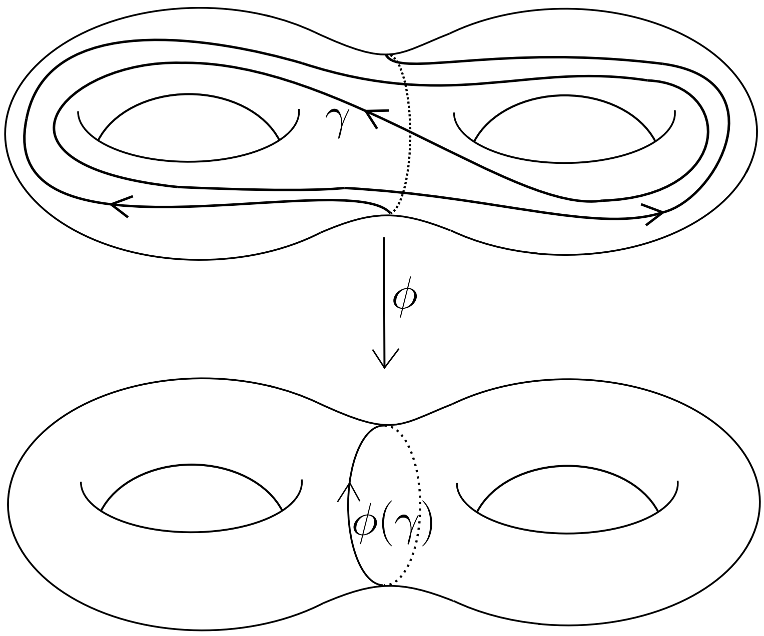

Fig. 2.5: Illustration of the change of coordinates principle.

2.5 Some Important Results

In this section, we will be stating some results that are central to the theory of mapping class groups.

Definition 2.13. Let ϕ and ψ be homeomorphisms of S then they are said to be *isotopic* if there is a homotopy

$$H : S \times [0,1] \longrightarrow S,$$

from ϕ to ψ with the property that, for every $t \in [0,1], H(x,\{t\})$ is a homeomorphism. The homotopy here is called the isotopy of homeomorphisms. Similar notions of isotopy of curves, loops and points can be defined.

In 1928, Baer [17, 18] related the homotopy and isotopy of essential simple closed curves.

Theorem 2.14 (Baer)**.** *Let α and β be two essential simple closed curves in a surface S. Then α is isotopic to β if and only if α is homotopic to β.*

Proof. By definition, an isotopy is a homotopy making the forward implication trivial. For the other case, we will be proving it for the genus $g \geq 2$ case. Let us assume that α is homotopic to β, therefore we have that $i(a, b) = 0$, where a is the class of α and b is the class of β. Now, α can be isotoped to a curve which is transverse to β. Abusing notation let us call this curve also to be α. Now, if α and β are not disjoint then by the bigon criterion, they will form a bigon. So, by shrinking the bigon we can reduce the intersection between them, and we can continue this process to the stage where α and β are at the minimal position, that is they are disjoint.

Now, we choose lifts $\hat{\alpha}$ and $\hat{\beta}$ of α and β respectively onto the universal cover of S, $\mathbb{H}^2$ such that they have same end points in $\partial\mathbb{H}^2$.

Moreover, $\exists$ a hyperbolic isometry ϕ, acting by translation on all lifts keeping these invariant. Further quotient $R' = R/\langle\phi\rangle$ gives an annulus. The image R'' of R in S is a further quotient of R'. However, since the covering map $R' \longrightarrow R''$ is single-sheeted on the boundary, it follows that $R' \approx R''$. So, the annulus R'' between α and β gives an isotopy. $\square$

Moreover, if we have an isotopy of simple closed curves, we can extend that isotopy to an isotopy of the surface.

Theorem 2.15 (Isotopy Extension Theorem)**.** *Let S be any surface. If*

$$F : S^1 \times I \longrightarrow S,$$

is a smooth isotopy of simple closed curves, then there is an isotopy

$$H : S \times I \longrightarrow S,$$

so that $H|_{S\times 0}$ is the identity and $H|_{F(S^1\times\{0\})\times I} = F$.

The next natural question is whether the result by Baer can be extended in the case of homeomorphisms.

Theorem 2.16. *Let S be any compact surface and let f and g be homotopic homeomorphisms of S. Then f and g are isotopic unless they are one of $S = D^2$*

and $S = A$. In particular, if f and g are orientation-preserving, then they are isotopic.

In 1950, Munkres [19] proved a remarkable result relating surface homeomorphisms to diffeomorphisms. The idea was to take a sequence of homeomorphism converging to a diffeomorphism and using this sequence as an isotopy.

Theorem 2.17 (Munkres)**.** *Let S be a compact surface. Then every homeomorphism of S is isotopic to a diffeomorphism of S.*

This theorem is often used to jump from the category of homeomorphism to the category of diffeomorphisms. Lastly, we conclude this chapter by stating the following result,

Theorem 2.18. *Let S be a compact surface, possibly minus a finite number of points from the interior. Assume that S is not homeomorphic to $S^2, \mathbb{R}^2, D^2, T^2$, the closed annulus, the once-punctured disk, or the once-punctured plane. Then the space $Homeo_0(S)$ is contractible.*

3. MAPPING CLASS GROUPS

In this chapter, we define the mapping class groups of surfaces and compute some explicit examples like the closed disk, the sphere with punctures and the torus.

3.1 Mapping Class Group

Definition 3.1. Let S be a surface and $\mathrm{Homeo}(S)$ be the group of all homeomorphisms of $S \longrightarrow S$. Then $\mathrm{Homeo}^+(S, \partial S)$ is the subgroup of $\mathrm{Homeo}(S)$ containing all orientation-preserving homeomorphisms of S to S which are identity on the boundary and preserve the set of punctures.

Remark 3.2. $\mathrm{Homeo}^+(S, \partial S)$ forms a group under composition and also notice that it forms a topological group under the compact-open topology.

Following this remark, we have that the notion of path-components in $\mathrm{Homeo}^+(S, \partial S)$ make sense, and here, the paths are isotopies of homeomorphisms. Thus we attempt to formalise the description of the group given before as follows:

Definition 3.3. Let S be a connected, orientable surface. The *mapping class group* of S, $\mathrm{Mod}(S)$ is defined to be the group of the path components of $\mathrm{Homeo}^+(S, \partial S)$, i.e.,

$$\mathrm{Mod}(S) = \pi_0(\mathrm{Homeo}^+(S, \partial S)).$$

Let $\mathrm{Homeo}_0(S, \partial S)$ be the path component of identity in $\mathrm{Homeo}^+(S, \partial S)$ then equivalently,

$$\begin{aligned}\mathrm{Mod}(S) &= \mathrm{Homeo}^+(S, \partial S)/\mathrm{Homeo}_0(S, \partial S)\\ &= \mathrm{Homeo}^+(S, \partial S)/\sim,\end{aligned}$$

where " $\sim$ " is the isotopy relation. Furthermore, by Theorem 2.17, we have that,

$$\mathrm{Mod}(S) = \mathrm{Diff}^{+}(S, \partial S)/\sim .$$

The elements of the mapping class group are called *mapping classes*. In the study of mapping classes we often treat the boundaries and punctures as the same while dealing with essential curves, though they are fundamentally different objects. A boundary is a result of removing an open disk from the interior of the surface whereas a puncture is a result of removing just a singleton point which is closed, from the surface. Further, the representatives of the mapping classes are allowed to permute the punctures but fix the boundary pointwise. We notice that although not drastically different they have subtle differences which make them largely different.

There are some examples like the annulus where homotopy is not the same as isotopy and thus the different definitions of $\mathrm{Mod}(S)$ give rise to unequivalent notions of the group. In the next section, we will see some computations of the mapping class group and we will lay down some foundational objects in the process.

3.2 Some explicit computations of mapping class groups

The first example we look at is that of a closed disk.

Theorem 3.4 (Alexander's Trick)**.** *The space* $\mathrm{Homeo}^{+}(D^2)$ *is contractible. In particular, we have* $\mathrm{Mod}(D^2) = \{1\}$.

Proof. Choose a representative ϕ of a non-trivial mapping class f of D^2 embedded in $\mathbb{R}^2$.

$$F(x,t) = \begin{cases} (1-t)\phi(\frac{x}{1-t}), & 0 \le |x| < 1-t, \text{ and} \\ x, & 1-t \le |x| \le 1, \end{cases}$$

for $0 \le t < 1$, and we define $F(x,1)$ to be the identity map of D^2. Therefore, F is an isotopy (Figure 3.1) from ϕ to identity, and a choice of ϕ is arbitrary, any

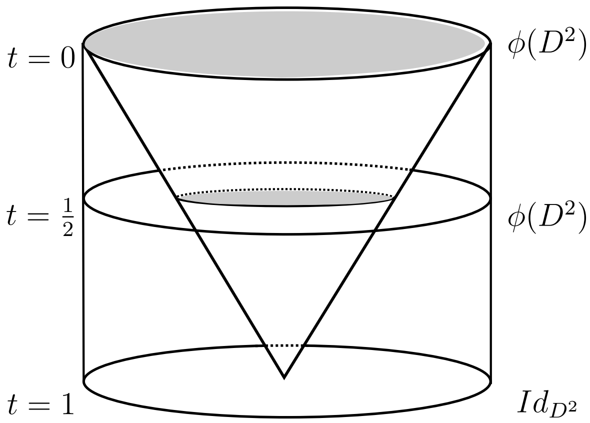

Fig. 3.1: Pictorial representation of the isotopy: The grey shaded area represents the part of the disk where ϕ acts nontrivially which is continuously shrunk to a point.

homeomorphism (in $\text{Homeo}^+(D^2, S^1)$) of the disk is isotopic to the identity.

□

Remark 3.5. The same homotopy works with the once-punctured disk. Since there is only one puncture it is identity there and by pasting lemma, we have that the mapping class group for the once-puncture disk is also trivial. So is the case for the once-punctured sphere, as it is contractible to a point.

Theorem 3.6. *The mapping class group of the* $2-$*sphere is trivial.*

Proof. Let $S^2 = S_{0,0,0}$ be the 2-sphere. Let f be a representative of a mapping class of S^2. Let γ be an oriented simple closed curve on S^2. But S^2 is simply connected, therefore $f(\gamma)$ is isotopic to γ. Now if we cut through $f(\gamma)$ then we get a disk with $f(\gamma)$ as its boundary. But, by Theorem 3.4 $\text{Mod}(D^2)$ is trivial. So, $f|_{(D^2, f(\gamma))}$ is isotopic to the identity of D^2. Thus, we have that the isotopy which takes $f(\gamma)$ to γ can be extended using the isotopy extension leaving us with an isotopy which takes f to Id_{S^2}. □

Lemma 3.7. *Any two essential simple proper arcs in* $S_{0,0,3}$ *with the same endpoints are isotopic. Any two essential arcs that both start and end at the same marked point of* $S_{0,0,3}$ *are isotopic.*

Proof (a sketch). If α and β are two essential simple closed curves in $S_{0,0,3}$ connecting two of the marked points. Remove the third marked point, and the resultant space becomes a disk with 2 punctures which are connected by α and β, then by the Theorem 2.4 we can find a disk embedded in between them and contract it. Thus α is isotopic to β. $\square$

Theorem 3.8. *The mapping class group of the thrice-punctured sphere is the symmetry group of three elements.*

Proof. Let ϕ be a representative of a mapping class of $\mathrm{Mod}(S_{0,0,3})$. Consider α to be a simple closed arc connecting two of the punctures. Now, by the definition of ϕ it preserves punctures and thus, $\phi(\alpha)$ is also a simple closed arc connecting the punctures. Now, if we cut through $\phi(\alpha)$, then we get a disk with a puncture and boundary $\phi(\alpha)$. But, $\mathrm{Mod}(D^2) = \{1\}$, therefore, any homeomorphism there is isotopic to the identity and is identity at the boundary. So, we extend this isotopy taking $\phi(\alpha)$ to α to the surface, and we are done. $\square$

The same proof also works for the twice punctured case. So, we conclude that:

Theorem 3.9. *Let S be the 2-sphere, and n denote the number of punctures on S the sphere then, for $n \leq 3$ we have that $Mod(S) = \Sigma_n$.*

An alternative way to look at the case of the thrice-punctures sphere case is through Möbius transformations. As 3 points are sufficient to completely characterise such a transformation. The fixing of the set of punctures gives rise to atmost 3 combinations giving rise to the permutation group.

Our first example of a mapping class group of infinite order is the annulus ($S^1 \times [0,1] \approx S_{0,2,0}$)

Theorem 3.10. *The mapping class group of the annulus is $\mathbb{Z}$.*

Proof. Let $f \in \mathrm{Mod}(A)$ and ϕ be a orientation preserving homeomorphism which is represented by f. The universal cover of A is the infinite strip $\tilde{A} \approx \mathbb{R} \times [0,1]$, and ϕ has a preferred lift $\tilde{\phi} : \tilde{A} \longrightarrow \tilde{A}$ which fixes the origin. Let $\phi|_{\mathbb{R}\times\{1\}} = \tilde{\phi}_1 : \mathbb{R} \longrightarrow \mathbb{R}$. Identifying $\mathbb{Z}$ to the group of integer translations of $\mathbb{R}$ we get that, $\tilde{\phi}_1$ is an integer translation. Define

$$\rho : \mathrm{Mod}(A) \longrightarrow \mathbb{Z},$$

such that, $\rho(f) = \tilde{\phi}_1$.

The map sends the mapping class $[f]$ to the number k such that $[-x + f(x)] = k \cdot a$, where a is a group isomorphism. The matrix,

$$M = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

induces a linear transformation ($\mathbb{R} \longrightarrow \mathbb{R}$) which preserves $\mathbb{R} \times [0, 1]$. This transformation is equivariant with respect to the deck transformations forcing the restriction of M on $\mathbb{R} \times [0, 1]$ to descend to a homeomorphism at the surface level. By the way we have constructed the map, we have $\rho([\phi]) = n$, giving us surjectivity.

Let $f \in \mathrm{Mod}(A)$ be an element of the kernel of ρ. Further, let us assume ϕ to be a homeomorphism representing f having a preferred lift $\tilde{\phi}$. As $\rho(f) = 0$ we have that $\tilde{\phi}$ acts as identity on $\partial\tilde{A}$.

Therefore, we have that the straight line homotopy descends to a homotopy between ϕ and Id_A fixing the boundary pointwise. Thus, f is identity and hence ρ is injective. $\square$

In this proof, we have implicitly used the fact that the straight line homotopy from $\tilde{\phi}$ to $Id\big|_{\tilde{A}}$ is equivariant.

Remark 3.11. The twist maps which generate $\mathrm{Mod}(A)$ are called *Dehn twists.* We will study Dehn twists in detail in Section 4.1.

Definition 3.12. Let α and β be two oriented simple closed curves in minimal position. We associate to each intersection of α and β an index $+1$ (resp. -1) if the orientation of the tangents to the two curves at the point of intersection agrees (resp. does not agree) with the orientation of the surface. The *algebraic intersection number* $\hat{i}(\alpha, \beta)$ is defined to be the sum of the indices taken over all intersections of α and β. Further, if a denotes the isotopy class of α and b denotes the same of β then we define

$$\hat{i}(a, b) := \min(\hat{i}(\alpha, \beta)).$$

Remark 3.13. The algebraic intersection number remains invariant under orientation-preserving homeomorphisms.

Theorem 3.14.

$$\mathrm{Mod}(S_1) \cong \mathrm{SL}(2, \mathbb{Z}).$$

Proof. Firstly we define ϕ to be the map

$$\mathrm{Mod}(S_1) \xrightarrow{\phi} \mathrm{SL}(2, \mathbb{Z}),$$

such that, if f is a representative of the mapping class $[f] \in \mathrm{Mod}(S_{1,0,0})$, then f induces a map at the homology level

$$(\mathbb{Z}^2 \cong)\mathrm{H}_1(S_1) \xrightarrow{f_*} \mathrm{H}_1(S_1)(\cong \mathbb{Z}^2),$$

and thus $[f] \overset{\phi}{\longmapsto} f_*$. But, f preserves the orientation and the intersection number and thus tells us that f_* we have that $\phi([f]) \in \mathrm{SL}(2, \mathbb{Z})$. So we have that,

$$\phi([f]) = \begin{pmatrix} \hat{\imath}(f_*(a), b) & \hat{\imath}(f_*(b), b) \\ \hat{\imath}(-f_*(a), a) & \hat{\imath}(-f_*(b), a) \end{pmatrix},$$

where $\hat{\imath} : \mathrm{H}_1(S_g; \mathbb{Z}) \times \mathrm{H}_1(S_g; \mathbb{Z}) \longrightarrow \mathbb{Z}$ is the algebraic intersection number.

Now we check the injectivity and surjectivity of the map. The surjectivity of ϕ can be proved as follows. We realise $S^1 \times S^1$ as $\mathbb{R}^2/\mathbb{Z}^2$, in such a way that the loop $S^1 \times \{1\}$ lifts to $[0,1] \times \{0\}$ and $\{1\} \times S^1$ lifts to $\{0\} \times [0,1]$. Any matrix $T \in \mathrm{SL}(2, \mathbb{Z})$ defines a linear homeomorphism $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, which leaves $\mathbb{Z}^2$ globally invariant and so induces an (orientation-preserving) homeomorphism $f : \mathbb{R}^2/\mathbb{Z}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$. It is easily checked that $\phi([f]) = T$.

To prove the injectivity, let us consider a homeomorphism $f : T^2 \longrightarrow T^2$ such that $\phi([f])$ is trivial. Since $\pi_1(T^2)$ is abelian, this implies that f acts trivally at the level of the fundamental group. The canonical projection $\mathbb{R}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$ gives the universal covering of $S^1 \times S^1 \approx T^2$. Thus, f can be lifted to a unique homeomorphism $\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(0) = 0$ and, by assumption on f, f is $\mathbb{Z}$-equivariant. Therefore, the "affine" homotopy

$$\begin{aligned} H : \mathbb{R}^2 \times [0,1] &\longrightarrow \mathbb{R}^2, \text{ such that,} \\ (x,t) &\longmapsto t \cdot f(x) + (1-t) \cdot x, \end{aligned}$$

between $Id_{\mathbb{R}^2}$ and f, descends to a homotopy from Id_{T^2} to f. Since homotopy coincides with isotopy in dimension two, we deduce that $[f] = 1 \in \phi(T^2)$.

$\square$

Remark 3.15. The mapping class group of the torus with a puncture is also isomorphic to the $\mathrm{SL}(2, \mathbb{Z})$.

4. DEHN TWISTS

Recall that in Theorem 3.10 for computing the mapping class group of the annulus we generated by twist maps known as Dehn twists. Max Dehn in 1938 [20] was the first one to give the idea of a Dehn twists. He showed these simplest infinite order elements of the mapping class groups generate the mapping class group of a surface. In this chapter, we will be studying the properties of Dehn twists.

4.1 Dehn twists

Definition 4.1. Let S be a connected oriented surface. Every simple closed curve in S has a regular annular neighbourhood in S. Let α be a simple closed curve in S and let N be the annular neighbourhood. Choose a homeomorphism from A to N, and call it ϕ. The *Dehn twist* about α is given by

$$T_\alpha : S \longrightarrow S,$$

such that

$$T_\alpha(x) = \begin{cases} \phi \circ T \circ \phi^{-1}, \text{if } x \in N, \text{ and} \\ x, \qquad\qquad\quad \text{if } x \in S \setminus N, \end{cases}$$

where $T : A \longrightarrow A$ such that, $(\theta, t) \mapsto (\theta + 2\pi t, t)$.

There are several ways of looking at a Dehn twists namely, via cutting and gluing, via inclusion and via surgery. We will briefly discuss these perspectives.

- **Cutting and Gluing:** Given a simple closed curve γ, in our surface S. We cut along the curve, then twist the curve by integral rotation. After that we glue back the curve. Since, the twist is a complete twist the gluing requires no extra details. This is what we term as a cutting and gluing realisation of the Dehn Twist. Figure 4.1 gives an illustration of the cutting and gluing perspective.

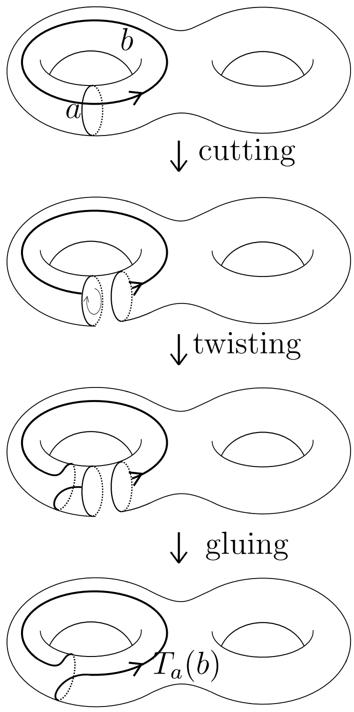

Fig. 4.1: Cutting and gluing after twisting.

- **Inclusion:** Given a surface S, by the Collar neighbourhood theorem [28] we can find an annular neighbourhood of a curve. The inclusion of annulus induces the homomorphism $\mathrm{Mod}(A) \longrightarrow \mathrm{Mod}(S)$, and the image of the generator of $\mathrm{Mod}(A)$ is a Dehn twist in $\mathrm{Mod}(S)$.

- **Surgery:** With some abuse of notation let us consider that a is a representative of the isotopy class of simple closed curves a, and similarly b be a representative of b. Now, $T_b(a)$ will be twisting a along b. In other words, we will be cutting the intersection point. This is illustrated in the Figure 4.2.

These are all well-established ways to view a Dehn twist, but it does depend on the parametrisation t and the choice of the simple closed curve α. Thus, a Dehn twist about α is not a well-defined element in the $\mathrm{Homeo}^+(S)$ group, but it is well

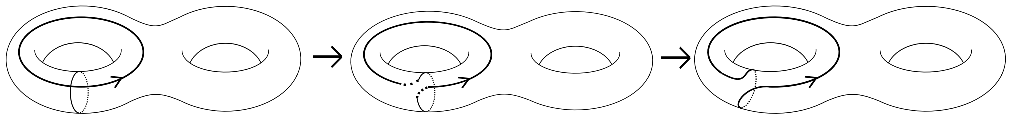

Fig. 4.2: Realizing a Dehn twist using surgery.

defined in Mod(S). We now show that Dehn twists are non-trivial infinite order elements in Mod(S).

Theorem 4.2. *Dehn twists are non-trivial elements in the mapping class group of a surface.*

Proof. Let b be a nonseparating simple closed curve on the surface. By the change

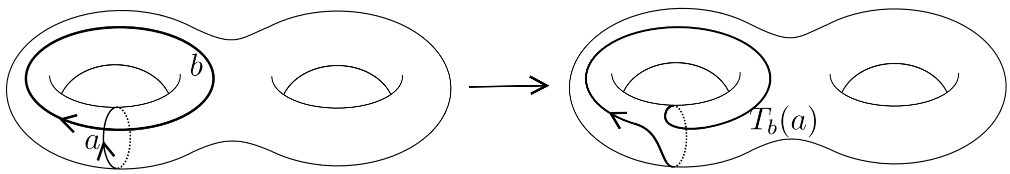

Fig. 4.3: A Dehn twist along a nonseparating curve in S_2.

of coordinates principle we can bring b to a homology generator say b_1. Further we can always find a curve a_1 such that $i(a_1, b_1) = 1 \Rightarrow i(a_1, b) = 1$. Thus, we see that $i(T_b(a), a) = 1$ where $i(a, a) = 0$. Thus, it is a non-trivial element. The argument

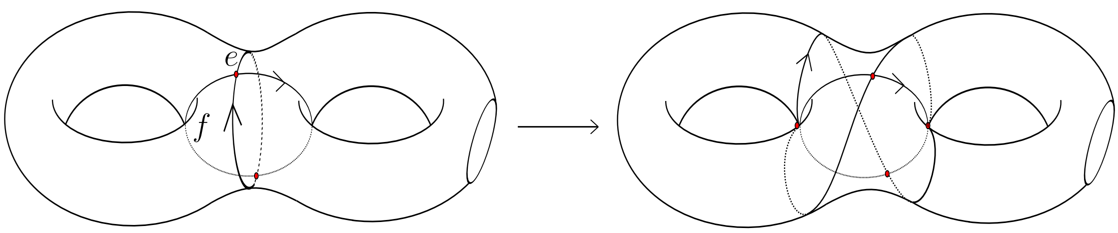

Fig. 4.4: A twist along a separating curve in S_2.

for the separating curve is similar (see Figure 4.4). □

4.2 Properties of Dehn twists

Proposition 4.3. *Let a and b be arbitrary isotopy class of essential simple closed curves in a surface and $k \in \mathbb{Z}$, then,*

$$i\left(T_a^k(b), b\right) = |k| i(a,b)^2.$$

Proof. We choose minimal representatives of a and b, say α and β respectively, then we take $ki(a,b)$ parallel copies of β with one copy of α and then perform surgery to perform the twists, then by counting we get the result. □

Corollary 4.4. *Dehn twists are infinite order elements in $Mod(S)$.*

Proposition 4.5. *Let a and b be unoriented isotopy classes of simple closed curves on a surface then*

$$T_a = T_b \Leftrightarrow a = b.$$

Proof. Since Dehn twists are well-defined we have that if $a = b$ then $T_a = T_b$. On the contrary, assume that $a \neq b$. By Theorem 2.11, we have that there is a simple closed curve c such that $\hat{i}(a,c) = 0$ and $\hat{i}(b,c) \neq 0$. Now if $\hat{i}(a,b) \neq 0$, then choose $b = c$. If that is not the case, we can again use change of coordinates principle to find c. Thus, we get that,

$$i\left(T_a(c), c\right) = i(a,c)^2 = 0 \neq i(b,c)^2 = i\left(T_b(c), c\right),$$

$\Rightarrow T_a \neq T_c \Rightarrow T_a \neq T_b$. Thus, we get a contradiction. □

Proposition 4.6. *For any $f \in Mod(S)$ and any isotopy class of simple closed curves a in S we have*

$$T_{f(a)} = f T_a f^{-1}.$$

Proof. Let ϕ denote a representative of f, let α denote a representative of a, and let ψ_α denote a representative of T_a whose support is an annulus. Note that ϕ^{-1} takes a regular neighbourhood of $\phi(\alpha)$ to a regular neighbourhood of α (preserving the orientation), then ψ_α twists the neighbourhood of α, and ϕ takes this twisted neighbourhood of α back to a neighbourhood of $\phi(\alpha)$ (again preserving the orientation). So the net result is a Dehn twist about $\phi(\alpha)$. □

Proposition 4.7. *For any $f \in Mod(S)$ and any isotopy class a of simple closed curves in S, we have*

$$f \text{ commutes with } T_a \iff f(a) = a.$$

Proof. We have that,

$$\begin{aligned} fT_a = T_a f &\iff fT_a f^{-1} = T_a \\ &\iff T_{f(a)} = T_a \\ &\iff f(a) = a. \end{aligned}$$

□

Proposition 4.8. *For any two isotopy classes a and b of simple closed curves in a surface S, we have*

$$i(a,b) = 0 \iff T_a(b) = b \iff T_aT_b = T_bT_a. \tag{4.1}$$

Proof. If $T_a(b) = b$, then $i(T_a(b), b) = i(b,b) = 0$. So, $i(T_a(b), b) = i(a,b)^2$, and it follows that $i(a,b) = 0$. □

Theorem 4.9. *If a and b are such that $i(a,b) = 1$ then,*

$$T_aT_bT_a = T_bT_aT_b. \tag{4.2}$$

This is called the braid relation.

Proof. Note that

$$T_aT_bT_a = T_bT_aT_b \Rightarrow (T_aT_b)\, T_a\, (T_aT_b)^{-1} = T_b.$$

But by Proposition 4.6 we have that $T_{T_aT_b(a)} = T_b$. So it reduces to checking that if $i(a,b) = 1$ then $T_aT_b(a) = b$. So, by change of coordinates principle stated in Theorem 2.11 we can choose the isotopy class representatives to be a and b in Figure 4.5. So, by the illustration in Figure 4.5, we are done. □

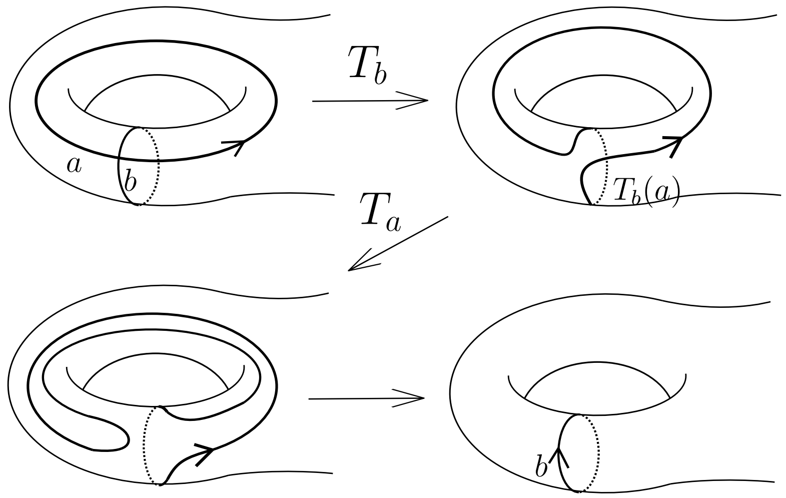

Fig. 4.5: An illustration of the braid relation.

4.3 Groups generated by two Dehn twists

Recall that a Dehn twist is an infinite order element in the mapping class group. So, a group generated by a Dehn twist is a free abelian group generated by the twist. But, what about the group generated by two Dehn twists. As discussed above we see that there may be relations between two Dehn twists. Thus, the groups generated depend on the isotopy classes of the curves along which the twists are defined. Again, if they do not have any geometric intersection, then it would be the free abelian group generated by the twists. This brings us to the following lemma.

Lemma 4.10 (Ping Pong lemma)**.** *Let G be a group acting on a set X. Let $g_1, \ldots, g_n$ be elements of G. Suppose that there are nonempty, disjoint subsets $X_1, \ldots, X_n$ of X with the property that, for each i and each $j \neq i$, we have $g_i^k(X_j) \subset X_i$ for every nonzero integer k. Then the group generated by the g_i is a free group of rank n.*

Proof. Consider a freely reduced word g starting with and ending with a nontrivial power of g_1. Then for every $x \in X_2$ we have that $g(x) \in X_1$, but $X_1 \cap X_2 = \emptyset$, so $g(x) \neq x$.

Since any other freely reduced word in the g_i is conjugate to a word that starts and ends with g_1, every freely reduced word in the g_i represents an element of G that is conjugate to a nontrivial element and hence is itself nontrivial. □

With this helpful lemma in place, we return to our discussion of the group generated by two Dehn twists.

Theorem 4.11. *Let a and b be two isotopy classes of simple closed curves in a surface S. If $i(a, b) \geq 2$, then the group generated by T_a and T_b is isomorphic to the free group F_2 of rank 2.*

Proof. Let us assume a and b are two isotopy classes of oriented simple closed curves having $i(a, b) \geq 2$. Let $F = \langle T_a, T_b \rangle$. Let X be the set of isotopy classes of simple closed curves in S.

Define X_a and X_b as follows,

$$\begin{aligned} X_a &= \{c \in X : i(c, b) > i(c, a)\} \neq \emptyset \\ X_b &= \{c \in X : i(c, a) > i(c, b)\} \neq \emptyset \\ X_a \cap X_b &= \emptyset. \end{aligned}$$

By the Lemma 4.10 it suffices to show that $T_a^k(X_b) \subset X_a$ for $k \neq 0$. Therefore,

$$\left| i\left(T_a^k(c), b\right) - \right| k|i(a, b)i(a, c)| \leq i(b, c),$$

and so

$$-i(b, c) \leq i\left(T_a^k(c), b\right) - |k|i(a, b)i(a, c) \leq i(b, c). \tag{4.3}$$

If $c \in X_b$, then $i(a, c) > i(b, c)$. Since $k \neq 0$, the inequality in 4.3 implies

$$\begin{aligned} i\left(T_a^k(c), b\right) &\geq |k|i(a, b)i(a, c) - i(b, c) \\ &\geq 2|k|i(a, c) - i(b, c) \\ &> 2|k|i(a, c) - i(a, c) \end{aligned}$$

$$\begin{aligned} &= (2|k|-1)i(a,c) \\ &\geq i\left(T_a^k(a), T_a^k(c)\right) \\ &= i\left(a, T_a^k(c)\right). \end{aligned}$$

Thus $i(T_a^k(c), b) > i(T_a^k(c), a)$, and so $T_a^k(c) \in X_a$. Hence the proof. $\square$

Thus we have that the group formed by two Dehn twists can be categorised by the intersection number of the curves along which the twist takes place.

5. SYMPLECTIC REPRESENTATION

In this chapter we discuss the natural action of $\mathrm{Mod}(S_g)$ on $\mathrm{H}_1(S_g : \mathbb{Z})$, given by $[f] \cdot \alpha = [f_*(\alpha)]$, which yields a representation into the integral symplectic group. We conclude by showing that the kernel of this representation is torsion-free.

5.1 Symplectic vector spaces

A *symplectic structure* on a vector space is a structure induced by a skew-symmetric alternating bilinear form. Let $(a_1, b_1, a_2, b_2, \ldots, a_g, b_g)$ be a basis for the vector space $\mathbb{R}^{2g}$. Now define our form to be as follows:

$$\omega = \sum_{i=1}^{g} da_i \wedge db_i.$$

Given two vectors $v = (v_1, w_1, \ldots, v_g, w_g)$ and $v' = \left(v'_1, w'_1, \ldots, v'_g, w'_g\right)$ in $\mathbb{R}^{2g}$, we compute

$$\omega\left(v, v'\right) = \sum_{i=1}^{g} \left(v_i w'_i - v'_i w_i\right).$$

The vector space formed by putting the above structure is called the *symplectic vector space* of order g, formally defined as

$$\mathrm{Sp}(2g, \mathbb{R}) = \left\{A \in \mathrm{GL}(2g, \mathbb{R}) : A^{\star}\omega = \omega\right\}.$$

Moreover, this form is given by a matrix J, so this vector space can be also looked at as,

$$\mathrm{Sp}(2g, \mathbb{R}) = \left\{A \in \mathrm{GL}(2g, \mathbb{R}) : A^T J A = J\right\},$$

where,

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$

In the case of surfaces, we always look at the generators of the first homology group as the basis and try to associate them to the bases of the vector spaces. Thus, we shift out interest from the *symplectic group* to the *integral symplectic group* which is

$$\mathrm{Sp}(2g, \mathbb{Z}) = \mathrm{Sp}(2g, \mathbb{R}) \cap \mathrm{GL}(2g, \mathbb{Z}) \subset \mathrm{SL}(2, \mathbb{Z}).$$

Burkhardt in 1890 [23], gave the explicit generators for the integral symplectic group and showed that it is finitely generated. To see this, we should have a basic idea of the elements in the space.

Theorem 5.1. *Let $A \in \mathrm{Sp}(2g, \mathbb{R})$, then λ is an eigenvalue of A if and only if λ^{-1} is also an eigenvalue of A. Further, the* elementary symplectic matrices *of the integral symplectic group are the matrices of the form*

$$SE_{ij} = \begin{cases} I_{2g} + e_{ij}, & \text{if } i = \sigma(j), \text{ and} \\ I_{2g} + e_{ij} - (-1)^{i+j} e_{\sigma(j)\sigma(i)}, & \text{otherwise,} \end{cases}$$

where, $i \neq j$ and σ is an element of the symmetry group acting on $2g$ elements transposing every $2i$ and $2i-1$.

Remark 5.2. For the case $g = 1$ we have $\mathrm{Sp}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$.

Theorem 5.3 (Burkhardt)**.** *The following linear maps form a finite generating set for $\mathrm{Sp}(4, \mathbb{Z})$:*

Trasvection

$$(a_1, b_1, a_2, b_2) \mapsto (a_1 + b_1, b_1, a_2, b_2),$$

Factor Rotation

$$(a_1, b_1, a_2, b_2) \mapsto (b_1, -a_1, a_2, b_2),$$

Factor Mix

$$(a_1, b_1, a_2, b_2) \mapsto (a_1 - b_2, b_1, a_2 - b_1, b_2), \ \ \textit{and}$$

Factor Swap

$$(a_1, b_1, a_2, b_2) \mapsto (a_2, b_2, a_1, b_1).$$

For a more general case of $\mathrm{Sp}(2g, \mathbb{Z})$ *we require* 1 *transvection,* 1 *factor rotation,* 1 *factor mix and* $g - 1$ *factor swaps.*

5.2 Algebraic intersection number

In the last section, when we referred to the basis being the generators of the first homology group we see it as the following diagram.

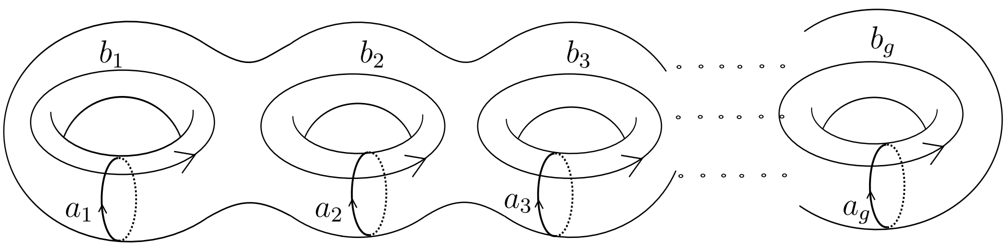

Fig. 5.1: Geometric symplectic basis for $\mathrm{H}_1(S_g, \mathbb{Z})$.

Recall the *algebraic intersection number* defined in Chapter 3. By defintion, it is apparent $\hat{i}(a, b) = -\hat{i}(b, a)$. the algebraic intersection number gives a symplectic form

$$\hat{i} : \mathrm{H}_1\left(S_g;\mathbb{Z}\right) \wedge \mathrm{H}_1\left(S_g;\mathbb{Z}\right) \longrightarrow \mathbb{Z}, \tag{5.1}$$

which further extends to

$$\hat{i} : \mathrm{H}_1\left(S_g;\mathbb{R}\right) \wedge \mathrm{H}_1\left(S_g;\mathbb{R}\right) \longrightarrow \mathbb{R}, \tag{5.2}$$

thereby giving a symplectic structure on $\mathrm{H}_1\left(S_g;\mathbb{R}\right)$. A basis of $\mathrm{H}_1(S_g;\mathbb{Z})$ will be termed as a *geometric symplectic basis.* This symplectic structure can be given as:

$$\hat{i} = \sum_{i=1}^{g} [a_i]^\star \wedge [b_i]^\star \in \triangle^2\left(\mathrm{H}_1\left(S_g;\mathbb{R}\right)^\star\right), \tag{5.3}$$

where $[a_i]^\star$ and $[b_i]^\star$ denote the vectors in $\mathrm{H}_1\left(S_g;\mathbb{R}\right)^\star$ dual to $[a_i]$ and $[b_i]$, respectively.

Example 5.4. Consider the surface S_2. Notice that, $\hat{i}(a_1, b_1) = 1, \hat{i}(b_1, d_1) = -1,$

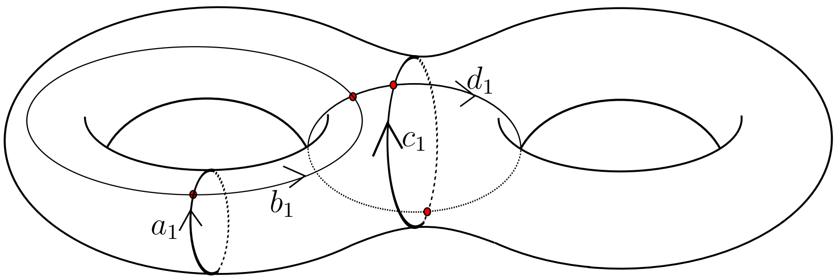

Fig. 5.2: Intersection points between essential simple closed curves in S_2,

and $\hat{i}(c_1, d_1) = 0$. So the algebraic intersection number can be 0 even when the geometric intersection is positive. Thus, in general, $\hat{i}(a, b) \leq i(a, b)$. The matrix J induced by this form would be given by,

$$J_{i,j} = \hat{i}(a_i, b_j).$$

Therefore, for a genus g surface having geometric symplectic basis in Figure 5.1 we have the $2g \times 2g$ matrix J introduced earlier. Moreover, the intersection can also be 0 when the geometric intersection is positive.

5.3 Symplectic Representation

The symplectic form defined in 5.1 leads to the representation,

$$\Psi : \mathrm{Mod}(S_g) \longrightarrow \mathrm{Aut}(\mathrm{H}_1(S_g;\mathbb{Z})),$$

such that $[f] \mapsto f_*$ where $f_* : \mathrm{H}_1(S_g;\mathbb{Z}) \longrightarrow \mathrm{H}_1(S_g;\mathbb{Z})$ is the induced isomorphism. This is well-defined as homotopic maps induce the same automorphisms at the level of homology. Since $\mathrm{H}_1(S_g;\mathbb{Z}) \cong \mathbb{Z}^{2g}$, we can extend this to

$$\Psi : \mathrm{Mod}(S_g) \longrightarrow \mathrm{Aut}(\mathrm{H}_1(S_g;\mathbb{Z})) \xrightarrow{\cong} \mathrm{Aut}(\mathbb{Z}^{2g}) \xrightarrow{\cong} \mathrm{GL}(2,\mathbb{Z}).$$

Moreover, any representative of a mapping class preserves the algebraic intersection number. Therefore, the image of $\mathrm{Mod}(S_g)$ under this representation sits inside the integral symplectic group, in other words,

$$\mathrm{Mod}(S_g) \xrightarrow{\Psi} \mathrm{Sp}(2g;\mathbb{Z}).$$

We shall look at some examples before we proceed. For ease of computation we will be looking at genus 2 surfaces.

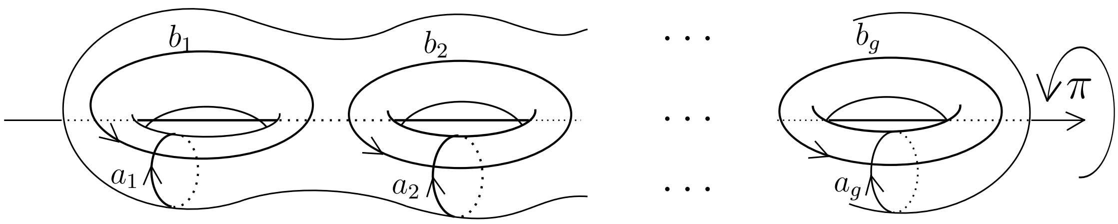

Fig. 5.3: Hyperelliptic involution.

Example 5.5. This hyperelliptic involution indicated in the Figure 5.3 $f \in \mathrm{Mod}(S_g)$ sends $a_i \mapsto -a_i$ and $b_i \mapsto -b_i$ for a genus g surface. This gives us the matrix, Thus, at the homology level we get f_* gives the matrix,

$$\Psi(f) = \begin{pmatrix} -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

This is the negative identity $-I_{2g\times 2g}$. This in fact is a symplectic matrix.

Example 5.6. For the Dehn twist $T_{b_1} \in \mathrm{Mod}(S_2)$ shown in Figure 5.4,

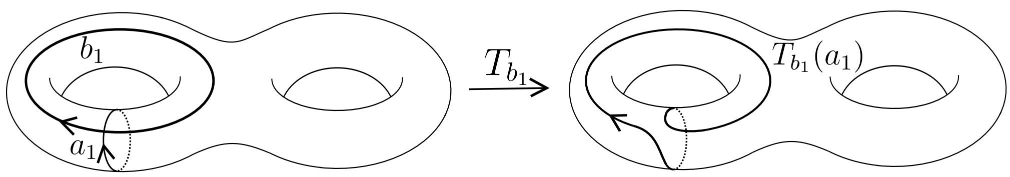

Fig. 5.4: Dehn twist along b_1 in $\mathrm{Mod}(S_2)$.

$$\Psi(T_{b_1}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In the general case of a genus g surface, a Dehn twist along the basis curve b_1 takes $a_1 \mapsto a_1 + b_1$, and the rest remains identity.

Theorem 5.7. *Let a and b be isotopy classes of oriented simple closed curves in S_g. For any $k \geq 0$, we have*

$$\Psi(T_b^k)([a]) = [a] + k \cdot \hat{i}(a,b)[b].$$

Proof. When b is separating, by Theorem 2.11, we have a geometric symplectic basis of $\mathrm{H}_1(S_g;\mathbb{Z})$ such that, $i(a_i,b)=i(b_i,b)=0$, for each i. Therefore, we get

$$\Psi(T_b^k)([a]) = [a] + 0 = [a].$$

If b is nonseparating, again by Theorem 2.11, we have a geometric symplectic basis say $\{a_i,b_i\}_{i=1}^g$ of $\mathrm{H}_1(S_g;\mathbb{Z})$ such that, $b_1 = b$. A direct check gives us,

$$\Psi\left(T_b^k\right)([c]) = \left[T_b^k(c)\right] = \begin{cases} [a_1] + k\,[b_1]\,, & \text{if } c = a_1, \text{ and} \\ [c], & \text{if } c \in \{b_1, a_2, b_2, \ldots, a_g, b_g\}\,. \end{cases}$$

Now let a be the isotopy class of an arbitrary oriented simple closed curve in S_g. The $[a_1]-$coefficient of $[a]$ in the basis $\{[a_i],[b_i]\}$ is $\hat{i}(a,b)$. The assertion now follows from the linearity of Ψ. $\square$

We will now show the surjectivity of $\Psi : \mathrm{Mod}(S_g) \longrightarrow \mathrm{Sp}(2g,\mathbb{Z})$.

Theorem 5.8 (Surjectivity of Ψ)**.** *For genus $g \geq 1$, the representation*

$$\Psi : Mod(S_g) \longrightarrow Sp(2g,\mathbb{Z}),$$

is surjective.

Proof. We give the following Table 5.1 giving the words in Dehn twists in $\mathrm{Mod}(S_g)$ which form the Burkhardt's generators [23] under Ψ.

Transvection	$(a_1,b_1,a_2,b_2) \mapsto (a_1+b_1,b_1,a_2,b_2)$	T_{b_1}
Factor Rotation	$(a_1,b_1,a_2,b_2) \mapsto (b_1,-a_1,a_2,b_2)$	$T_{b_1}T_{a_1}T_{b_1}$
Factor Mix	$(a_1,b_1,a_2,b_2) \mapsto (a_1-b_2,b_1,a_2-b_1,b_2)$	$T_{b_1}^{-1}T_{b_2}^{-1}T_c$
Factor Swap	$(a_1,b_1,a_2,b_2) \mapsto (a_2,b_2,a_1,b_1)$	$(T_{a_{i+1}}T_{b_{i+1}}T_{d_i}T_{a_i}T_{b_i})^3$

Tab. 5.1: The words in Dehn twists realising the Burkhardt generators.

Transvections can be seen to be generated by Dehn twist as given above. We claim that, these generators can be written in terms of Dehn twists.

Let $\{a_i,b_i\}_{i=1}^g$ be the standard geometric symplectic basis for $H_1(S_g,\mathbb{Z})$ (as shown in Figure 5.1), and $c = b_2 - b_1$ and $d_i = a_{i+1} + b_i$.

We obtain Burkhardt's factor rotation generator as follows. Let N be a closed regular neighbourhood of $a_1 \cup b_1$ in S_g. The subsurface N is homeomorphic to

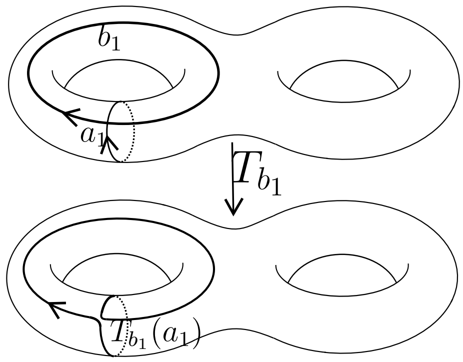

Fig. 5.5: Realising a transvection by T_{b_1} in genus 2.

a torus with one boundary component. We think of N as a square with sides identified and an open disk removed from the center.

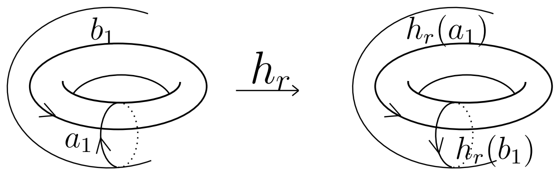

Fig. 5.6: Geometric realisation of factor rotation.

Consider the homeomorphism of N obtained by rotating the boundary of the square by $\frac{\pi}{2}$ and leaving the boundary of N fixed. Extending by the identity map gives a homeomorphism of S_g, hence a mapping class $h_r \in \mathrm{Mod}(S_g)$ called a handle rotation (see Figure 5.6). This handle rotation represents a mapping class which equals the product of Dehn twists:

$$T_{b_1} T_{a_1} T_{b_1}.$$

This tells us $\Psi(h_r)$ is the factor rotation generator.

We next realise Burkhardt's factor mix generator by a mapping class. Consider a closed annular neighborhood of b_1 and push the left-hand boundary component of this annulus along a path in the surface that intersects a_2 once (from the left of a_2) and misses the other curves in the geometric symplectic basis; The resulting mapping class h is called a handle mix (see Figure 5.7).

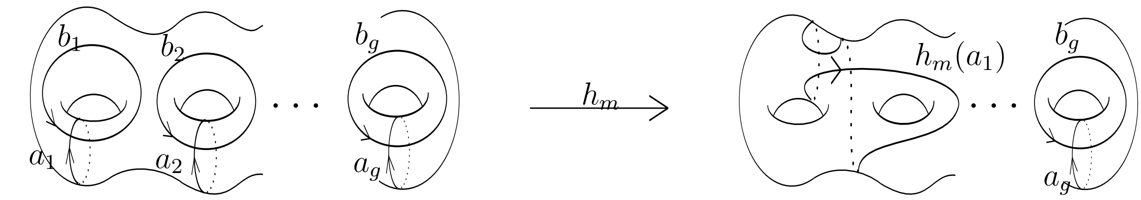

Fig. 5.7: Geometric realsiation of factor mix in a genus g surface: Here, the image of a_1 under h_m is $a_1 - b_2$, similarly, the image of a_2 would be $a_2 - b_1$.

We can also describe h as the mapping class obtained by cutting S_g along b_1, pushing one of the new boundary components through the $(a_2, b_2)-$handle, and then regluing. Alternatively, h_m is a product of three commuting Dehn twists:

$$h_m = T_{b_1}^{-1} T_{b_2}^{-1} T_c,$$

where c is a simple closed curve in the homology class $[b_2] - [b_1]$. A direct check gives that $\Psi(h)$ is Burkhardt's factor mix generator.

Finally, we have Burkhardt's $g - 1$ factor swaps. These are obtained as the images under Ψ of handle swaps. The i^{th} handle swap h_{s_i} for $1 \leq i \leq g - 1$ is

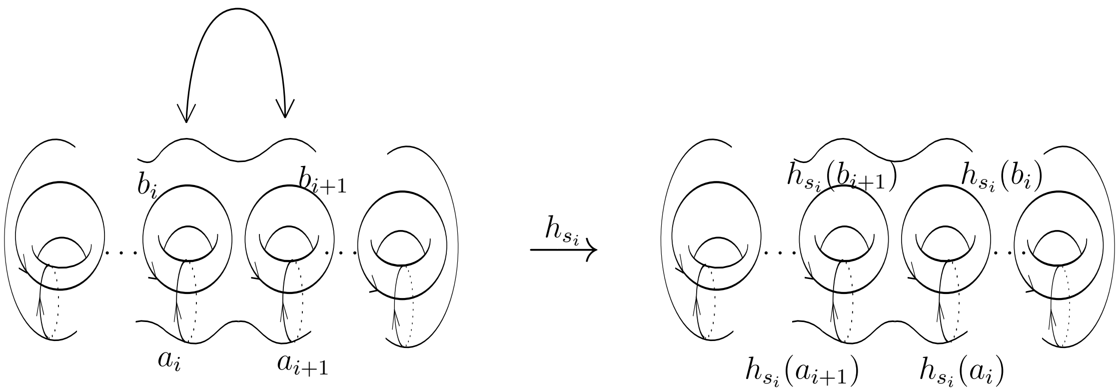

Fig. 5.8: Geometric realisation of Factor swap in a genus g surface: Here we can see that the positions of a_i, b_i is swapped with a_{i+1}, b_{i+1}.

easily visualized (see Figure 5.8), but we can also write it as a product of Dehn twists:

$$h_{s_i} = \left(T_{a_{i+1}} T_{b_{i+1}} T_{d_i} T_{a_i} T_{b_i}\right)^3,$$

where d_i is a simple closed curve in the homology class $[a_i + 1] + [b_i]$.

Therefore each of Burkhardt's generators can be realised in terms of Dehn twists. $\square$

Once we have this representation it is only natural to ask about its kernel. The *Torelli group* ($\mathcal{I}(S_g)$) is a normal subgroup of $\mathrm{Mod}(S_g)$ defined to be the kernel of Ψ, and so we have the exact sequence.

$$0 \longrightarrow \mathcal{I}(S_g) \longrightarrow \mathrm{Mod}(S_g) \xrightarrow{\Psi} \mathrm{Sp}(2g, \mathbb{Z}) \longrightarrow 0.$$

We conclude this chapter by stating these results.

Theorem 5.9. *For $g \geq 1$, $\mathcal{I}(S_g)$ is torsion-free.*

The proof of this result uses some concepts from Riemannian geometry, algebraic topology, and the following result by Fenchel and Nielsen [25],

Theorem 5.10. *Let $S = S_{g,n}$ and suppose $\chi(S) < 0$. If $f \in Mod(S)$ is an element of finite order k, then there is a representative $\phi \in Homeo^+(S)$ so that ϕ has order k. Further, ϕ can be chosen to be an isometry of some hyperbolic metric on S.*

This was further genralised by Kerckhoff saying that every finite subgroup of $\mathrm{Mod}(S)$ comes from a finite subgroup of $\mathrm{Homeo}^+(S)$.

Proof of Theorem 5.9. Let $f \in \mathrm{Mod}(S_g)$ be a finite ordered mapping class group with order n, then there is a diffeomorphism ϕ representing f such that $\phi^n = Id_{S_g}$. The existence of this equality is due to Theorem 5.10. Let h be a Riemannian metric on S_g. Then we have that

$$h + \phi^* h + (\phi^2)^* h + \ldots + (\phi^{n-1})^* h,$$

also forms a metric of S_g such that ϕ is isometric under this metric. If there is a fixed point of ϕ say $x \in S_g$, then the derivative $D\phi_x$ at x is a 2×2 orthogonal matrix. But, it is also orientation preserving, so determinant of $D\phi$ is 1.

Again, ϕ is nontrivial, so $D\phi_x$ is a nontrivial rotation, with x as an isolated fixed point of degree 1. So, by Lefschetz – Hopf formula [14]

$$\begin{aligned} L(\phi) &= \sum_{i=0}^{2} (-1)^i \operatorname{Trace}\left(\phi_\star : H_i\left(S_g;\mathbb{Z}\right) \to H_i\left(S_g;\mathbb{Z}\right)\right) \\ &= 1 - \operatorname{Trace}\left(\phi_\star : H_1\left(S_g;\mathbb{Z}\right) \to H_1\left(S_g;\mathbb{Z}\right)\right) + 1 \\ &= M(\phi), \end{aligned}$$

where $M(\phi)$ is the sum of indexes of fixed points of ϕ. Each fixed point has degree 1, thus $M(\phi) \geq 0$. Now, if ϕ_* is identity then, trace would be at least 4 as we have taken $g \geq 2$. Thus, $\Psi(f) = \phi_* \neq Id_{S_g}$. $\square$

APPENDIX

A. REVIEW OF HYPERBOLIC GEOMETRY

Any topological surface can be endowed with a geometric structure. This means that one can find a metric on the surface which in small regions looks like one of the geometries namely, Euclidean, spherical or hyperbolic such that the 'transition maps' are isometries of the appropriate geometry [15]. Hyperbolic geometry is the study of geometry on Riemann surfaces having constant negative Gaussian curvature [15]. There are four well known models to study this geometry, namely:

1. Upper half plane ($\mathbb{H}^2$),
2. Poincaré disk model ($\mathbb{D}$),
3. Hyperboliod model and
4. Klein model ($\mathbb{K}$).

The upper half plane model is of interest to us.

Definition A.1. Let $\mathbb{C}$ be the complex plane then, we define the upper half plane to be

$$\mathbb{H}^2 := \{z \in \mathbb{C} : Im(z) > 0\},$$

where $Im(z)$ is the imaginary part of z.

Further due to the one point compactification of $\mathbb{C}$, we define the *boundary of* $\mathbb{H}^2$ to be

$$\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}.$$

Lines are arcs of circles (or lines) which meet $\partial\mathbb{H}^2$ orthogonally. Note that arcs must be semicircles with centres on $\mathbb{R}$. The metric on $\mathbb{H}^2$ is given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{A.1}$$

$\mathbb{H}^2$ endowed with the above metric in A.1 is a space with Gaussian curvature of -1 [15]. If two lines $l, l' \subset \mathbb{H}^2$ meet at $\partial\mathbb{H}^2$ then, the infimum of the distances between their points is 0, however, if they do not meet at the boundary then there is a non-zero shortest distance between them which is realised by by the unique common perpendicular [15].

Definition A.2. A map $f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ is said to be an *orientation-preserving isometry of* $\mathbb{H}^2$ if it is differentiable as a function $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, with $\det Df(z) > 0, \forall z \in \mathbb{H}^2$, and if

$$d(f(z_1), f(z_2)) = d(z_1, z_2) \text{ for all } z_1, z_2 \in \mathbb{H}^2.$$

We denote the set of all orientation-preserving isometries of $\mathbb{H}^2$ with $\mathrm{Isom}^+(\mathbb{H}^2)$

Theorem A.3. $\mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}(2, \mathbb{R})$.

Definition A.4. If $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$, then the trace of T is defined by:

$$\mathrm{Tr}(T) = a + d.$$

Since $\mathrm{Tr}(AD) = \mathrm{Tr}(DA)$ for all $A, D \in \mathrm{GL}(2, \mathbb{C})$, we have that $\mathrm{Tr}(STS^{-1}) = \mathrm{Tr}(T)$.

Theorem A.5. *Let T be a nontrivial element in $SL(2, \mathbb{C})$ then T has either 1 or 2 fixed points. [15]*

Definition A.6. Let $m(z) = \frac{az+b}{cz+d}$ be an element of $\mathrm{Isom}^+(\mathbb{H}^2)$. Then we say that m is *parabolic* if m has exactly one fixed point in $\bar{\mathbb{R}}$. Any such m is conjugate to

$$q(z) = z + 1, \text{ and } \mathrm{Tr}(m) = 2.$$

We say that m is *hyperbolic* or *loxodromic* if m has exactly two fixed points in $\bar{\mathbb{R}}$. Any such m is conjugate to

$$m(z) = a^2 z, \text{ for } a \in \mathbb{R} \setminus \{0, \pm 1\}, \text{ and } \mathrm{Tr}(m) > 2.$$

We say m is *elliptic* if m has exactly one fixed point in $\mathbb{H}^2$. Such an m is conjugate to

$$q(z) = \frac{z\, cos\theta + sin\theta}{-z\, sin\theta + cos\theta}, \text{ for } \theta \in \mathbb{R}, \text{ and } \mathrm{Tr}(m) < 2.$$

Definition A.7. Let S be a surface. A *hyperbolic structure* on S is a maximal collection of coordinate charts, that is open sets $U_i \subset S$ and maps $\phi_i : U_i \longrightarrow \mathbb{H}^2$ such that:

1. the sets U_i cover S,
2. $\phi_i : U_i \longrightarrow \phi_i(U_i)$ is a homeomorphism, and
3. $\phi_i \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$ is an isometry of $\mathbb{H}^2$, i.e., the transition map is in $\mathrm{Isom}^+(\mathbb{H}^2)$.

Remark A.8. A surface S with hyperbolic structure on it, is called a *hyperbolic surface.*

It is impossible to embed an infinite simply connected surface of constant negative curvature isometrically into Euclidean 3-space [16].

Definition A.9. A *Fuchsian group* is a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$.

For hyperbolic surfaces the *Uniformisation theorem* stated in Theorem 2.2 takes the following form [15]:

Theorem A.10. *Let X be a complete hyperbolic surface. Then $X = \mathbb{H}^2/G$ for some torsion-free Fuchsian group G.*

B. MORE ON DEHN TWISTS

So far we have realised the importance of Dehn twists in the theory of the mapping class groups. In this section, we explore some additional properties of Dehn twists.

The inclusion homomorphism

Consider a closed subsurface S of a surface S'. The inclusion map $S \longrightarrow S'$ will induce a map at the level of the mapping class group, let $\mathrm{Mod}(S) \xrightarrow{\eta} \mathrm{Mod}(S')$.

Theorem B.1. *Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ denote the boundary components of S that bound once-punctured disks in $S' - S$ and let $\{\beta_1, \gamma_1\}, \{\beta_2, \gamma_2\}, \ldots, \{\beta_n, \gamma_n\}$ denote the pairs of boundary components of S that bound annuli in $S' - S$. Then the kernel of η is injective.*

Proof (a sketch). Consider a representative of a mapping class of the subsurface and extend it to the surface. Now we choose a simple closed curve and look at its image under the homeomorphism. Further, we isotope it back to extend this isotopy later. Lastly, we smartly choose a family of curves and apply Alexander's method. □

The cutting homomorphism

This is a geometric operation used heavily in theory. Given a simple closed curve on a surface, if we cut along it the resultant cut surface will be having two more punctures than the original surface. The idea is that a mapping class of the cut surface can be extended nicely to the original surface.

Theorem B.2 (The cutting homomorphism)**.** *Let S be a closed surface with finitely many marked points. Let $\alpha_1, \ldots, \alpha_n$ be a collection of pairwise disjoint,* *homotopically distinct essential simple closed curves in S. There is a well-defined homomorphism*

$$\zeta : \mathrm{Mod}\left(S, \{[\alpha_1], \ldots, [\alpha_n]\}\right) \to \mathrm{Mod}\left(S - \cup\alpha_i\right),$$

with kernel $\langle T_{\alpha_1}, \ldots, T_{\alpha_n}\rangle$.

Proof (a sketch). We consider an open regular neighbourhood of the unions of all α_is. Then we look at the induced inclusion from the surface having the neighbourhood removed to the surface. We have its kernel from the previous theorem, and we compare the kernels of this inclusion to the inclusion from the cut surface to its closure. Using that we conclude the well-definedness of the map. Then the homomorphism property follows from the construction of the map. □

The capping homomorphism

The capping tells us that if there is a boundary component of the surface, then we can cap it by a disk with a puncture with $\mathrm{Mod}(D^2, \{x_0\}) = \{1\}$. To demonstrate the strength of some tools we would be proving it after we introduce the Birman exact sequence. The result is as follows:

Theorem B.3. *Let S' be the surface obtained from a surface S by capping the boundary component β with a once-marked disk; call the marked point in this disk p_0. Denote by $Mod(S, \{p_1, p_2, ..., p_k\})$ the subgroup of $Mod(S)$ consisting of elements that fix the punctures $p_0, ..., p_k$, where $k \geq 0$. Let*

$$\mathcal{C}ap : Mod(S, \{p_1, ..., p_k\}) \longrightarrow Mod(S', \{p_0, ..., p_k\}),$$

be the induced homomorphism. Then the following sequence is exact:

$$1 \longrightarrow \langle T_\beta \rangle \longrightarrow Mod(S, \{p_1, ..., p_k\}) \xrightarrow{\mathcal{C}ap} Mod(S, \{p_0, ..., p_k\}) \longrightarrow 1.$$

These three ideas play important roles in making arguments precise. Once we discuss in detail about the capping in the next section we will compute the mapping class group of a pair of pants, which are the topological building blocks of any compact surface.

Generating the mapping class group

In 1938, Dehn [20] proved that $\mathrm{Mod}(S_g)$ is generated by $2g(g-1)$ Dehn twists. In 1967, Mumford [21] improved Dehn's work by showing that we require twists only around nonseparating curves. In 1964, Lickorish [22] independently proved that $\mathrm{Mod}(S_g)$ is generated by $3g-1$ Dehn twists about nonseparating curves. These results are remarkable and have given the topic, new dimensions.

Theorem B.4 (Dehn–Lickorish)**.** *For $g \geq 0$, the mapping class group Mod(S_g) is generated by finitely many Dehn twists about nonseparating simple closed curves.*

The proof shows that the subgroup of $\mathrm{Mod}(S_g)$ fixing all the punctures denoted by $\mathrm{PMod}(S_g)$ is finitely generated by Dehn twists, and then extend it to the general case. It is a double induction process on the genus and number of punctures, and it also assumes the connectedness of the complex of non-separating curves [11], and the exactness of the Birman exact sequence [11] given by

$$1 \longrightarrow \pi_1(S,x) \xrightarrow{\mathcal{P}ush} \mathrm{Mod}(S,x) \xrightarrow{\mathcal{F}orget} \mathrm{Mod}(S) \longrightarrow 1.$$

We conclude this thesis with the following result.

Theorem B.5. *Let $S_{g,n}$ be a surface of genus $g \geq 1$ with $n \geq 0$ punctures. Then the group PMod($S_{g,n}$) is finitely generated by Dehn twists about nonseparating simple closed curves in $S_{g,n}$.*

Proof (a sketch). Let $g \geq 1$ and $n \geq 0$. Further assume for the inductive step that, $\mathrm{PMod}(S_{g,n})$ is generated by finitely many Dehn twists about nonseparating simple closed curves $\{\alpha_i s\}$ in $S_{g,n}$. Now, by the Birman exact sequence we get that,

$$1 \longrightarrow \pi_1(S_{g,n},x) \xrightarrow{\mathcal{P}ush} \mathrm{PMod}(S_{g,n+1}) \xrightarrow{\mathcal{F}orget} \mathrm{PMod}(S_{g,n}) \longrightarrow 1.$$

Now note that as we have a positive genus (non-zero) thus, the fundamental group would be generated by classes of finitely many non-separating simple closed curves. Further the $\mathcal{P}ush$ map takes these loops to mapping classes ($\neq 1$). Moreover, for every non-separating simple closed curve in $S_{g,n}$ there is a non-separating simple closed curve in $S_{g,n+1}$ such that it is the image under the $\mathcal{F}orget$ map. Then the

Dehn twists along these curves will be having pre-images in $S_{g,n+1}$ which completes the inductive step.

Now we will discuss the inductive step on the genus. Let $g \geq 2$, and assume that our claim is true for genus $g-1$ that is, $\mathrm{PMod}(S_{g-1,n})$. Here we require the connectedness of the complex of curves of non-separating simple closed curves.

Let $\mathrm{Mod}(S_g, \overline{a})$ be the subgroup of $\mathrm{Mod}(S_g, a)$ consisting of elements that preserve the orientation of a. We have the short exact sequence

$$1 \to \mathrm{Mod}\left(S_g, \vec{a}\right) \to \mathrm{Mod}\left(S_g, a\right) \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

Now, we smartly look at the twists which change the orientation of a by changing coordinates as well. This gives us a nontrivial coset of $\mathrm{Mod}(S_g, \overline{a})$ and look at the sequence,

$$1 \to \langle T_a \rangle \to \mathrm{Mod}\left(S_g, \vec{a}\right) \to \mathrm{PMod}\left(S_g - \alpha\right) \to 1.$$

where α is a representative of a. The surface $S_g - \alpha$ is homeomorphic to $S_{g-1}, 2$ which is finitely generated. Thus $\mathrm{PMod}(S_g - \alpha)$ is finitely generated by Dehn twists along nonseparating simple closed curves. By construction of this surface we get our desired result for the inductive step. $\square$

BIBLIOGRAPHY

[1] M. Dehn. Papers in Group Theory and Topology. Translated and Introduced by John Stillwell. Springer (1987).

[2] M. Dehn. On curve systems on two-sided surfaces, with application to the mapping problem. Lecture (supplemented) to the math, colloquium, Breslau 11-2-1922. In: [1], 234-252.

[3] M. Dehn. Die Gruppe der Abbildungsklassen. Acta Math. 69 (1938), 135-206; English transl.: [1],256- 362.

[4] J. Nielsen. Untersuchungen zur Theorie der geschlossenen zweiseitigen Fldchen I. Acta Math. 50 (1927), 189-358.

[5] J. Nielsen. Untersuchungen zur Theorie der geschlossenen zweiseitigen Fldchen II. Acta Math. 53 (1929), 1-76.

[6] J. Nielsen. Untersuchungen zur Theorie der geschlossenen zweiseitigen Fldchen III. Acta Math. 58 (1932), 87-167

[7] W.J. Harvey. Geometric structure of surface mapping-class groups, Homological Group Theory. C.T.C. Wall, ed., London Math. Soc. Lecture Notes Series, Vol. 36. Cambridge University Press (1979), 255-269.

[8] W.J. Harvey. Boundary structure of the modular group, Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference. I. Kra and B. Maskit, eds, Ann. of Math. Stud., Vol. 97. Princeton University Press (1981). 245-251.

[9] A. Fathi, F. Laudenbach and V. Poenaru, Travaux de Thurston sur les surfaces, Seminaire Orsay, Asterisque. Vol. 66-67, Soc. Math, de France (1979).

[10] W.P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. Bull. Amer. Math. Soc. 19 (2) (1988), 417-431.

[11] Benson Farb, Dan Margalit. *A primer on mapping class groups(pms 49)*. Princeton University Press 2011.

[12] Nikolai V. Ivanov . *A Handbook of Geometric Topology*, pages 523-633. Elsevier 2001.

[13] James Anderson. *Hyperbolic Geometry*. Springer Undergraduate Text Series 2005.

[14] Albrecht Dold. *Lectures on Algebraic Topology*. Ed 2. Springer-Verlag Classics in Mathematics 1995.

[15] Series. Caroline Hyperbolic geometry. Lecture Notes. MA 448.2008; Revised 2010. Last edited: January 4, 2013. http://homepages.warwick.ac.uk/~masbb/Papers/MA448.pdf

[16] do Carmo. Manfredo Differential Geometry of Curves and Surfaces. Prentice Hall. 1976.

[17] Baer. R. Isotopie con Kurven auf orientierbaren geschlossenen Flächen und ihr Zusammenhang mit der topologischen Deformation der Flächen. J. f. Math. 159 (1928). $101 - 116$.

[18] Stillwell. John. Classical Topology and Combinatorial Group Theory. Ed. 2. Springer Graduate Texts in Mathematics. GTM 72. 1993

[19] Munkres. James Obstructions to the smoothing of piecewise-differentiable homeomorphisms. Ann. of Math. (2), $72 : 521$–554, 1960.

[20] Dehn, M. Die Gruppe der Abbildungsklassen: Das arithmetische Feld auf Flächen. Acta Math. 69 (1938), $135 - -206$. doi:10.1007/BF02547712. https://projecteuclid.org/euclid.acta/14858882

[21] Mumford, David. Abelian quotients of the Teichmüller modular group. J. Analyse Math., $18 : 227$–244, 1967.

[22] Lickorish. W. B. R. “A finite set of generators for the homeotopy group of a 2 manifold”. Proc. Cambridge Philos. Soc. Corrigendum in 62 (1967), 679 – 681.

[23] Burkhardt. H. Grundz*ü*ge einer allgemeinen Systematik der hyperelliptischen Functionen I. Ordnung. Mathematische Annalen, 35 : 198 – 296, 1890.

[24] Massuyeau, Gwenael. A short introduction to mapping class groups. pages 17. http://massuyea.perso.math.cnrs.fr/notes/MCG.pdf

[25] Kerckhoff. Steven P. The Nielsen realization problem. Ann. of Math. (2), 117(2) : 235 – 265, 1983.

[26] Carsten Thomassen. The Jordan–Sch*ö*nflies theorem and the classification of surfaces. Amer. Math. Monthly, 99(2):116–130, 1992.

[27] Chan. Kevin. Uniformaization of Riemann surfaces. Harvard Mathematics Department Senior Thesis. 2004. http://www.math.harvard.edu/theses/senior/chan/fulldraft7.pdf

[28] Kapovitch. Vitali, Notes on Collar neighbourhood theorem and tensors. 2015. http://www.math.toronto.edu/vtk/1300Fall2015/lecture-nov2.pdf

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