TOPOLOGICAL GRAPH THEORY AND THE HEAWOOD PROBLEM

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CERTIFICATE

This is to certify that Sreekanth D, BS-MS (Dual Degree) student in Department of Mathematics, has completed bonafide work on the dissertation entitled **'Topological Graph Theory and The Heawood Problem'** under my supervision and guidance.

April 2018 **Dr. Kashyap Rajeevsarathy**

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Sreekanth D

ABSTRACT

Topological graph theory is a branch of graph theory that studies graphs as topological spaces, their embeddings on surfaces and other properties alongside the combinatorial and algebraic definition. The primary objective of topological graph theory is to study graph embeddings on surfaces, which in layman's terms, pertains to understanding whether a given graph can be drawn on a surface without crossings.

We will be focusing on graph embedding in closed orientable surfaces. We begin by understanding planar embedding and the Kuratowski's theorem, one of the well-known results in topological graph theory [3] and its application. We will then learn about graph embeddings on higher genus surfaces. Finally, we will discuss the Heawood problem and its solution for orientable surfaces. In this direction, we will derive the complete graph orientable embedding inequality which gives a relation for the genus of a complete graph.

CONTENTS

1. PRELIMINARIES

In this introductory chapter, we provide the background to the material that we present more formally in later chapters.

1.1 Graphs

Definition 1.1. A *graph* G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of *vertices* and a set $E(G)$ of *edges*, together with an incidence function ψ_G that associates with each edge an unordered pair of distinct vertices called the *endpoints* of that edge.

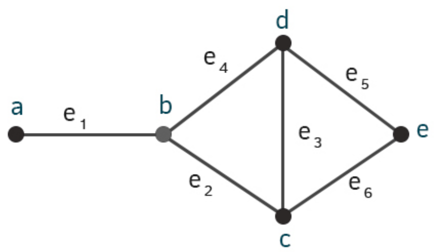

Fig. 1.1: Graph G

Example 1.2. Let G (see Figure 1.1) be a graph with vertex set $V := \{a, b, c, d, e\}$, edge set $E := \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $\psi_G(e_1) = \{a, b\}$, denoted simply by $e_1 = ab$.

For a graph G, the *cardinality* of $V(G)$ is called it's *order*, denoted by $\#G$. Throughout this thesis, we will denote the cardinality of a set A by $\#A$.

Definition 1.3. Let G be a graph, consider $v_1, v_2 \in V(G)$ and $e_1, e_2 \in E(G)$.

(i) The vertices v_1 and v_2 are said to be *adjacent* if there exists $e \in E(G)$ such that $\psi_G(e) = \{v_1, v_2\}$.

(ii) The edges e_1 and e_2 are said to be *adjacent* if e_1 and e_2 have a common endpoint.

(iii) The *neighbourhood* of a vertex v_1 is defined to be the set $N_G(v_1) := \{v \in V|$ v is adjacent to v_1 $\}$.

(iv) The *valence* of a vertex v_1 is defined to be the cardinality of its neighbourhood, that is $valence(v_1) = \#N_G(v_1)$.

Definition 1.4. Let G be a graph and $v \in V(G)$.

(i) An edge e of form $\{v\}$ is called a *loop*.

(ii) If $E(G)$ is a multiset, then G is called a *multigraph*.

(iii) G is called *simple* if G is not a multigraph and contains no loop.

(iv) If $\#G < \infty$ then G is called a *finite* graph.

(v) G is called *k-regular* if $valence(v) = k$ for all $v \in V(G)$.

In this thesis, a graph G is assumed to be undirected, finite and simple unless stated otherwise.

Definition 1.5. Let G be a graph.

(i) A graph G' is called a *subgraph* of G (denoted by $G' \subseteq G$) if $V(G') \subseteq V(G)$, $E(G') \subseteq E(G)$ and $\psi_{G'}$ is a restriction of ψ_G on $E(G')$.

(ii) If $V(G') \subsetneq V(G)$ and $E(G') \subsetneq E(G)$, then G' is called a *proper* subgraph of G (denoted by $G' \subsetneq G$).

(iii) If $V(G') = V(G)$ then G' is called a *spanning* subgraph of G.

(iv) The subgraph G' is called an *induced* subgraph of G if $E(G') := \{e \in E(G) \mid \psi_G(e) = \{u, v\} \ \forall \ u, v \in V(G')\}$.

Definition 1.6. A graph K_n of n vertices is called a *complete graph* if for each vertex $v \in V(K_n)$, $N_{K_n}(v) = V(K_n)$ (see Figure 1.2).

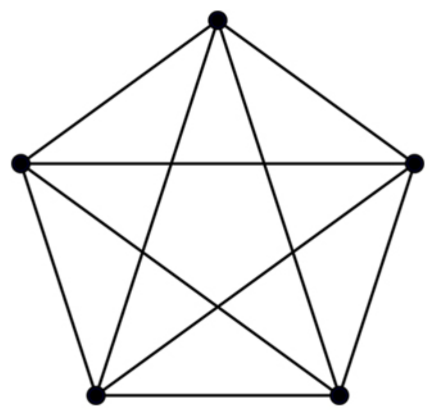

Fig. 1.2: Complete graph K_5

Definition 1.7. A graph $K_{m,n}$ with vertex set $V(K_{m,n}) = X \bigsqcup Y$ where $\#X = m$ and $\#Y = n$, such that $E(K_{m,n}) := \{uv \mid u \in X, v \in Y\}$ is called a *complete bipartite graph* (see Figure 1.3).

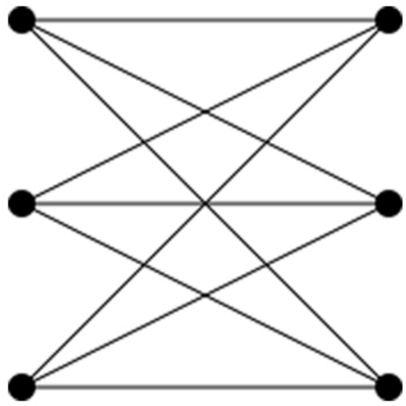

Fig. 1.3: Complete bipartite graph $K_{3,3}$

Definition 1.8. A *cycle* C_n is a graph of n vertices with $V(C_n) := \{v_1, v_2, ..v_n\}$ such that $E(C_n) := \{v_i v_{i+1} \mid 1 \leq i \leq n-1\} \cup \{v_1 v_n\}$.

Definition 1.9. For $n \geq 4$, a *wheel* W_n is defined to be a graph of n vertices with $V(W_n) := V(C_{n-1}) \cup \{uv \mid u = V(K_1)\}$ and $v \in V(C_{n-1})\}$ (See Figure 1.4).

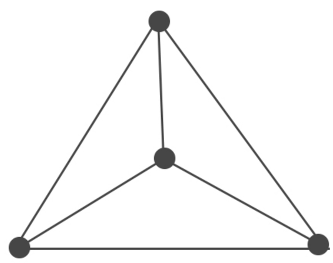

Fig. 1.4: K_4 or W_4

Definition 1.10. Two graphs G and H are said to be *isomorphic* (written as

$G \cong H$), if there are bijections $\theta : V(G) \longrightarrow V(H)$ and $\phi : E(G) \longrightarrow E(H)$ such that $\psi_G(e) = uv$ if and only if $\psi_H(\phi(e)) = \theta(u)\theta(v)$ (see Figure 1.5).

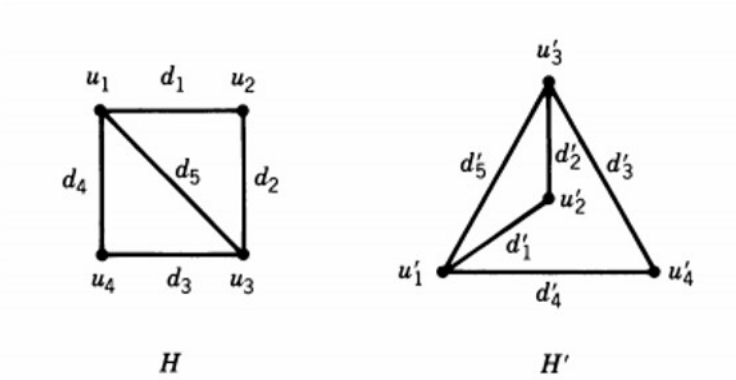

Fig. 1.5: Isomorphic graphs

1.2 Walk, Path and Connectivity

Definition 1.11. Let G be a graph.

(i) A *walk* is a sequence of vertices $v_0, v_1, ..., v_i, ..., v_n$ such that $v_i \in V(G)$ for $1 \leq i \leq n$ and $v_i v_{i+1} \in E(G)$ for $1 \leq i \leq n-1$. The vertices v_0 and v_n are called the *initial* and *terminal* points, respectively.

(ii) A walk is said to be *closed* if $v_0 = v_n$.

(iii) A *path* is a walk $v_0, ..., v_n$ such that $v_i \neq v_j$ for $0 < i, j < n$.

The number of edges transversed in a path (or a walk) is called the *length* of the path (or the walk).

Definition 1.12. A *tree* T is a graph in which any two vertices of T are connected by exactly one path.

Remark 1.13. let T be a tree such that $\#V(T) = n$ (see [4]).

(i) $\#E(T) = n - 1$.

(ii) T is bipartite.

(iii) T has no simple cycles.

Definition 1.14. A graph G is called *connected* if there exists a path from u to v, for all $u, v \in V(G)$ (see Figure 1.6).

Definition 1.15. A vertex $v \in V(G)$ of a graph G is called a *cutpoint* of G if removing v along with its incident edges from G disconnects G (see Figure 1.6).

Definition 1.16. Let G be a graph and $x, y \in V(G)$.

(i) The maximum number of pairwise internally disjoint paths from x to y in G is called the *local connectivity*, which we denote by $\rho(x, y)$.

(ii) G is called k-*connected* (see Figure 1.6) if $\rho(x, y) \geq k$ for all $x, y \in V(G)$. k is called the *connectivity* $\kappa(G)$ of G.

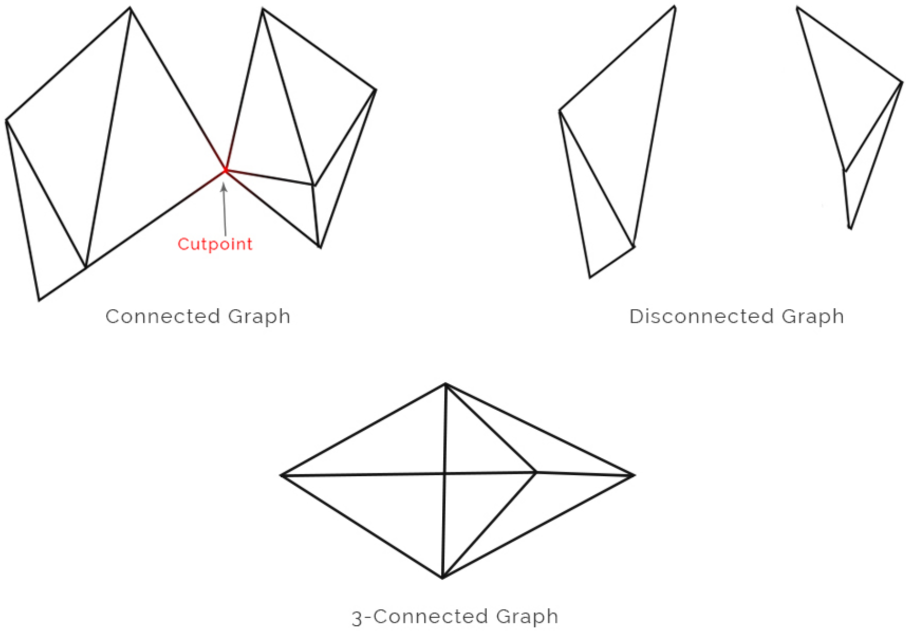

Fig. 1.6: Examples for connectedness.

1.3 Graph Operations

Let G be a graph, $v \in V(G)$ and $e = uv \in E(G)$.

Definition 1.17. The subgraph $G - v$ with $V(G - v) := V(G) \setminus v$ and $E(G - v) := E(G) \setminus \{uv \in E(G) \mid u \in N_G(v)\}$ is called the graph obtained by *deleting vertex* v from G (see Figure 1.7).

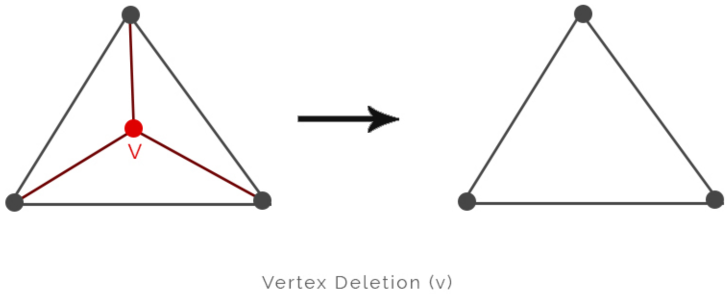

Fig. 1.7: Vertex deletion at v.

Definition 1.18. The subgraph $G-e$ with $V(G-e) := V(G)$ and $E(G-e) := E(G) \setminus e$ is called the graph obtained by *deleting edge* e from G (see Figure 1.8).

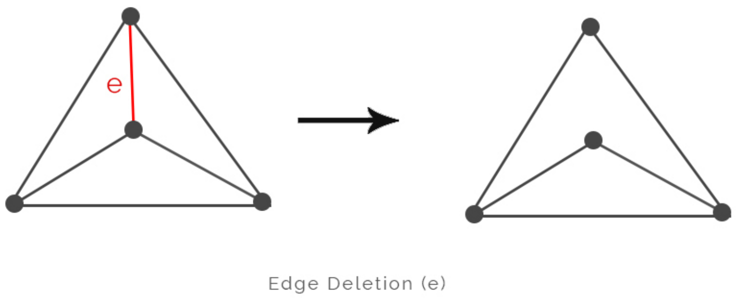

Fig. 1.8: Edge deletion at e.

Definition 1.19. The *contraction of an edge* e of the graph G, denoted by G/e (see Figure 1.9) is the graph obtained from G by the following steps:

(i) delete vertices u and v from G.

(ii) insert a new vertex u' such that $u'v \in E(G/e)$, for all $v \in N_G(u) \cup N_G(v)$.

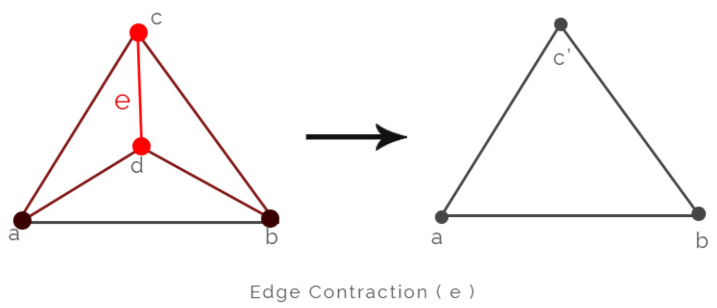

Fig. 1.9: Contracing edge e.

Definition 1.20. An *n-subdivision of an edge* e of a graph G (see Figure 1.10) is the graph H obtained by adding n vertices to edge e, that is, $H = (V(H), E(H))$ where $V(H) := V(G) \cup \{v_1, ..., v_n\}$ and $E(H) := (E(G) \setminus e) \cup \{v_i v_{i+1} | 1 \leq i \leq n-1\}$. The reversal of subdivision is called *smoothing*.

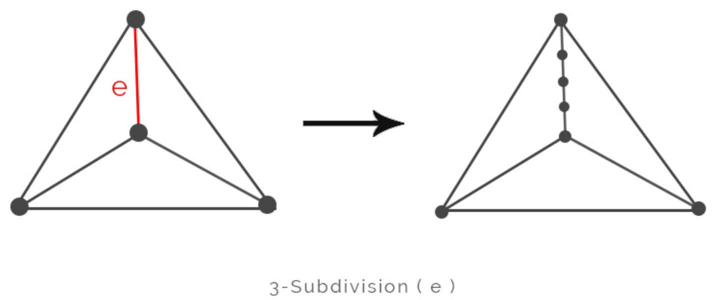

Fig. 1.10: Subdividing edge e.

1.4 Surfaces and Simplicial Complexes

Definition 1.21. A topological space M is called n-manifold if M is Hausdorff and each point of M has an open neighborhood homeomorphic to the n-dimensional open ball or the n-dimensional half-ball.

Definition 1.22. Let M be an n-manifold.

(i) The *boundary* of M is the collection of all points in M that do not have a neighbourhood homeomorphic to the n-dimensional open ball;

(ii) M is called *closed* if it is compact and its boundary is empty.

A *surface* is a 2-dimensional manifold. In this thesis we will be dealing with closed, orientable surfaces of genus $g \geq 0$.

Definition 1.23. A *(geometric) k-simplex* is the convex hull of $k+1$ affinely independent points or vertices in Euclidean n-space $\mathbb{R}^n$, denoted by $[v_0, v_1, ..., v_k]$. The simplex determined by any subset of $\{v_0, ..., v_k\}$ is called a *face* of $[v_0, v_1, ..., v_k]$.

Definition 1.24. A *simplicial complex* K (see Figure 1.11) is a finite collection of simplexes in $\mathbb{R}^n$ satisfying the two conditions.

(i) Every face of every simplex in K is a simplex in K.

(ii) The intersection of any two simplexes in K is a simplex in K.

Definition 1.25. Let K be a simplicial complex.

(i) The point set $|K| := \bigcup_{s \in K} S$ is called the *carrier* of K.

(ii) If m is the largest integer such that K contains an m-simplex, then K is called a m-complex.

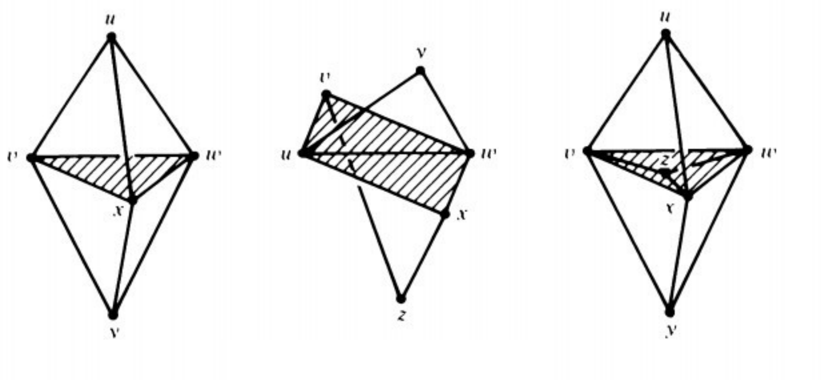

Fig. 1.11: Example of simplicial complexes.

(iii) The collection of all k-simplexes of K for $k \leq r$ is called the r-skeleton of K. Denoted by $K^{(r)}$.

Definition 1.26 (Triangulation)**.** A *triangulation* of a topological space X is a homeomorphism h from the carrier of some simplical complex K to the space X. The image of a simplex of K under h is called a simplex of triangulation.

1.5 Graph Embedding

Remark 1.27. Any graph G can be represented by a topological space in the following sense:

(i) $V(G)$ is represented by a collection of distinct point in $\mathbb{R}^3$.

(ii) $E(G)$ is represented by a collection of distinct, internally disjoint arcs, homeomorphic to the closed interval [0, 1] such that boundary points of the arcs represent the endpoints of the corresponding edge

Definition 1.28. Let G be a graph (topological representation) and S_g, a surface of genus g. A *graph embedding* is a continuous one-to-one function $i : G \longrightarrow S_g$ such that the function $i' : G \longrightarrow i(G)$ obtained by restricting the range of i is a homeomorphism (see Figure 1.12).

Definition 1.29. Let G be a graph and Let $i : G \longrightarrow S$ be an embedding on G on surface S.

(i) The set $F(G) = S \setminus i(G)$ is called the *set of regions* (or faces) of the embedding i. Each element of F represents a maximal connected component of $S \setminus i(G)$ and is called a *region (or face)*.

(ii) The graph embedding i is called a *2-cell embedding* if each region is homeomorphic to an open disk.

(iii) Two graphs are said to be *homeomorphic* if both can be obtained from the same graph by a sequence of subdivisions of edges.

Definition 1.30. Consider the graph G and surfaces S, T. The two embeddings $i : G \longrightarrow S$ and $j : G \longrightarrow T$ are called *weakly equivalent* if there exists a homeomorphism $h : S \longrightarrow T$ such that $h(i(G)) = j(G)$.

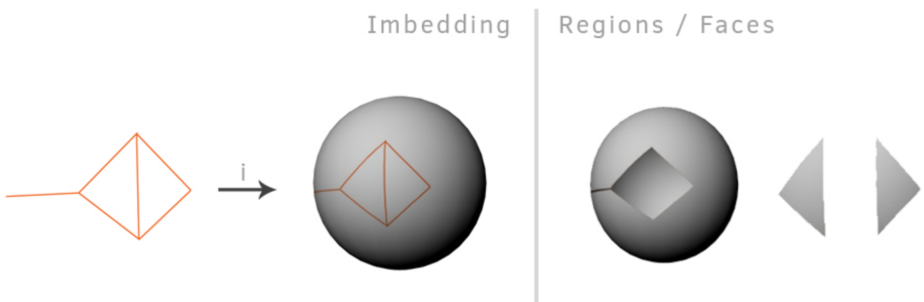

Fig. 1.12: Graph embedding on S_0 and corresponding regions.

Definition 1.31. The *genus* γ_G of a graph G is defined to be the smallest number g such that the graph G embeds in the orientable surface S_g.

Theorem 1.32 (Euler)**.** *The sum of the valences of the vertices of a graph G equals twice the number of edges.*

$$\sum_{v \in V(G)} valence(v) = 2\#E$$

Proof. Let $T = \sum_{v \in V(G)} valence(v)$. Observe that each $e \in E(G)$ gets added exactly twice in the sum T. Hence, $T = \sum_{e \in E(G)} 2 = 2\#E$. This concludes the proof. $\square$

2. PLANAR EMBEDDINGS OF GRAPHS

In this chapter we will study graph embeddings on genus 0 surfaces.

2.1 Planarity

Definition 2.1. A graph G is said to be *planar* if and only if it can be embedded on a sphere S_0.

Remark 2.2. Let G be a graph. The following are equivalent.

(i) G is planar.

(ii) It can be embedded on a plane.

(iii) Genus $\gamma_G = 0$ (see Lemma 2.4).

The inequality in the following lemma is known as the *edge-region inequality* and it gives a relation for the number of faces and the number of edges for any graph embedding.

Lemma 2.3. Let $i : G \longrightarrow S$ be an embedding of a connected simple graph which is not a tree, with atleast three vertices into any surface S. Then

$$2\#E \geq 3\#F$$

generally,

$$2\#E \geq girth(G)\#F$$

where $girth(G)$ is the length of the minimum cycle.

Proof. Observe that, $\sum_{f \in F} s_f = 2\#E$, where s_f denote the number of sides of the region f (see Figure 2.1). Since G is simple, $s_f \geq 3$ for each $f \in F$ and hence the conclusion. □

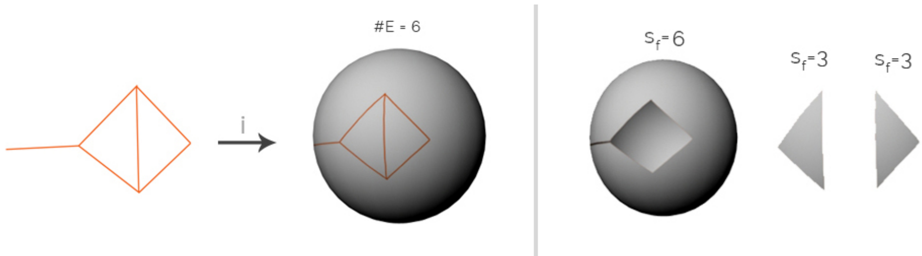

Fig. 2.1: An instance of edge-region inequality.

Lemma 2.4. Let $i : G \longrightarrow S_0$ be an embedding of a connected graph G in the sphere. Then $\#V(G) - \#E(G) + \#F(G) = 2$.

Proof. This proof proceeds by induction on the number $\#F(G)$ of regions. First, observe that if $\#F(G) = 1$, then G must be a tree, since the Jordan curve theorem implies that any cycle would separate the sphere. Thus, $\#E(G) = \#V(G) - 1$ (by Remark 1.13) , from which it follows that $\#V(G) - \#E(G) + \#F(G) = 2$.

Now suppose that the Euler's formula holds when the number of regions is at most n, and suppose that $\#F(G) = n + 1$. Then some edge e lies in the boundary walk of two distinct regions. Since the two regions are distinct, the subgraph G' obtained by removing the edge e is connected. Then $\#F(G') = \#F(G) - 1 = n$, so by induction, $\#V(G') - \#E(G') + \#F(G') = 2$. Since

$\#V(G') = \#V(G)$, $\#E(G') = \#E(G) - 1$, and $\#F(G') = \#F(G) - 1$, it follows that $\#V(G) - \#E(G) + \#F(G) = 2$. □

Proposition 2.5. If graph G is n-connected, $n \geq 2$, then every set of n points of G lie in a cycle.

Proof. By the definition of n-connected graph, G has no cutpoints and there must be a maximum of atleast n number of pairwise internally disjoint paths between any two vertices x, y in $V(G)$. Thus, for any set $P = \{p_1, p_2, ..., p_n\} \subset V(G)$, we can find two internally disjoint paths between p_0 and p_n such that all p_i $(i \neq 1, n)$ lie in either of the two path. This gives a cycle containing P. □

2.2 Kuratowski's Theorem

Kuratowski's theorem give the criterion for a graph to be planar. The graphs $K_{3,3}$ and K_5 forms the complete set of obstruction in planar embedding. These graphs are called the *Kuratowski's graphs*.

Lemma 2.6. K_5 and $K_{3,3}$ are non-planar.

Proof. For K_5 , (see Figure 2.2)

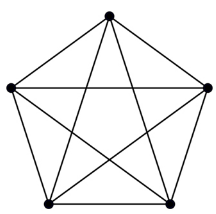

Fig. 2.2: $K_5, \#E = 10$ and $\#F = 7 = (2 + \#E - \#F)$

The edge-region inequality yields, 2 . 10 (=20) $\ngeq$ 3 . 7 (=21). Hence, K_5 is non-planar.

Now for $K_{3,3}$, (see Figure 2.3)

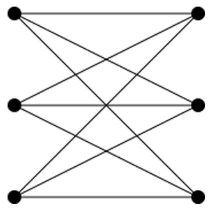

Fig. 2.3: $K_{3,3}, \#E = 9$ and $\#F = 5$

Again by edge-region inequality, since $K_{3,3}$ is bipartite $girth(G) \geq 4$. Thus we get, 2 . 9 (=18) $\ngeq$ 4 . 5 (=20). Hence, $K_{3,3}$ is non-planar.

□

Theorem 2.7 (Kuratowski)**.** *A graph G is planar if and only if G has no subgraph homeomorphic to K_5 or $K_{3,3}$.*

Proof. From lemma 2.6, K_5 and $K_{3,3}$ are non-planar. So any graph containing a homeomorph of K_5 or $K_{3,3}$ are non-planar. Thus the converse is proved.

Now we want to prove that, a graph G is non-planar then G has a subgraph homeomorphic to K_5 or $K_{3,3}$. Consider the (edge) minimal counter example H such that H is non-planar and does not contain a homeomorph of $K_{3,3}$ or K_5 . The minimality ensures that removal of any edge makes H planar. We also assume that H does not have any vertex of valence 2, since a valence 2 vertex can be considered as the 1-subdivision of some edge and hence, we can smooth out valence 2 vertices to obtain H.

First, we claim that H is at least 2-connected or in other words, H does not have any cutpoints. To show this, suppose H has a cutpoint v, then removal of v disconnects H. Since H is non-planar, some component of H must be non-planar. This contradicts the minimality of H, and the claim follows.

Next, we claim that there exists edge e such that $H - e$ has no cutpoints. To show this, assume that H has a no such edge. Then, for any edge $e' \in E(H)$, $H - e'$ has a cutpoint. This means that $H - e$ is 1-connected for all e, which in turn shows that H is 1-connected. This contradicts the first claim, and hence the claim holds.

Now, choose an edge $e = \{u, v\}$ of H whose removal does not affect connectivity. Consider $H - e = H'$ which is planar and 2-connected. Thus we can find a planar embedding of H' with a cycle C (by Proposition 2.5) such that C contains u and v and the number of regions enclosed by C is maximal among other embeddings. Let $C = v_0, v_1, ...v_k = v, v_{k+1}, ..., v_l, v_0$ and consider path P (see Figure 2.4). From the maximality of C we can see that there is no path connecting two vertices in the set $\{v_0, v_1, ...v_k\}$ that lies exterior to C and furthermore, there is no path connecting two vertices in the set $\{v_{k+1}, ..., v_l, v_0\}$ that lies exterior to C.

The non-planarity of H implies that there is some structure inside cycle C that restricts the insertion of edge e between u and v. While taking all the possible structures we finally arrive at the structures shown in Figure 2.4 such that any obstructing structure is homeomorphic to one of the graphs in Figure 2.4.

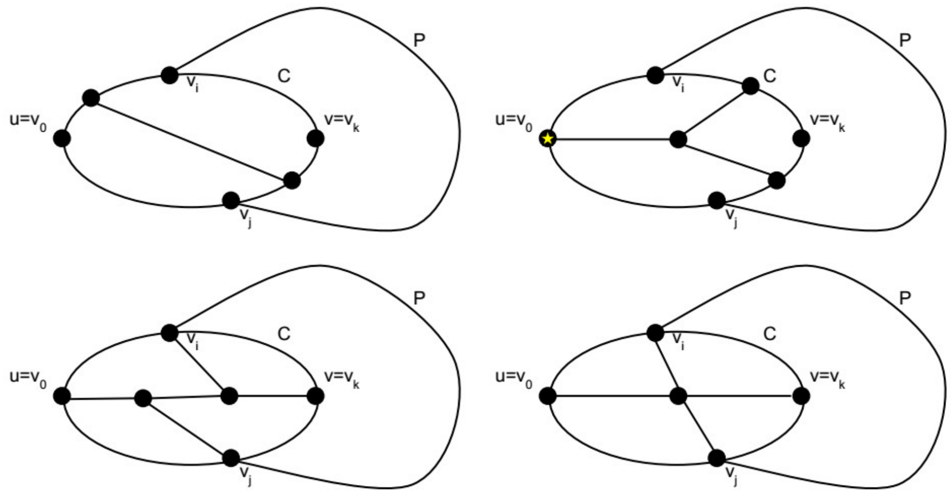

Fig. 2.4: All possible structures of H.

In Figure 2.4, the bottom right graph is homeomorphic to K_5 and the rest are homeomorphic to $K_{3,3}$.

□

The above proof can be developed into the naive planarity algorithm which will be discussed in the next section. A much more efficient planarity testing algorithm can be obtained from the following proof of Kuratowski's theorem due to Thomassen [3]. The idea of Thomassen's proof relies on the following result by Tutte which we state without proof [4].

Proposition 2.8 (Tutte)**.** A graph G is 3-connected then it is a wheel or can be obtained from a wheel by a sequence of operations of the following two types:

(i) The addition of new edge

(ii) The replacement of a vertex v having valence (≥ 4) by two adjacent points v' and v'' such that each point formerly joined to v is joined to exactly one of v' and v'' so that in the resulting graph, $valence(v') \geq 3$ and $valence(v'') \geq 3$

The above proposition ensures the existence of wheel structure for the 3-connected graph in the following proof of Kuratowski's theorem.

Theorem 2.9. *Let G be a 3-connected graph with five or more vertices. Then there is some edge e of G such that the graph G/e is also 3-connected.*

Proof. (Thomassen) Suppose for every edge e, the contracted graph G/e has a set of two vertices that disconnects it. One of those two vertices must be the vertex obtained by identifying the two endpoints of the edge e or else the same set of two vertices would also disconnect G, thereby contradicting the 3-connectivity of G. Thus for every edge $e = uv$ together with some third vertex w disconnect G. Accordingly, let us choose an edge e and a vertex w such that the largest component H of the graph $G - \{u, v, w\}$ is the largest for any disconnecting set consisting of three vertices, two of which are adjacent.

Let x be a vertex adjacent to w such that x lies in a component of $G - \{u, v, w\}$ other than the maximum component H. Since vertices w and x are the endpoints of an edge of G, it follows that G has a disconnecting set of the form $\{w, x, y\}$. Now claim that some component of $G - \{w, x, y\}$ is larger than H, a contradiction. To see this, let H' be the subgraph of G induced by the vertices of H together with u and v. Since both u and v are adjacent to vertices of H (otherwise G would not be 3-connected), the subgraph H' is connected. On one hand, perhaps the vertex y is not in H'. Since w and x are not in H' either, it follows that H' is contained in a component of $G - \{w, x, y\}$, contradicting the maximality of H. On the other hand, perhaps y is in H'. If $H' - y$ is connected, then there is again a contradiction of the maximality of H, since $H' - y$ has one more vertex than H. If $H' - y$ were not connected, then one component of $H' - y$ would contain both the vertices u and v, since u is adjacent to v and hence all the other components of $H' - y$ are connected to the rest of the graph G through the vertices y and w. This would imply that $\{y, w\}$ disconnects G, contradicting the 3-connectivity of G. We conclude that for some edge e, the contracted graph G/e is 3-connected.

□

Corollary 2.10. A graph G contains no homeomorph of K_5 or $K_{3,3}$ then G is planar.

Proof. We prove by induction on the number of vertices. The statement is vacuously true of all graphs with four or fewer vertices. We assume that the statement is true for all graphs with fewer than n vertices, for $n \geq 5$.
Consider the $n+1$ case. By Theorem 2.9, we can choose an edge $e = \{u, v\}$ such that G/e with the identified vertex v' is still 3-connected. This means that G-v' is 2-connected. Now consider the cycle C containing all the neighbours of v' (see Figure 2.5).

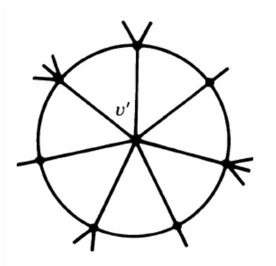

Fig. 2.5: Wheel stucture with cycle C.

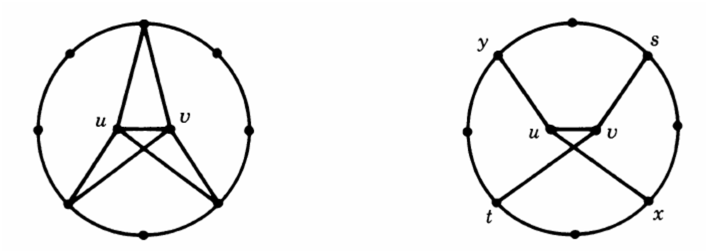

Fig. 2.6: Homeomorphs of K_5 and $K_{3,3}$ respectively.

Now expand v' back to u and v. By induction hypothesis, G will not contain the graphs in Figure 2.6 since those graphs are homeomorphic to K_5 and $K_{3,3}$ as noted. Hence, G is planar.

□

2.3 Planarity Algorithms

Planarity checking has applications in many fields such as in VLSI designing, architecture etc. Thus, an efficient planarity algorithm is desirable. In this section we will discuss two planarity algorithms.

2.3.1 Naive Planarity Algorithm

Naive planarity algorithm [3] (see Figure 2.7) is an exponential time algorithm and hence very inefficient. But it is a much straight forward algorithm.

Algorithm Run Time $= O(2^{\#E})$

(i) $G = (V, E)$;Input graph.

(ii) For each subset of E whose removal leaves only one nontrivial component H. $O(2^{\#E})$

(i) $H_{smooth} = H$ - { vertices v_t of valence 2 + the incendent edges } + {new edges between vertices in $N_G(v_t)$ }

(ii) if $H_{smooth} \cong K_5$ or $K_{3,3}$.

G is non-planar.

else

G is planar.

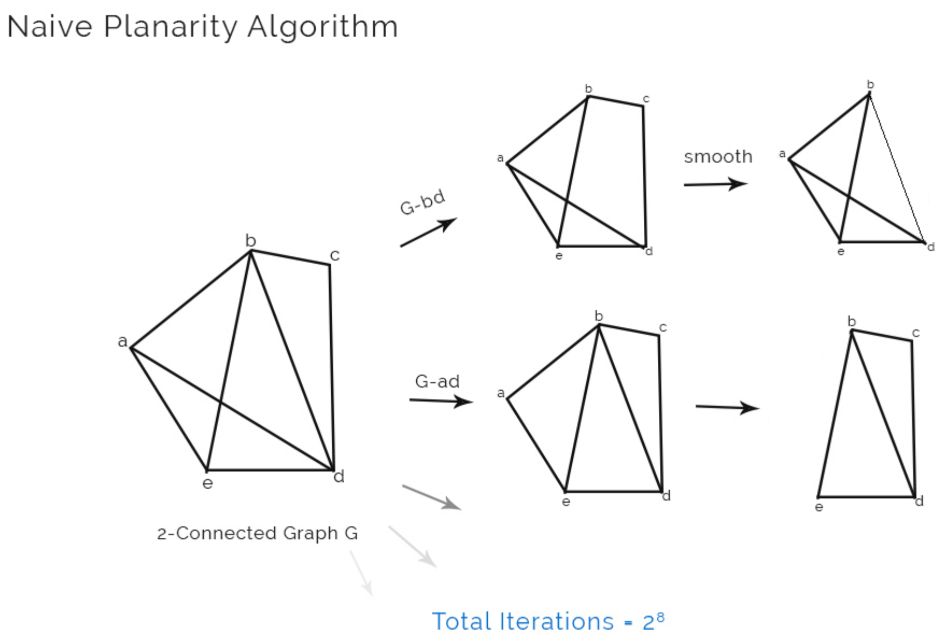

Fig. 2.7: An instance of naive planarity algorithm.

2.3.2 Polynomial time Planarity Algorithm

The algorithm we discuss here has a worst case running time of polynomial order. Thus it is much more efficient than the naive algorithm. The algorithm is as follows (see Figure 2.8) :

(i) Let $G = (V, E)$ a 3-connected graph and $\#V_G = n$.

(ii) Choose edge e such that G/e is 3-connected. Now take $G/e = H$.

(iii) Now repeat step 2 with each H obtained after simplical contraction of another similar edge in H till a Graph H' of 4 vertices is obtained. Note that, $H' = K_4$.

(iv) Now Reverse each contraction of H'. and check whether there exist a cycle C such that

- 3 vertices in C adjacent to both u and v.
- There exists a path $v_{i_1}u_{j_1}v_{i_2}u_{j_2}$ in C such that $v_i \in N_G(v)$ and $u_j \in N_G(u)$

(v) If any of the above test returns true $\Rightarrow$ the graph is non-planar.

Pseudocode

$G = (V, E)$ a 3-connected graph with $\#V = n$.

if $(\#E \geq 3n - 6)$ then
  ;Graph is non-planar. Exit

$e = e_i \in E$ such that G/e_i is 3-connected

;Defining any edge array for tracking the edge contraction

$k = 0$

$BK[k] = e$

$H = G$;initializing H

do while $(\#V_H > 4)$ $T_1 \approx O((n-4)(3n-6).n.n)$

$H = H/e$;contracting e in H

$e = e_i \in H_G$ such that H/e_i is 3-connected

$BK[++k] = e$

;Now we have $H = K_4$, since K_4 is the only 3-connected four vertex graph.

;Reversing the contraction on H.

do while $(k \geq 0)$ $T_2 \approx O(n^4)$

$H =$ restore(H, $BK[k]$)

;The restore function restores the edge $BK[k]$ on H.

$C =$ cycle with all vertices in $N_G(BK[k])$

test1 $=$ check that 3 vertices in C adjacent to both u and v .

test2 $=$ check $\exists$ path $v_{i_1}u_{j_1}v_{i_2}u_{j_2}$ in C, $v_i \in nbhd(v)$ and $u_j \in nbhd(u)$

if (test1 $=$ true or test2 $=$ true) then

;G is non-planar.

else

;G is planar.

$k = k - 1$

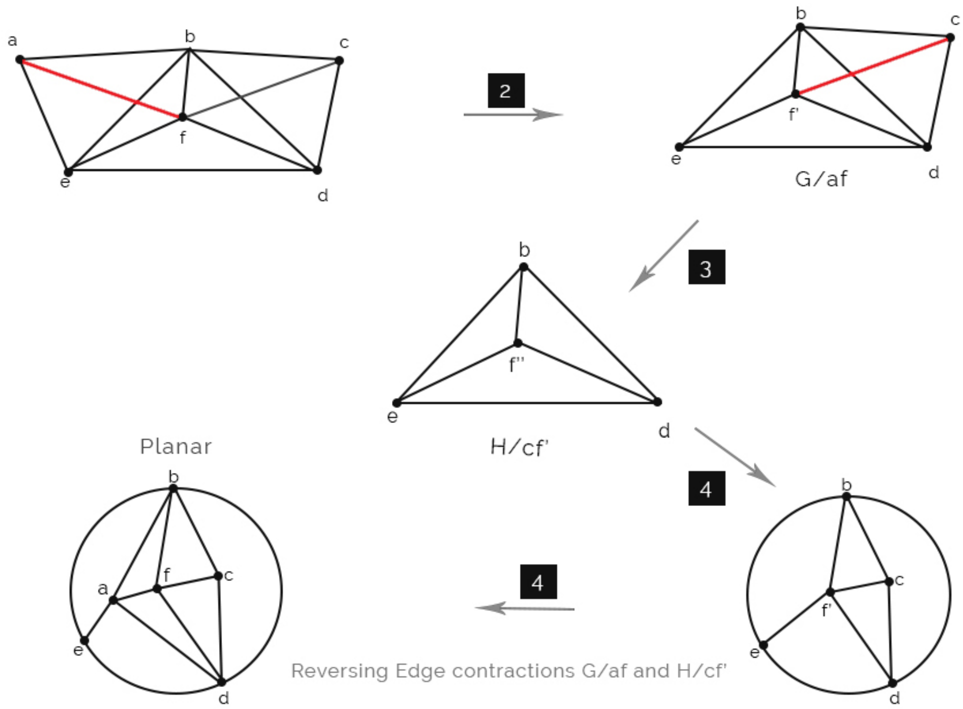

Fig. 2.8: An instance of $O(n^k)$ planarity algorithm.

3. NON-PLANAR EMBEDDINGS OF GRAPHS

3.1 Band Decompositions

Let $G \longrightarrow S$ be a 2-cell embedding of a graph in a surface. We can surround each vertex by a small disk in the surface S and each edge by a thin band such that the union of all disks and bands is a neighbourhood of G in S whose shape preserves that of the graph itself. The complement of this neighbourhood in S gives a family of open disks, one just inside each region of the embedding. Thus we define band decomposition of a surface which is the two-dimensional version of the topological construction known as a *handle decomposition* of an n-manifold.

Definition 3.1. For n=0, 1 and 2.

(i) A *1-band* is a topological space b together with a homeomorphism $h : I \times I \to b$, where $I = [0, 1]$.
$h(I \times \{j\})$ and $h(\{j\} \times I)$ for $j = 0, 1$ are called the *ends* and the *sides* of b respectively.

(ii) A *0-band* or *2-band* is simply a homeomorph of the unit disk (see Figure 3.1).

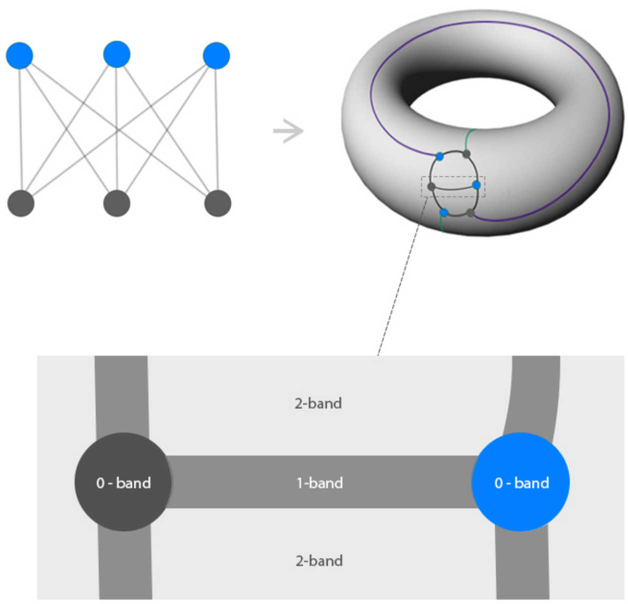

Fig. 3.1: Band Decomposition of $K_{3,3} \longrightarrow S_1$

Definition 3.2. A *band decomposition* of a surface S is a collection B of 0-bands, 1-bands and 2-bands satisfying these conditions:

(i) Different bands intersect only along arcs in their boundaries.

(ii) The union of all the bands is S.

(iii) Each end of each 1-band is contained in a 0-band.

(iv) Each side of each 1-band is contained in a 2-band.

(v) The 0-bands are pairwise disjoint and the 2-bands are pairwise disjoint.

Definition 3.3. *Reduced Band decomposition* of a graph embedding is the band decomposition with the 2-bands removed (see Figure 3.2).

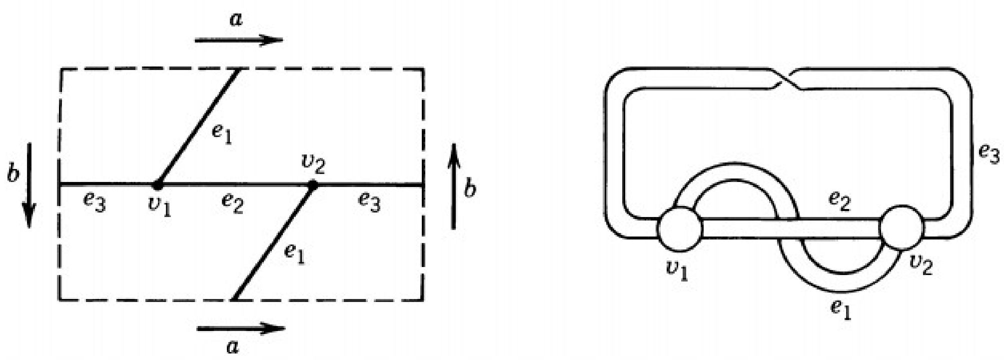

Fig. 3.2: Reduced band decomposition of graph imbedding in Klein bottle.

3.2 Orientability

Definition 3.4. Let B be a band decomposition. Then

(i) B is called *locally oriented* if each 0-band is assigned an orientation.

(ii) A 1-band is called *orientation-preserving* if the directions induced on its ends by adjoining 0-bands are the same as those induced by one of the two possible orientations of the 1-band, otherwise it is called *orientation-reversing* (see Figure 3.3).

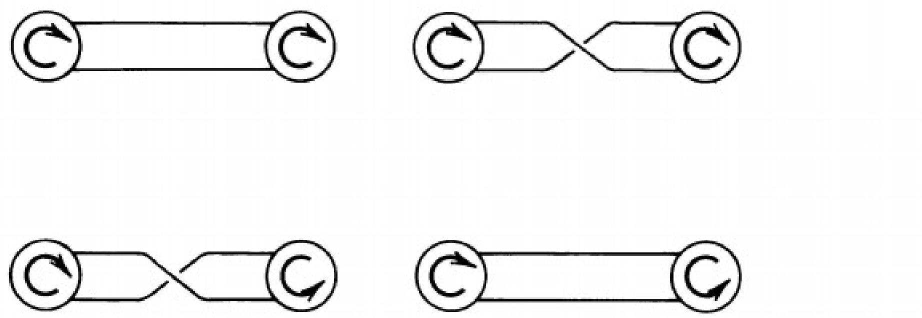

Fig. 3.3: Orientation of bands.

Definition 3.5. Consider the graph imbedding $G \to S$ with locally oriented band decomposition and edge e.

(i) Type 0: if the corresponding 1-band of e is orientation preserving.

(ii) Type 1: if the corresponding 1-band of e is orientation reversing.

3.3 Rotation System

Definition 3.6. A *rotation* at a vertex v of a graph is an ordered list, unique up to cyclic permutation, of the edges incident on that vertex.

Definition 3.7. A *rotation system* on a graph is an assignment of a rotation to each vertex and a designation of orientation type for each edge.

Remark 3.8. Representing a rotation system as a diagram).

(i) Draw dots representing each vertex with spokes radiating from the dot labelled in clockwise order according to the rotation at the vertex.

(ii) Draw curves joining spokes with the same label. Finally, all type-1 edges are marked with a cross.

These kind of diagrams are called *rotation projections* (see Figure 3.4).

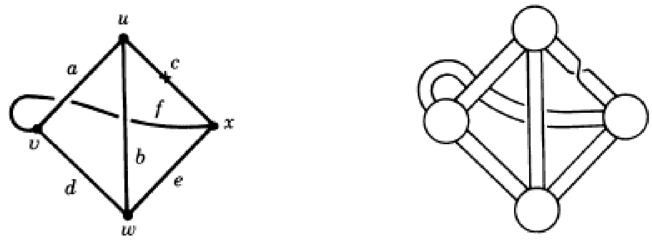

Fig. 3.4: Rotation projection of K_4 (left) and its reduced band decomposition (right).

Example 3.9. The rotation system for K_4 shown in figure 3.4 can be given a *list format* in edge form.

$$u.c^1ba$$
$$v.fad$$
$$w.dbe$$
$$x.efc^1$$

By tracing along the boundary of the reduced band decomposition surface, we can verify that that the embedding has two faces $afebcedfc$ and adb.

The following theorem which we state without proof tells the existence and uniqueness of a rotation system for every locally oriented graph embedding (see [3]).

Theorem 3.10. *Every pure rotation system (all edges are of type 0) for a graph G induces (up to orientation-preserving equivalence of embeddings) a unique embedding of G into an oriented surface. Conversely, every embedding of a graph G into an oriented surface induces a unique pure rotation system for G.*

3.4 Edge-Deletion Surgery

Definition 3.11. Let $G \to S$ be a cellular embedding of a graph in a surface, and $e \in E(G)$. Consider the band decomposition B obtained by the following operation on the band decomposition for $G \to S$:

(i) delete the 2-bands that meet the e-band

(ii) delete the e-band

(iii) close the holes with one or two 2-bands as needed.

The operations performed to obtain B is called the edge-deletion surgery.

Remark 3.12. The effect of the surgery depends on three cases that arises when the edge e is deleted.

(i) The two sides of e lie in different faces, f_1 and f_2. Then deleting $f_1 - band$, $f_2 - band$ and $e - band$ leaves one hole, which can be closed off by one new 2-band.

(ii) Face f is pasted to itself along edge e without a twist. Deleting the $f - band$ and $e - band$ leaves two distinct holes. These holes can be closed off with two new 2-bands.

(iii) Face f us pasted to itself with a twist along edge e, so that the union of the $f - band$ and the $e - band$ is a Mobius band. Deleting the $f - band$ and $e - band$ leaves one hole, that can be closed with one new 2-band.

3.5 Orientable interpolation theorem

Definition 3.13. The *genus range* of a graph, denoted $GR(G)$, is defined to be the set of numbers g such that the graph G can be cellularly embedded in the surface S_g.

Let $\gamma(G)$ and $\gamma_M(G)$ denote the minimum and maximum genus of G, respectively.

Definition 3.14. The graph embeddings $G \to S$ and $G \to T$ are called *adjacent* if there is an edge e in G such that the two embeddings $(G - e) \to S'$ and $(G - e) \to T'$ are equivalent.

Remark 3.15. Adjacent embedding surfaces differ in genus by at most one. For a graph G with two embeddings $(G - e) \to S'$ and $(G - e) \to T'$ equivalent, then the most reasonable way to create embedding of G from the embedding of $G - e$ is to attach a handle and insert e through it or simply inserting e in the same embedding if the resulting embedding remains cellular. This implies that the respective adjacent embeddings of G differ by at most genus 1.

Theorem 3.16. *Let G be a connected graph. Then the genus range $GR(G)$ is an unbroken interval of integers, that is, $GR(G) = [\gamma(G), \gamma_M(G)]$.*

Proof. Let G_1 and G_n be the cellular embeddings with genus $\gamma(G)$ and $\gamma_M(G)$ respectively. From Remark 3.11 we can find unique rotation systems R_1 and R_n for G_1 and G_n respectively. Let L_i denote the list format for rotation system R_i. Then, there exists a permutation π that takes L_1 to L_n. Then take the decomposition of π into transpositions. For instance, let L_i and L_j be list formats in a transposition, then L_j may be obtained from L_i by moving one edge symbol at a time. These consecutive list formats represent adjacent embeddings. This means that there exists a sequence of adjacent cellular embeddings of the G from $S_{\gamma(G)}$ $(= G_1)$ to $S_{\gamma_M(G)}$ $(= G_n)$. But we know that, the adjacent embedding surfaces differ by at most one in genus and hence the conclusion. $\square$

3.6 Maximum Genus of a Graph

Objective of this section is to calculate the maximum genus $\gamma_M(G)$ of graph G. The most reasonable approach is to find if there exists a one-face embedding of G on some surface S_g.

Definition 3.17. The *Betti number* $\beta(G)$ of a connected graph $G = (V, E)$, is defined by the equation

$$\beta(G) = 1 - \#V + \#E$$

and it is equal to the number of edges in the complement of a spanning tree.

Lemma 3.18. Let d and e be adjacent edges in a connected graph G such that $G - d - e$ is a connected graph having an orientable one face embedding. Then the graph G has a one face orientable embedding.

Proof. Let $d = uv$ and $e = vw$ be the two adjacent edges. Extend the one face embedding $(G - d - e) \longrightarrow S$ to a two-face embedding $(G - e) \longrightarrow S$ by placing the image of d across the single face. Observe that the vertex v lies in both faces. Now, attach a handle from one face of $(G - e) \longrightarrow S$ to the other and place the image of e via the handle. This create a one-face embedding (see Figure 3.5). $\square$

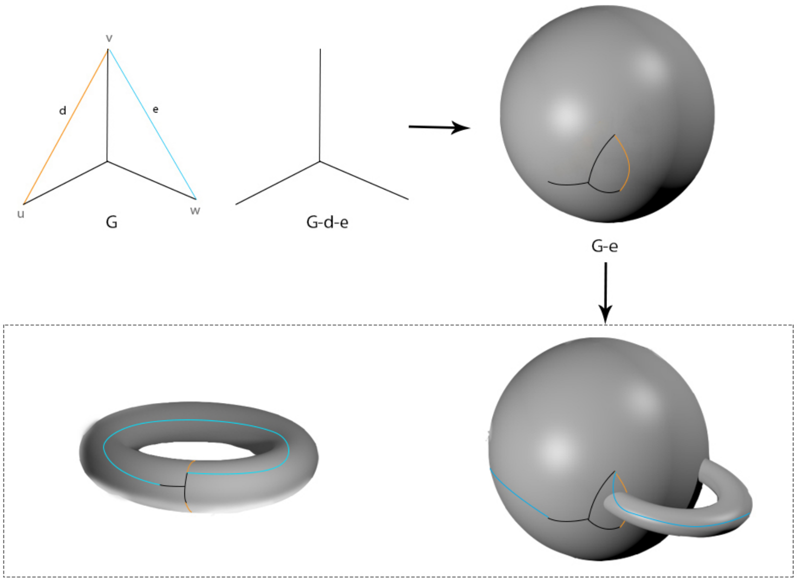

Fig. 3.5: An instance of the proof

Lemma 3.19. Let $G = (V, E)$ be a connected graph such that every vertex has valence atleast 3, and let G have a one-face orientable embedding $G \longrightarrow S$. Then there exists adjacent edges d and e in G such that $G - d - e$ has a one-face orientable embedding.

Proof. Let $d \in E$ whose two occurrences in the single boundary walk of the embedding $G \longrightarrow S$ are the closest together, among all other edges. Let the boundary walk be $dAdB$ where no edge appear twice in the sub walk. The edge deletion surgery on d in the embedding $G \longrightarrow S$ yields a two-face embedding $(G - d) \longrightarrow S'$. The boundary walks of the two faces are A and B and the edge e appears in both A and B. Thus the result of edge-deletion surgery on e in the embedding $(G - d) \longrightarrow S$ is a one-face embedding of $G - d - e$ (see Figure 3.6). $\square$

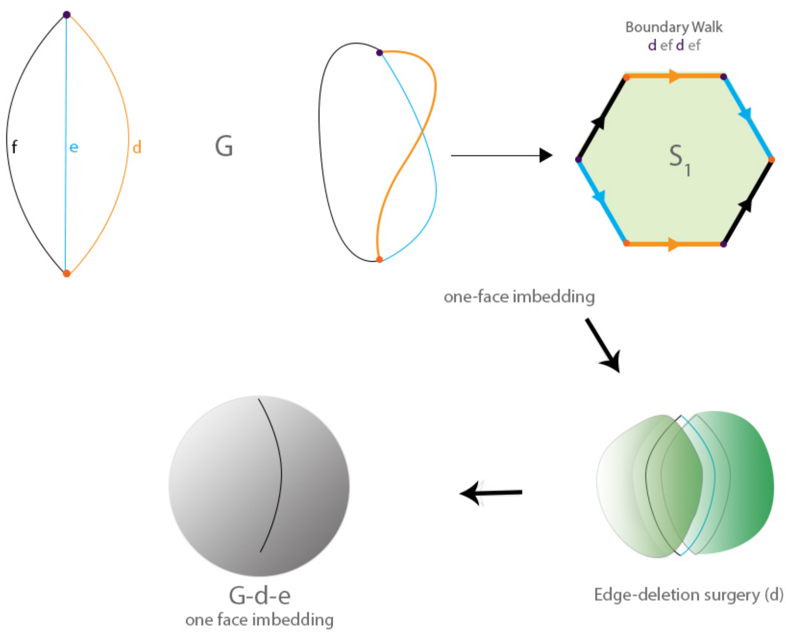

Fig. 3.6: An instance of the proof

Definition 3.20. The deficiency $\xi(G,T)$ of a connected graph G with respect to the spanning tree T is defined to be the number of components of $G-T$ that have an odd number of edges (see Figure 3.7).

Definition 3.21. The deficiency $\xi(G,T)$ of a connected graph G is defined by

$$\xi(G) := Min\{\xi(G,T) : T \in ST(G)\}$$

where ST is the collection of all spanning trees of G.

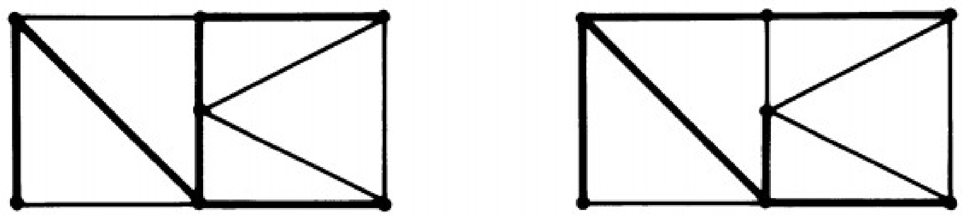

Fig. 3.7: Left: deficiency = 3 , Right: deficiency = 1

Lemma 3.22. Let T be a spanning tree for graph G, and let d and e be pair of adjacent edges in $G-T$. Then $\xi(G-d-e,T)=0$ if and only if $\xi(G,T)=0$.

Proof. Every component of $G-d-e-T$ that meets either of the edges d or e has an even number of edges, since $\xi(G-d-e,T)=0$. It follows that the number of edges in the components of $G-T$ that contains the edges d and e is even, and all other components of $G-T$ has evenly many edges as in $G-d-e-T$. This implies that $\xi(G,T)=0$. By a similar argument we can prove the converse.

□

Theorem 3.23 (Xuong, 1979)**.** *Let G be a connected graph. Then G has a one-face orientable embedding if and only if $\xi(G)=0$.*

Proof. We prove by induction. Assume that $\xi(G)=0$ for any graph G with n or fewer edges. Let $G=(V,E)$ be a graph with $n+1$ edges and $valence(v)\geq 3$, for all $v\in V$.
Suppose that G has one-face orientable embedding. Then by Lemma 3.19, there exists adjacent edges $d,e\in E$ such that $G-d-e$ has one face embedding. Then by the induction hypothesis $\xi(G-d-e)=0$. This implies by Lemma 3.22 that there exists a spanning tree T in $G-d-e$ such that $\xi(G-d-e,T)=0$. This implies that $\xi(G,T)=\xi(G)=0$, since T spans G.

Conversely, suppose $\xi(G) = 0$. Then there exists a spanning tree T such that $\xi(G,T) = 0$. It follows from the above lemmas that there exists adjacent edges $d, e \in E_{G-T}$ such that $\xi(G-d-e,T) = 0$. This means that the graph $G-d-e$ has a one-face orientable embedding by induction hypothesis. It follow by Lemma 3.18 that G has a one-face orientable embedding.

□

Theorem 3.24 (Xuong, 1979). *Let G be a connected graph. Then the minimum number of faces in any orientable embedding of G is exactly $\xi(G)+1$.*
For any embedding of G,

$$\#F \geq \xi(G)+1.$$

We can rephrase the above theorem as follows:

Theorem 3.25. *The graph G has an orientable embedding with $n+1$ or fewer faces if and only if $\xi(G) \leq n$.*

Proof. We prove this by inducting on the number of faces n. It holds for $n = 0$. Assume that the statement is true for all values of k less than or equal to n and $n > 0$. We will now be using the arguments in Lemmas 3.18, 3.19, 3.22 and Theorem 3.23 to prove the theorem.
Suppose $G \longrightarrow S$ is an orientable embedding with $\#F_G = n+1$. Perform edge-deletion surgery on an edge e common to two faces of the embedding. The resulting embedding $G - e \longrightarrow S'$ has n faces. Then, by induction hypothesis, $\xi(G-e) \leq n-1$ which implies that $\xi(G) \leq n$.
Conversely, Suppose $\xi(G) = n$, then there exists a spanning tree T in G such that $\xi(G,T) = n$. Let H be a component of $G-T$ with odd number of edges. Now choose an edge e from H that does not disconnect H or such that one end point of e has valence 1. It follows that $\xi(G-e,T) = n-1$. Thus, by the

induction hypothesis, the graph $G - e$ has an orientable embedding with at most n faces. Hence, G has an orientable embedding with at most $n + 1$ faces. □

Corollary 3.26 (Xoung, 1979)**.** Let G be a connected graph. Then

$$\gamma_M(G) = \frac{1}{2}(\beta(G) - \xi(G)).$$

Proof. Let $g = \gamma_M(G)$. Then by Theorem 3.25, we have

$$\begin{aligned} 2 - 2g &= \#V - \#E + (\xi(G) + 1) \\ \Rightarrow 2g &= (1 + \#E - \#V) - \xi(G) \\ \Rightarrow g &= \frac{1}{2}(\beta(G) - \xi(G)). \end{aligned}$$

□

3.7 Heffter-Edmonds Algorithm

An exponential time algorithm [3] for calculating the minimum genus γ_m of a graph, based on rotation system enumeration.

Algorithm

(i) List all the pure rotation systems of a graph.

(ii) Compute the number of faces for each rotation system.

(iii) Choose the one having the most faces and calculate the genus.

Run time is $O(2^k)$ because a regular $(r + 1)$ valent graph with n vertices has $(r!)^n$ pure rotation systems.

4. HEAWOOD PROBLEM

In this chapter we will review the Heawood problem [3] and its solution with respect to closed orientable surfaces S_g of genus $g > 0$. For a sphere S_0, the problem is popularly known as the Four Color Theorem [4].

Definition 4.1. A graph G is said to be *n-coloured* with a set C of n distinct elements (called colours) if there exists a surjective map $\zeta_n : V(G) \longrightarrow C$ such that $\zeta_n(v_1) \neq \zeta_n(v_2)$ for any adjacent vertices $v_1, v_2 \in V(G)$.

The n-colouring of a graph G can also be defined in terms of edge set $E(G)$ or by the face set $F(G)$ of an embedding of G. We will be using the above (vertex) definition throughout this thesis.

Definition 4.2. The *chromatic number* $chr(G)$ *of a graph* G is defined to be the smallest number n such that G has an n-colouring.

Definition 4.3. The *chromatic number of a surface* S is equal to the maximum of the set of chromatic numbers of simplicial graphs that can be embedded in S.

Definition 4.4. A graph G is called *chromatically critical* if $chr(G-e) < chr(G)$ for any edge $e \in E(G)$.

Remark 4.5. For complete graph K_n, the following holds.

(i) $chr(K_n) = n$, since any two edges are adjacent in K_n.

(ii) K_n is chromatically critical. If we remove any edge $e = uv$ from K_n then we can colour u and v with the same colour, i.e. $chr(K_n - e) < chr(K_n)$.

4.1 Heawood problem

For a surface S with Euler characteristic $c \leq 1$, Percy John Heawood [3] showed there is a finite maximum for $chr(S)$, that is, there exists $H(S)$ such that $chr(S) \leq H(S)$. The finite maximum $H(S)$ is called the *Heawood number* of the surface S.

The determination of the chromatic numbers of the surfaces other than the sphere is called the *Heawood Problem*. The solution of the Heawood problem is that $chr(S) = H(S)$, except for the Klein bottle. Due to the complexity of the original solution of the Heawood problem [3], we will narrow down our exploration of the Heawood problem to a basic overview of the implementation of the Ringel-Young solution [3] for closed orientable surfaces. The initial step in this pursuit is to derive the *Heawood inequality*. We will then use the Heawood inequality to reduce the Heawood problem to finding the genus of the complete graphs.

4.2 Heawood inequality

Lemma 4.6. Let S be a closed surface of Euler characteristic c, and let G be a simplical graph embedded in S. Then

$$average\, valence(G) \leq 6 - \frac{6c}{\#V}$$

Proof. From the Euler's equation, we get

$$\#V - \#E + \#F \geq c$$

and by the edge-region inequality we get $\#F \leq \frac{2}{3}\#E$. Set $\#F$ to be $\frac{2}{3}\#E$ in the above inequality. Then,

$$\begin{aligned} \#V - \frac{1}{3}\#E &\geq c \\ \Rightarrow \frac{\#E}{\#V} + \frac{3c}{\#V} &\leq 3 \\ \Rightarrow \frac{2\#E}{\#V} + \frac{6c}{\#V} &\leq 6 \\ \Rightarrow average\, valence(G) &\leq 6 - \frac{6c}{\#V}, \end{aligned}$$

since $\sum_{v \in V} valence(v) = 2\#E$, By Theorem 1.32. □

Lemma 4.7. Let S be a closed surface, and let G be a chromatically critical graph such that $chr(G) = chr(S)$. Then for every vertex v, $chr(S) - 1 \leq valence(v)$.

Proof. Suppose v is a vertex in G with $valence(v) < chr(S) - 1$. Since G is chromatically critical, we get $chr(G - v) \leq chr(S) - 2$. This means that we can colour the neighbours of v in $G - v$ with $chr(S) - 2$ colours. This implies that we can colour v and its neighbourhood with atleast $chr(S) - 1$ colours in G, that is, $chr(G) \leq chr(S) - 1$. This contradicts the assumption $chr(G) = chr(S)$. □

Theorem 4.8 (Heawood, 1890). *Let S be a closed surface with Euler characteristic $c \leq 1$. Then*

$$chr(S) \leq \left\lfloor \frac{7 + \sqrt{49 - 24c}}{2} \right\rfloor = H(S) \tag{4.1}$$

Proof. If $c = 1$, $H(S) = 6$. S is the projective plane (see Figure 4.1) and $chr(S) = 6$. The inequality holds.

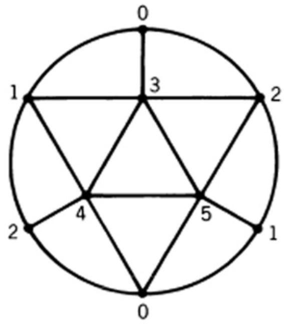

Fig. 4.1: Embedding of K_6 in projective plane.

For $c \leq 0$, let G be a graph embedded in S such that $chr(G) = chr(S)$ and G is chromatically critical. Then by Lemma 4.6 and 4.7, we get

$$chr(S) - 1 \leq average\ valence(G) \leq 6 - \frac{6c}{\#V},$$

which implies that

$$\begin{aligned} chr(S) - 1 &\leq 6 - \frac{6c}{\#V} \\ \Rightarrow chr(S) - 1 &\leq 6 - \frac{6c}{chr(S)} \\ \Rightarrow chr^2(S) - 7chr(S) + 6c \leq 0 \end{aligned}$$

Consider the inequality $chr^2(S) - 7chr(S) + 6c \leq 0$, the quadratic polynomial in the left side of the inequality yields

$$\left(chr(S) - \frac{7 - \sqrt{49 - 24c}}{2}\right)\left(chr(S) - \frac{7 + \sqrt{49 - 24c}}{2}\right) \leq 0$$

For $c \leq 0$,the first factor is a non zero positive number and $chr(S)$ is an integer. Hence, we have the required result.

$\square$

4.3 Complete Graph Embedding

For surface S_g of genus g, $H(S_g) = \left\lfloor \frac{7+\sqrt{1+48g}}{2} \right\rfloor$. The Figure 4.2 shows the Heawood number for surfaces of genus g where $1 \leq g \leq 23$.

genus g	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$H(S_g)$	7	8	9	10	11	12	12	13	13	14	15	15	16	16	16	17	17	18	18	19	19	19	20

Fig. 4.2: Table showing $H(S_g)$ for corresponding genus g.

Theorem 4.9.

$$\gamma(K_n) \geq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \tag{4.2}$$

Proof. By Euler equation, we have $\#V - \#E + \#F = 2 - 2\gamma(K_n)$ and from edge-region inequality, we have $\#F \leq \frac{2}{3}\#E$. Setting $\#F = \frac{2}{3}\#E$, we get

$$\begin{aligned} \#V - \frac{1}{3}\#E &\geq 2 - 2\gamma(K_n) \\ \Rightarrow \gamma(K_n) &\geq \frac{n^2 - 7n + 12}{12} \\ &\geq \frac{(n-3)(n-4)}{12}. \end{aligned}$$

Since $\gamma(K_n)$ is an integer, the conclusion follows. $\square$

Remark 4.10. It is possible to triangulate surfaces by complete graphs. By complete graph orientable embedding inequality,

$$\gamma(K_n) \geq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$$

construct minimum embeddings of the complete graphs. The general form of these embeddings seems to depend strongly on the residue class of $n \pmod{12}$.

Let $I(n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ then $\left(I(n) - \frac{(n-3)(n-4)}{12} \right)$ measures how much an orientable embedding $K_n \longrightarrow S_{I_n}$ fails to be a triangulation. If $((n-3)(n-4)) \pmod{12} = 0$ then $\left(I(n) - \frac{(n-3)(n-4)}{12} \right) = 0$. This implies that $K_n \longrightarrow S_{I_n}$ has a triangulation by K_n (see [3]).

Appendices

Appendix A

COMPUTATIONAL COMPLEXITY BASICS

RAM (Random-Access Machine) Model

- No concurrent operations
- Common instructions (arithmetic, memory) take constant time.
- Data Type - Integer and Floating point.

Definition A.1 (Run Time $T(n)$)**.** The number of primitive operations or steps executed, for an algorithm for a particular input (size n).

Asymptotic Efficiency of algorithm: Growth of run time with respect to increase in size of input.

Definition A.2 (Big-O Notation)**.** $f(n)$ is $O(g(n))$ if there exist positive number c and N such that $f(n) \leq cg(n) \; \forall \, n > N$.

Run time $T(n) = O(f(n))$ ordered in decreasing order of asymptotic efficiency.

$$O(1) > O(log_2(n)) > O(n) > O(nlog_2(n)) > O(n^c) > O(2^n).$$

For more details refer [2].

Appendix B

ADDITIONAL THEOREMS

Theorem B.1. *Let $G \longrightarrow S_g$ be a cellular embedding, for any $g = 0, 1, 2, ...$ Then $\chi(G \longrightarrow S_g) = 2 - 2g$.*

Proof. Omitted, see [3] p.112. $\square$

Corollary B.2. Let i and j be distinct non-negative integers. Then S_i and S_j are not homeomorphic.

Proof. A homeomorphism $f : S_i \longrightarrow S_j$ would carry a cellular embedding $G \longrightarrow S_i$ of relative Euler characteristic $\chi(G \longrightarrow S_i) = 2 - 2i$ to a cellular embedding $G \longrightarrow S_j$ of relative Euler characteristic $\chi(G \longrightarrow S_i) = 2 - 2i$, in violation of the invariance of Euler characteristic, which implies that $\chi(G \longrightarrow S_j) = 2 - 2j$. $\square$

Remark B.3. Let $G \longrightarrow S$ be a cellular graph embedding, and let e be an edge of the graph. Let F be the set of faces for $G \longrightarrow S$, and let F' be the set of faces of the embedding obtained by edge-deletion surgery at e. Then in cases (see Remark 3.11)

1. $\#F' = \#F - 1$, and the resulting surface is homeomorphic to S.

2. $\#F' = \#F + 1$.

3. $\#F' = \#F$.

Theorem B.4 (Classification of Surfaces)**.** *Every closed, connected, orientable surface is homeomorphic to one of the standard surfaces S_g with $g \geq 0$.*

Proof. Omitted, see [3] p.128 . $\square$

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