Second Largest Eigenvalues of Some Cayley Graphs on Alternating Groups

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Srabana Biswas

ABSTRACT

Let $\mathrm{Cay}(G, \Gamma)$ be the Cayley graph of the group G with respect to the generating set Γ, and let A_n be the alternating group on n symbols. The primary goal of this thesis is to understand second largest eigenvalues of some Cayley graphs on alternating groups, namely the alternating group graph $AG_n = \mathrm{Cay}(A_n, T_1)$, where $T_1 = \{(1, 2, i), (1, i, 2) \mid 3 \leq i \leq n\}$, extended alternating group graph $EAG_n = \mathrm{Cay}(A_n, T_2)$, where $T_2 = \{(1, i, j), (1, j, i) \mid 2 \leq i < j \leq n\}$, and the complete alternating group graph $CAG_n = \mathrm{Cay}(A_n, T_3)$, where $T_3 = \{(i, j, k), (i, k, j) \mid 1 \leq i < j < k \leq n\}$, as derived in [5]. To this end, we explore the necessary background from spectral and algebraic graph theory. In particular, we study the notions of a Cayley graph, the isoperimetric constant of a graph, and the Rayleigh-Ritz Theorem, which yields a bound on the isoperimetric constant of a k-regular graph. Furthermore, we study the Alon-Boppana theorem, which provides a lower bound on the second-largest eigenvalue of the adjacency matrix of a k-regular graph. Finally, we will discuss the concept of equitable partitions of graphs, and how they connect with the central theme of this thesis.

LIST OF FIGURES

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1. INTRODUCTION

Algebraic graph theory is a rapidly expanding area of mathematics where various methods from linear algebra and group theory are used to address problems in graph theory. In a broad sense, linear algebra is used to analyze the spectra of graphs and group theory is used for studying the symmetries of graphs. Moreover, these areas all connect with other branches of mathematics, such as logic and harmonic analysis. In spectral graph theory, the eigenvalues of matrices (i.e graph spectra) associated with graph such as the adjacency matrix, Laplacian matrix, etc. are studied to understand fundamental properties of graphs such as connectivity. In this regard, the *spectral gap* of a graph, which is the difference in magnitude of the two largest eigenvalues of its adjacency matrix, is one of the fundamental measures of graph connectivity. Thus, families of graphs which have large spectral gaps (also known as *expander families*) have been widely studied from the viewpoint of their applicability to communication networks.

One of the basic objects of study in algebraic graph theory are Cayley graphs, which are graphs that capture the algebraic structures of the groups. Thus, Cayley graphs possess an intrinsic symmetry that they inherit from the groups they represent. It is a natural pursuit to examine the expander properties of Cayley graphs yielded by infinite families of groups.

2. INTRODUCTION TO GRAPH THEORY

Graphs can be used to model many types of relations and processes in physical, biological, social and information systems where it has a wide range of useful application. In this chapter, we will introduce some basic notations and terminologies in graph theory that will use later. Also, we will see examples of graph spectra and with the help of the matrix representation of a graph, we will derive some important results in spectral graph theory. The references used for this chapter are [2, 3, 6, 7].

2.1 Graphs

Definition 2.1. An *undirected graph* G is a pair of sets (V, E) where V is a non-empty set of vertices and E is a multiset of edges of the form $\{v, w\}$ with $v, w \in V(X)$.

Definition 2.2. Let $X = (V(X), E(X))$ be a graph. Then:

(i) A *loop* is an edge of X whose endpoints are equal.

(ii) *Multi-edges* are edges having the same pair of endpoints.

(iii) Two vertices $u, v \in V(X)$ are said to be *adjacent or neighbours* if $\{u, v\} \in E(X)$.

(iv) The *degree* of a vertex $v \in V(X)$, denoted by $deg(v)$, is the number of vertices adjacent to v.

(v) X is said to be *k-regular* if $deg(v) = k$, for every $v \in V(X)$.

Example 2.3. In Figure 2.1, we give an example of a graph $G = (V, E)$, where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{\{v_1\}, \{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}$. The green coloured edge $\{v_1\}$ is a *loop*, the blue coloured edges $\{v_2, v_3\}$ and the red coloured edges $\{v_3, v_4\}$ are two pairs of *multi-edges*. Note that the *degree* of each vertex in the graph is four. Thus, the graph is a 4-*regular* graph.

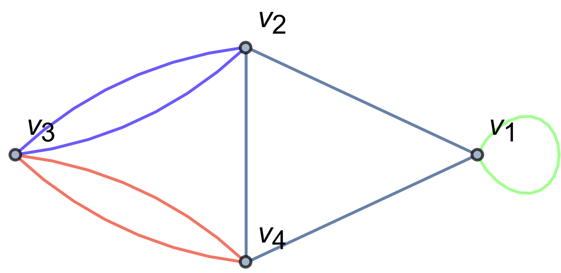


Fig. 2.1: A 4-regular graph G.

Definition 2.4. A *simple graph* is a graph having no loops or multi-edges.

For example, the graph obtained from the graph G in Figure 2.1 by removing the green coloured edge and one of each blue and red coloured edges is a simple graph.

Definition 2.5. Let $G = (V, E)$ be a graph.

(i) A *walk* of length k in G is an alternating sequence of vertices and edges $\{v_0, e_1, v_1, \ldots, e_k, v_k\}$, where $e_i = \{v_{i-1}, v_i\}$.

(ii) A walk in G with no repeated vertex is called a *path*.

(iii) A path in G of non-zero length from a vertex to itself is called a *cycle*.

(iv) A walk in G with no repeated edge is called a *trail*.

(v) A closed trail in G is called a *circuit*.

Definition 2.6. A graph that is a cycle of length n is called a n-*cycle*, which we will denote by C_n.

Definition 2.7. A graph G is said to be *connected* if there is a walk in between any two vertices of G.

Definition 2.8. Let $G = (V, E)$ be a graph. For $x, y \in V$, the *distance* $\text{dist}(x, y)$ between x and y defined as the minimal length of any walk between x and y.

Definition 2.9. The *diameter* $\text{diam}(G)$ of a graph G is defined by

$$\text{diam}(G) = \max_{x,y \in V} \text{dist}(x, y).$$

Example 2.10. Consider the graph X shown in Figure 2.2. Observe that $\text{dist}(2, 5) = 2$, and $\text{diam}(X) = 3$, which is realized as $\text{dist}(2, 7)$.

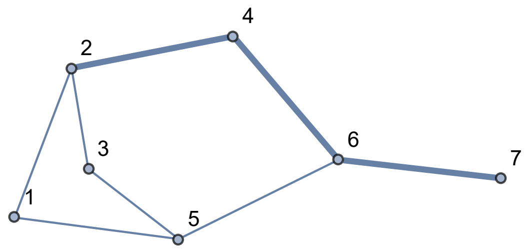


Fig. 2.2: The path realizing the diameter of a graph X.

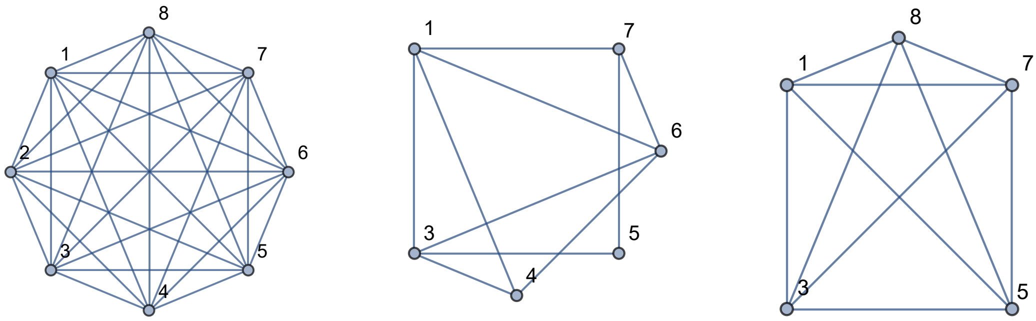
Definition 2.11. A graph H is said to be a *subgraph* of the graph G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$.

Definition 2.12. Let $G = (V, E)$ be any graph, and let $S \subset V$. Then the *induced subgraph* of G on S is the graph $G[S] = (V', E')$ defined as follows:

(i) $V' = S$, and

(ii) $E' = \{\{u, v\} \in E : u, v \in V'\}$.

Example 2.13. Consider the graph K_8 in Figure 2.3(a) below. Observe that the graph X in Figure 2.3(b) is a subgraph of K_8, and the graph Y in Figure 2.3(c) is an induced subgraph of K_8 on the subset $S = \{1, 3, 5, 7, 8\}$ of $V(K_8)$.



(a) K_8 (b) X (c) Y

Fig. 2.3: Subgraphs of a graph.

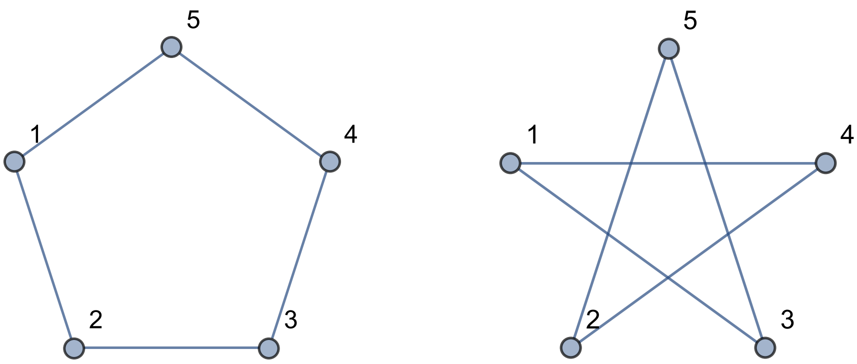
Definition 2.14. Let $G = (V, E)$ and $G' = (V', E')$ be graphs. A *graph homomorphism* between G and G' is a map $\phi : V \to V'$ such that $\{\phi(a), \phi(b)\} \in E'$, whenever $\{a, b\} \in E$.

Definition 2.15. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. The graphs are called *isomorphic* if

(i) $|V| = |V'|$, and

(ii) there exists a bijective map $\phi : V \to V'$ such that $\{a, b\} \in E \iff \{\phi(a), \phi(b)\} \in E'$.

Example 2.16. Consider the graphs $G = C_5$ and H in Figure 2.4. The graph in 2.4(b) is obtained by permuting the vertex set of 2.4(a).



(a) G (b) H

Fig. 2.4: Isomorphic graphs.

Some important graphs

Definition 2.17. A *directed graph* (or *digraph*) G is a pair (V, E), where V is a finite non-empty set of *vertices* of G and $E \subset V \times V$ is a set of *edges* in G.

We visualize the ordered pair (u, v) as an arrow incident at u and pointing towards v. We call (u, v) the directed edge from u to v. We call u the *initial point* and v the *terminal point* of the directed edge (u, v). A directed edge of the form (v, v) is called a *loop* at v.

Example 2.18. The graph in Figure 2.5 is a digraph. Here, 2 is the initial point and 3 is the terminal point of $(2, 3)$.

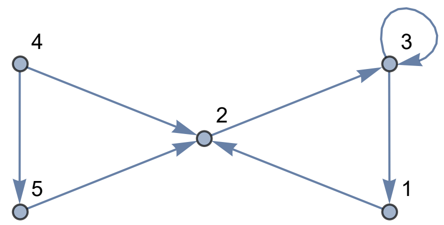
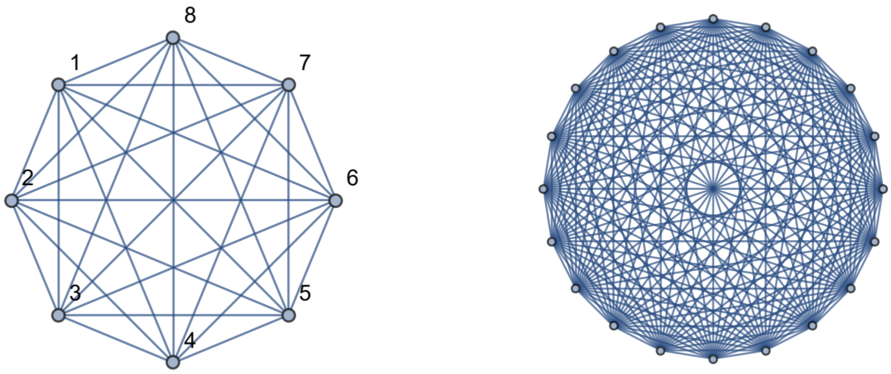


Fig. 2.5: A directed graph.

Definition 2.19. A graph is said to be *complete* if every distinct pair of vertices of the graph is connected by a unique edge.

A complete graph with n vertices is denoted by K_n.

Example 2.20. The graphs K_8 and K_{20} in Figure 2.6 are examples of complete graphs.



(a) K_8 (b) K_{20}

Fig. 2.6: Two examples of complete graphs.

Definition 2.21. A graph $G = (V, E)$ is called *bipartite* if there exists $V_1, V_2 \subset V$ such that:

(i) $V = V_1 \sqcup V_2$, and

(ii) $E = \{\{v, w\} : v \in V_1 \text{ and } w \in V_2\}$.

Example 2.22. The graph in Figure 2.7 is an example of a bipartite graph with the partition $V_1 = \{1, 4, 6, 7\}$ and $V_2 = \{2, 3, 5, 8\}$ coloured in black and white, respectively.

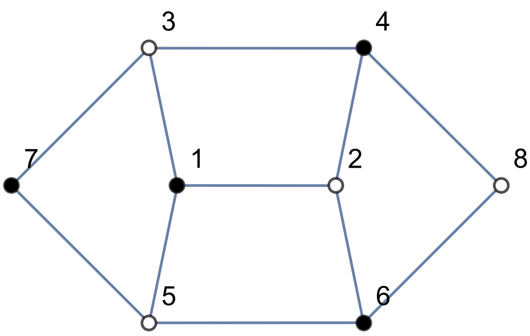


Fig. 2.7: A bipartite graph.

Remark 2.23. Note that C_n is bipartite when n is even.

Definition 2.24. Let $G = (V, E)$ be a bipartite graph, with vertex partition V_1 and V_2, such that $E = \{\{v_i, v_j\} \,|\, v_i \in V_1, v_j \in V_2, \text{ and } i \neq j\}$. Then G is called a *complete bipartite graph*.

2.2 Matrix representation

Definition 2.25. Let X be a finite graph with vertex set $V(X) = \{v_1, v_2, \ldots, v_n\}$. Then the *adjacency matrix* of X is defined as $A(X) = (a_{ij})_{n\times n}$, where a_{ij} = the number of edges incident to v_i and v_j.

Since $A(X)$ is a symmetric matrix, its eigenvalues are real, and we can order these as:

$$\lambda_{n-1}(X) \le \lambda_{n-2}(X) \le \ldots \le \lambda_1(X) \le \lambda_0(X).$$

Remark 2.26. The adjacency matrix of a bipartite graph is of the form

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}.$$

Definition 2.27. Let X be a graph, and let $\mu_1, \mu_2, ..., \mu_r$ be distinct eigenvalues of $A(X)$ with multiplicity $m_1, m_2, ..., m_r$, respectively. Then *spectrum* of X is defined by

$$\text{Spec}(X) = \begin{bmatrix} \mu_1 & \mu_2 \ldots & \mu_r \\ m_1 & m_2 \ldots & m_r \end{bmatrix}.$$

Example 2.28. Consider the graph $K_{3,2}$ shown in Figure 2.8 below.

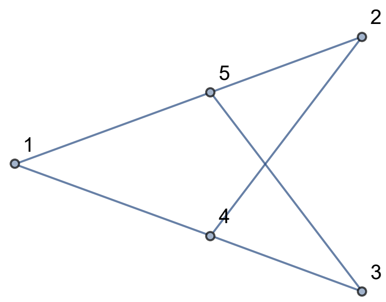


Fig. 2.8: The graph $K_{3,2}$.

Then:

$$A(K_{3,2}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix},$$

and $\mathrm{Spec}(K_{3,2}) = \begin{bmatrix} \sqrt{6} & -\sqrt{6} & 0 \\ 1 & 1 & 3 \end{bmatrix}$.

Definition 2.29. Let X be a simple graph with vertex set $V(X) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(X) = \{e_1, e_2, \ldots, e_m\}$. The *incidence matrix* of X is given by: $M(X) = (m_{i,j})_{n\times m}$, where,

$$m_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ is an endpoint of } e_j, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.30. For the graph $K_{3,2}$ in Figure 2.8, we have:

$$M(K_{3,2}) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

2.3 Adjacency operator

For a finite set S, we define complex vector space $L^2(S)$ by $L^2(S) = \{f : S \to \mathbb{C}\}$. Given $f, g \in L^2(S)$ and $\alpha \in \mathbb{C}$ the vector space $L^2(S)$ has the following properties:

(i) $(f + g)(x) = f(x) + g(x)$,

(ii) $(\alpha f)(x) = \alpha f(x)$,

(iii) $\langle f, g\rangle_2 = \sum_{x\in S} f(x)\overline{g(x)}$, and

(iv) $\left\|f\right\|_2 = \sqrt{\langle f, f\rangle_2} = \sqrt{\sum_{x\in S} |f(x)|^2}$.

Definition 2.31. Let $X = (V, E)$ be a graph with $V = \{v_1, \dots v_n\}$ and $A = A(X)$. Given $f \in L^2(X)$, we may think of f as a vector in $\mathbb{C}^n$, and we define:

$$Af = \begin{bmatrix} A_{v_1,v_1} & A_{v_1,v_2} & \dots & A_{v_1,v_n} \\ A_{v_2,v_1} & A_{v_2,v_2} & \dots & A_{v_2,v_n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{v_n,v_1} & A_{v_n,v_2} & \dots & A_{v_n,v_n} \end{bmatrix} \begin{bmatrix} f(v_1) \\ f(v_2) \\ \vdots \\ f(v_n) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} A_{v_1,v_j} f(v_j) \\ \sum_{j=1}^{n} A_{v_2,v_j} f(v_j) \\ \vdots \\ \sum_{j=1}^{n} A_{v_n,v_j} f(v_j) \end{bmatrix}.$$

Thus, we view Af as a linear transformation from $L^2(X)$ to itself by

$$(Af)(v) = \sum_{w\in V} A_{v,w} f(w).$$

Here, A is called the *adjacency operator* of X and f is called the *eigenfunction* of A.

Proposition 2.32. *Let $X = (V, E)$ be a k-regular graph with n vertices. Then:*

(i) k is an eigenvalue of X,

(ii) $|\lambda_i(X)| \le k$, for $0 \le i \le n-1$, and

(iii) $\lambda_1(X) < \lambda_0(X)$ if and only if X is a connected graph.

Proof. Let $A(X) = A = (a_{ij})_{n\times n}$, and f be the eigenfunction associated to some eigenvalue λ, and let $v \in V$ be such that $|f(v_i)| = \max_{v_j\in V} |f(v_j)|$, for $1 \le j \le n$.

(i) Let $f_0 \in L^2(V)$ such that $f_0(v_i) = 1$, for $1 \le i \le n$. Then we have:

$$(Af_0)(v_i) = \sum_{j=1}^{n} a_{i,j} f_0(v_j)$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} a_{i,j} \\
&= k \\
&= k.f_0(v_i).
\end{aligned}
$$

Hence, k is an eigenvalue of X.

(ii) By definition, we have:

$$
\begin{aligned}
|\lambda||f(v_i)| &= |Af(v_i)| \\
&= |\sum_{j=1}^{n} a_{i,j} f(v_j)| \\
&\leq \sum_{j=1}^{n} |a_{i,j}||f(v_j)| \\
&\leq |f(v_i)| \sum_{j=1}^{n} |a_{i,j}| \\
&= k.|f(v_i)|.
\end{aligned}
$$

Thus, the assertion follows, as $f \neq 0$.

(iii) ($\Leftarrow$) Let X be connected. We will show that $\lambda_1 < d$.
Without loss of generality, we may assume that $f(v_i) > 0$. Then by definition, we have:

$$
\begin{aligned}
kf(v_i) &= (Af)(v_i) = \sum_{j=1}^{n} a_{i,j} f(v_j) \\
\implies f(v_i) &= \sum_{j=1}^{n} \frac{a_{i,j}}{k} f(v_j).
\end{aligned}
$$

Now, suppose that $f(u) < f(v_i)$ for some u adjacent to v_i. Then as $f(v_j) \leq f(v_i)$, for all $v_j \in V$, it follows that:

$$
f(v_i) = \frac{\sum_{j=1}^{n} a_{i,j}}{k} f(v_j)
$$

$$< f(v_i)\frac{\sum_{j=1}^{n} a_{i,j}}{k}$$
$$= f(v_i),$$

which is a contradiction.
Hence, $f(v_j) = f(v_i)$, for each v_j adjacent to v_i and since X is connected, $v_j = v_i$, for all $1 \leq j \leq n$.

($\Rightarrow$) Let X be a disconnected graph. Pick a vertex $v_i \in V$ and define V_1 as the set of vertices connected to v_i and $V_2 = V \setminus V_1$. Thus by definition, V can be divided into two non-empty disjoint k-regular subgraphs. Hence, the multiplicity of k is more than 1.

□

Theorem 2.33. *Let X be a bipartite graph. Then the spectrum of X is symmetric about 0, that is, if λ is an eigenvalue of X then $-\lambda$ is also an eigenvalue of G.*

Proof. First, we observe that the

$$A(X) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}.$$

By assumption, λ is an eigenvalue of $A(X)$, and let $[u, v]^t$ be the eigenvector associated to λ. Then:

$$\begin{aligned} A(X)\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \\ &= \begin{bmatrix} Bv \\ B^T u \end{bmatrix} \\ &= \lambda \begin{bmatrix} u \\ v \end{bmatrix} \\ &= \begin{bmatrix} \lambda u \\ \lambda v \end{bmatrix}. \end{aligned}$$

Hence,

$$Bv = \lambda u \text{ and } B^T u = \lambda v.$$

Now for the eigenvector $[-u, v]^T$, we have:

$$A(X)\begin{bmatrix}-u\\ v\end{bmatrix} = \begin{bmatrix}0 & B\\ B^T & 0\end{bmatrix}\begin{bmatrix}-u\\ v\end{bmatrix} = \begin{bmatrix}Bv\\ -B^T u\end{bmatrix} = \begin{bmatrix}\lambda u\\ -\lambda v\end{bmatrix} = -\lambda\begin{bmatrix}-u\\ v\end{bmatrix}.$$

Thus, $-\lambda$ is also an eigenvalue of $A(X)$. □

2.4 Laplacian operator

Let $X = (V, E)$ be a graph. In order to orient the edges in E, for an edge $e \in E$, we label one end point of e by e^+, and the other one by e^- as shown in Figure 2.9 below.

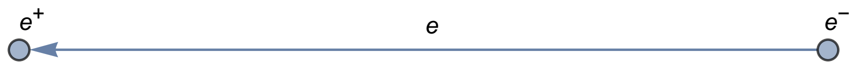


Fig. 2.9: An orientation of an edge.

Definition 2.34. The *gradient operator* $d : L^2(V) \to L^2(E)$ is defined by

$$df(e) = f(e^+) - f(e^-), \text{ for } f \in L^2(V).$$

Definition 2.35. Let *divergence operator* $d^* : L^2(E) \to L^2(V)$ is defined by

$$(d^* f)(v) = \sum_{\substack{e \in E\\ v = e^+}} f(e) - \sum_{\substack{e \in E\\ v = e^-}} f(e), \text{ for } f \in L^2(E).$$

Definition 2.36. The *Laplacian operator* $\Delta : L^2(V) \to L^2(V)$ is defined by $\Delta = d^* \circ d$.

Lemma 2.37. *Let X be a k-regular graph with $A(X) = A$. Then, we have:*

$$\Delta = kI - A.$$

Proof. Let $f \in L^2(V)$. Then by definition, for $v \in V$, we have:

$$\begin{aligned}
(\Delta f)(v) &= (d^*(df))(v) \\
&= \sum_{\substack{e\in E \\ v=e^+}} (df)(e) - \sum_{\substack{e\in E \\ v=e^-}} (df)(e) \\
&= \left(\sum_{\substack{e\in E \\ v=e^+}} (f)(v) - \sum_{\substack{e\in E \\ v=e^+ \& u=e^-}} (f)(u)\right) \\
&- \left(\sum_{\substack{e\in E \\ v=e^- \& u=e^+}} (f)(u) - \sum_{\substack{e\in E \\ v=e^-}} (f)(v)\right) \\
&= kf(v) - \sum_{u\in V} A_{v,u} f(u) \\
&= ((kI - A)f)(v).
\end{aligned}$$

□

Corollary 2.38. *If A has eigenvalue λ with a eigenfunction f then Δ has $k - \lambda$ as an eigenvalue.*

Proof. By assumption, $Af(v) = \lambda f(v)$. By Lemma 2.37, we have:

$$\begin{aligned}
\Delta f(v) &= (kI - A)f(v) \\
\implies \Delta f(v) &= (kI)f(v) - Af(v) \\
&= kf(v) - \lambda f(v) \\
&= (k - \lambda)f(v).
\end{aligned}$$

Hence, Δ has eigenvalue $k - \lambda$. □

3. INTRODUCTION TO ALGEBRAIC GRAPH THEORY

In this chapter we will explore some foundational topics in algebraic graph theory such as Cayley graphs and expander families of graphs. The reference used for this chapter is [6, Chapters 1-3].

3.1 Cayley graph

Definition 3.1. Let G be a group and Γ be a multi-subset of G. We call Γ to be a *symmetric subset* of G if for each $\gamma \in \Gamma$ with multiplicity n, there exists $\gamma^{-1} \in \Gamma$ with equal multiplicity.

Definition 3.2. Let G be a group and Γ be a symmetric subset of G. Then the *Cayley graph of G with respect to* Γ, is defined to be the graph $\mathrm{Cay}(G, \Gamma) := (V, E)$, where:

(i) $V = G$,

(ii) for any two vertices, $x, y \in V$ $\{x, y\} \in E$ if and only if $x^{-1}y \in \Gamma$, that is, $x = \gamma y$, for any $\gamma \in \Gamma$, and

(iii) the multiplicity of an edge $\{x, y\} \in E$ equals the multiplicity of $x^{-1}y \in \Gamma$.

Remark 3.3. We choose Γ to be symmetric or else $\mathrm{Cay}(G, \Gamma)$ would be a directed graph.

Example 3.4. Consider the group S_3 and let $\Gamma = \{(1\,2), (1\,2\,3), (1\,3\,2)\}$. Then the Cayley graph $\mathrm{Cay}(S_3, \Gamma)$ is shown in Figure 3.1.

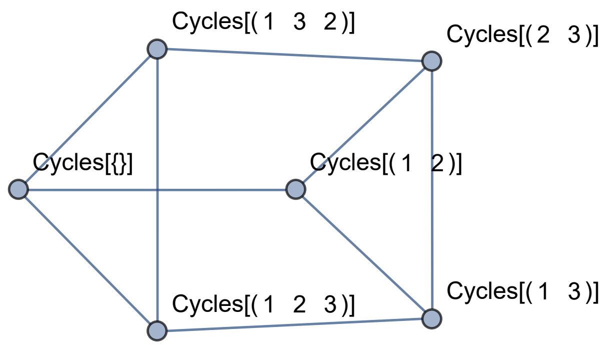


Fig. 3.1: $\text{Cay}(S_3, \{(1\,2), (1\,2\,3), (1\,3\,2)\})$.

Remark 3.5. Let, $X = \text{Cay}(G, \Gamma)$, A be the adjacency operator of X, and $f \in L^2(G)$. Then:

$$(Af)(x) = \sum_{\gamma \in \Gamma} f(x\gamma), \ \forall\, x \in G.$$

Proposition 3.6. *Let G be a group and Γ is symmetric subset of G. Then:*

(i) $\text{Cay}(G, \Gamma)$ *is $|\Gamma|$-regular, and*

(ii) $\text{Cay}(G, \Gamma)$ *is connected if and only if Γ generates G as a group.*

Proof. Let $g \in G$,and let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$.

(i) Then $|\Gamma| = k$, and by the definition $g\gamma_1, g\gamma_2, \ldots, g\gamma_k$ are the neighbours of an arbitrary vertex $g \in G$ (counted with multiplicity). Hence, $\text{Cay}(G, \Gamma)$ is $|\Gamma|$-regular.

(ii) Let 1_G be the identity element of the group G.Then:

$$\begin{aligned}
&\Gamma \text{ generates } G \\
\iff\ & \text{for every } g \in G, g = \gamma_1.\gamma_2 \ldots \gamma_k = 1_G.\gamma_1.\gamma_2 \ldots \gamma_k \\
\iff\ & \text{there is a walk from } 1_G \text{ to } g, \text{ which can be seen as a walk} \\
& (1_G \to 1_G\gamma_1 \to 1_G\gamma_1\gamma_2 \to \ldots \to 1_G\gamma_1\gamma_2 \ldots \gamma_k) \\
\iff\ & 1_G \text{ is connected to } g.
\end{aligned}$$

Now, given two distinct vertices $g, h \in G$, we have that both g, and h are connected to 1_G. Hence, $\mathrm{Cay}(G, \Gamma)$ is a connected graph.

□

Example 3.7. The Cayley graph in Figure 3.1 is 3-regular since $|\Gamma| = 3$.

3.2 Isoperimetric constant and Expander families

Definition 3.8. Let $X = (V, E)$ be a graph, and $F \subset V$ such that $|F| \leq \frac{|V|}{2}$. Then the *boundary of* F, denoted by δF, is defined to be the set of edges in E with one endpoint in F and another in $V \setminus F$.

In other words, δF is the set of edges connecting F to $V \setminus F$.

Definition 3.9. Let X be a graph with vertex set V and $F \subset V$. Then the *isoperimetric constant* of X is defined by

$$h(X) = \min \left\{ \frac{|\delta F|}{|F|} \mid F \subset V \text{ and } |F| \leq \frac{|V|}{2} \right\}.$$

Definition 3.10. Let $\{a_n\}$ be a sequence of nonzero real numbers. We say that $\{a_n\}$ is *bounded away from zero* if there exists a real number $\epsilon > 0$ such that $a_n \geq \epsilon$ for all n.

Definition 3.11. Let d be a positive integer. Let $\{X_n\}$ be a sequence of $d-$regular graphs such that $|X_n| \to \infty$ as $n \to \infty$. We say that $\{X_n\}$ is an *expander family* if the sequence $\{h(X_n)\}$ is bounded away from zero.

Example 3.12. Note that, $\{C_n\}$ is a sequence of 2-regular connected graphs. We will show that $\{C_n\}$ is not expander family.
Let V be the vertex set of C_n. For some $F \subset V$ with $|F| \leq \frac{|V|}{2}$, the ratio $\frac{|\delta F|}{|F|}$ will be minimum when $|\delta F|$ is minimum and $|F|$ is maximum. This happens precisely when there is a walk connecting any two vertices of the

induced subgraph of C_n on F. Moreover, $|F|$ is bounded above by $\frac{n}{2}$ or $\frac{n-1}{2}$ depending on whether n is even or odd. Therefore we have:

$$h(C_n) = \min\{\frac{|\delta F|}{|F|}\} = \begin{cases} \frac{4}{n}, & \text{if } n \text{ is even, and} \\ \frac{4}{n-1}. & \text{if } n \text{ is odd.} \end{cases}$$

Since, $h(C_n) \to 0$ as $n \to \infty$, we have that $\{C_n\}$ is not an expander family.

3.3 Rayleigh-Ritz Theorem

Definition 3.13. Let X be a finite set and f_0 be the function that is equal to 1 on all of X. We define

$$L^2(X, \mathbb{R}) = \{f : X \to \mathbb{R}\},$$

and

$$\begin{aligned} {L_0}^2(X, \mathbb{R}) &= \{f \in L^2(X, \mathbb{R}) : \langle f, f_0 \rangle_2 = 0\} \\ &= \{f \in L^2(X, \mathbb{R}) : \sum_{x \in X} f(x) = 0\}. \end{aligned}$$

Theorem 3.14 (Rayleigh-Ritz)**.** *Let $X = (V, E)$ be a k-regular graph. Then:*

$$\lambda_1(X) = \max_{f \in {L_0}^2(V,\mathbb{R})} \frac{\langle Af, f \rangle_2}{||f||_2} = \max_{\substack{f \in {L_0}^2(V,\mathbb{R}) \\ ||f||_2 = 1}} \langle Af, f \rangle_2. \tag{3.1}$$

Equivalently,

$$k - \lambda_1(X) = \min_{f \in {L_0}^2(V,\mathbb{R})} \frac{\langle \Delta f, f \rangle_2}{||f||_2} = \min_{\substack{f \in {L_0}^2(V,\mathbb{R}) \\ ||f||_2 = 1}} \langle \Delta f, f \rangle_2. \tag{3.2}$$

Proof. Let A be a $n \times n$ adjacency matrix of X. We know that there exist an orthonormal basis $\{f_0, f_1, f_2, ..., f_{n-1}\}$ for $L^2(V, \mathbb{R})$, such that each f_i is a real-valued eigenfunction of A associated with the eigenvalues $\lambda_i = \lambda_i(X)$ and $f_0 = 1$.

Now, let $f \in {L_0}^2(V, \mathbb{R})$ and $||f||_2 = 1$. Then, $f = c_0 f_0 + c_1 f_1 + \ldots + c_{n-1} f_{n-1}$ for some scalars $c_i \in \mathbb{R}$ so that

$$\langle f, f_0 \rangle_2 = c_0 \langle f_0, f_0 \rangle_2 + c_1 \langle f_1, f_0 \rangle_2 + \ldots + c_{n-1} \langle f_{n-1}, f_0 \rangle_2 = c_0.$$

By definition of ${L_0}^2(V, \mathbb{R})$, we have $\langle f, f_0 \rangle_2 = 0$, and so it follows that $f = c_1 f_1 + c_2 f_2 + \ldots + c_{n-1} f_{n-1}$.
Now,

$$\begin{aligned}
\langle Af, f \rangle_2 &= \langle A \sum_{i=1}^{n-1} c_i f_i, \sum_{j=1}^{n-1} c_j f_j \rangle_2 \\
&= \langle \sum_{i=1}^{n-1} c_i \lambda_i f_i, \sum_{j=1}^{n-1} c_j f_j \rangle_2 \\
&= \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_i c_j \lambda_i \langle f_i, f_j \rangle_2 \\
&= \sum_{i=1}^{n-1} {c_i}^2 \lambda_i \\
&\le \lambda_1 \sum_{i=1}^{n-1} {c_i}^2 \\
&= \lambda_1 ||f||_2^2 \\
&= \lambda_1.
\end{aligned}$$

Hence,

$$\lambda_1(X) \ge \max_{\substack{f \in {L_0}^2(V,\mathbb{R}) \\ ||f||_2 = 1}} \langle Af, f \rangle_2.$$

Since, $f_1 \in {L_0}^2(V, \mathbb{R})$ and $||f_1||_2 = 1$, we have:

$$\langle Af_1, f_1 \rangle_2 = \langle \lambda_1 f_1, f_1 \rangle_2 = \lambda_1 \langle f_1, f_1 \rangle_2 = \lambda_1.$$

Thus,

$$\lambda_1(X) = \max_{\substack{f \in {L_0}^2(V,\mathbb{R}) \\ ||f||_2 = 1}} \langle Af, f \rangle_2.$$

The equivalent statement follows directly from Lemma 2.37. □

With the help of Rayleigh-Ritz theorem we can bound the isoperimetric constant of a k-regular graph.

Lemma 3.15. *Let $X = (V, E)$ be a k-regular graph. Then:*

$$\frac{k - \lambda_1(X)}{2} \leq h(X). \tag{3.3}$$

Proof. Consider, $F \subset V$ such that $|F| \leq |V|/2$ and $h(X) = |\delta F|/|F|$. Let $a = |V \setminus F|$ and $b = |F|$.
Define

$$g(x) = \begin{cases} a, & \text{if } x \in F, \text{ and} \\ -b, & \text{if } x \in V \setminus F. \end{cases}$$

and $f = g/||g||_2$.
Observe that, $f, g \in {L_0}^2(V, \mathbb{R})$ as

$$\sum_{v \in V} g(v) = \sum_{v \in F} g(v) + \sum_{v \in V \setminus F} g(v) = 0.$$

Moreover, we have:

$$\langle \Delta g, g \rangle_2 = \sum_{e \in E} |g(e^+) - g(e^-)|^2 = \sum_{e \in \delta F} (b + a)^2 = |\delta F|(b + a)^2, \text{ and}$$

$${||g||_2}^2 = \langle g, g \rangle_2 = \sum_{x \in F} a^2 + \sum_{x \in V \setminus F} b^2 = a^2 b + b^2 a = ab(a + b).$$

Since by assumption $b \leq |V|/2$, we have $a \geq b$, and so it follows that:

$$\langle \Delta f, f \rangle_2 = \frac{1}{{||g||_2}^2} \langle \Delta g, g \rangle_2 = \frac{|\delta F|(b + a)}{ba} = (1 + \frac{b}{a}) h(X) \leq 2h(X).$$

By Theorem 3.14, we know that $k - \lambda_1(X) \leq \langle \Delta f, f \rangle_2$, from which our assertion follows. □

3.4 Alon-Boppana Theorem

Lemma 3.16. *Let X be a connected k-regular graph. If the* $\mathrm{diam}(X) \geq 4$, *then*

$$\lambda_1(X) > 2\sqrt{k-1} - \frac{2\sqrt{k-1}-1}{\lfloor \frac{1}{2}\mathrm{diam}(X) - 1 \rfloor}. \tag{3.4}$$

Proposition 3.17. *If $\{X_n\}$ is a sequence of connected k-regular graphs with $|X_n| \to \infty$ as $n \to \infty$, then*

$$\liminf_{n\to\infty} \lambda_1(X_n) \geq 2\sqrt{k-1}. \tag{3.5}$$

Proof. Consider a connected $k-$regular graph X. Fix a vertex v of X. Note that, the number of walks of length ℓ starting at v is k^ℓ and a walk of length ℓ contains at most $\ell + 1$ vertices. We can cover the entire graph by taking all walks of length $\mathrm{diam}(X)$ from the fixed vertex v. There are $k^{\mathrm{diam}(X)}$ such walks, each containing at most $\mathrm{diam}(X) + 1$ distinct vertices. Hence,

$$|X| \leq (\mathrm{diam}(X) + 1)k^{\mathrm{diam}(X)}. \tag{3.6}$$

Let (X_n) be a sequence of connected, $k-$regular graphs such that $|X_n| \to \infty$ as $n \to \infty$ so that $\mathrm{diam}(X_n) \to \infty$, as $n \to \infty$. Then $\frac{2\sqrt{k-1}-1}{\lfloor \frac{1}{2} diam(X) - 1 \rfloor} \to 0$, as $n \to \infty$.

Since the expressions in both sides of Equation 3.4 are bounded sequences, we have:

$$\begin{aligned}
\liminf_{n\to\infty} \lambda_1(X) &\geq \liminf_{n\to\infty} \left(2\sqrt{k-1} - \frac{2\sqrt{k-1}-1}{\lfloor \frac{1}{2}\mathrm{diam}(X) - 1 \rfloor}\right) \\
&= 2\sqrt{k-1} + \liminf_{n\to\infty} \left(- \frac{2\sqrt{k-1}-1}{\lfloor \frac{1}{2}\mathrm{diam}(X) - 1 \rfloor}\right) \\
&= 2\sqrt{k-1} + \lim_{n\to\infty} \left(- \frac{2\sqrt{k-1}-1}{\lfloor \frac{1}{2}\mathrm{diam}(X) - 1 \rfloor}\right) \\
&= 2\sqrt{k-1}.
\end{aligned}$$

□

Definition 3.18. Given a graph X, we define

$$\lambda(X) := \begin{cases} \max\{|\lambda_1(X)|, |\lambda_{n-1}|\}, & \text{if } X \text{ is non-bipartite, and} \\ \max\{|\lambda_1(X)|, |\lambda_{n-2}|\}, & \text{if } X \text{ is bipartite.} \end{cases}$$

Since by definition $\lambda(X) \geq \lambda_1(X)$, the next theorem follows immediately from Proposition 3.17.

Theorem 3.19 (Alon-Boppana Theorem)**.** *If $\{X_n\}$ is a sequence of connected k-regular graphs with $|X_n| \to \infty$ as $n \to \infty$, then*

$$\liminf_{n\to\infty} \lambda(X_n) \geq 2\sqrt{k-1}. \tag{3.7}$$

Definition 3.20. Let X be a k-regular graph. We say that X is *Ramanujan* if $\lambda(X) \leq 2\sqrt{k-1}$

Example 3.21. For the r-regular complete bipartite graph $K_{r,r}$, we have:

$$\mathrm{Spec}(K_{r,r}) = \begin{pmatrix} r & 0 & -r \\ 1 & 2r-2 & 1 \end{pmatrix}.$$

So, $\lambda = 0 \leq 2\sqrt{r-1}$, for any choice of r. Hence, $K_{r,r}$ is Ramanujan.

4. EQUITABLE PARTITIONS OF GRAPHS

Partitioning a graph equitably is a process of decomposing a graph into smaller graphs with strictly smaller adjacency matrices. Equitable decomposition can be used to bound the simple eigenvalues and the spectral gap of a undirected graph effectively. The references used for this chapter are [1], [4, Chapters 8-9].

Definition 4.1. Let $X = (V, E)$ be a graph. We say that a partition π : $V = C_1 \sqcup C_2 \sqcup \ldots \sqcup C_r$ of V is *equitable* if any vertex u of C_i has b_{ij} number of neighbours in C_j, where b_{ij} is constant that is independent of the choice of the vertex u.

The sets $C_1, C_2, \ldots, C_r$ are called the *cells* of the partition π.

Definition 4.2. Let π be an equitable partition of a graph X as in Definition 4.1. The directed graph X/π with the r cells of π as its vertex set and b_{ij} arcs from the i^{th} to the j^{th} cells of π as its edge set is called *the quotient graph of* X *over* π.

Note that the adjacency matrix is called the *quotient matrix* and the entries of that matrix are given by $A(X/\pi) = b_{ij}$.

Definition 4.3. Given an equitable partition of graph X as in Definition 4.1, the *characteristic matrix* χ of the partition π is a $|V| \times r$ matrix defined by:

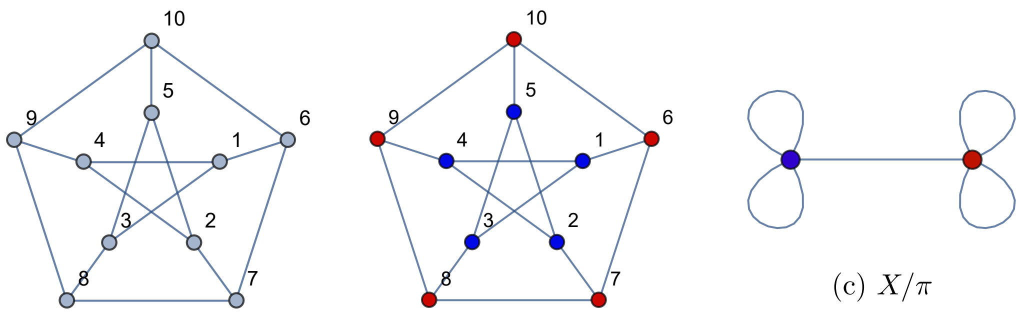
$$\chi_{ij} = \begin{cases} 1, & \text{if } i \text{ belongs to the cell } C_j \text{, and} \\ 0, & \text{otherwise.} \end{cases}$$

Example 4.4. Consider the Peterson graph $X = (V, E)$ in Figure 4.1(a). The partition $\pi : \{1, 2, 3, 4, 5\} \sqcup \{6, 7, 8, 9, 10\}$ is an equitable partition of X as indicated in Figure 4.1(b). The quotient graph X/π of this partition is shown in Figure 4.1(c) whose adjacency (or quotient) matrix is:

$$A(X/\pi) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The characteristic matrix is a 10×2 matrix given by

$$\chi = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t.$$



(a) X (b) Partition (c) X/π

Fig. 4.1: Equitable partition of a graph and its quotient graph.

In contrast with the quotient graph in Figure 4.1(c), quotient graphs are generally directed graphs.

Lemma 4.5. *Let π be an equitable partition of the graph X, with characteristic matrix χ, and let $B = A(X/\pi)$. Then:*

(i) $A\chi = \chi B$, *and*

(ii) $B = (\chi^T\chi)^{-l}\chi^T A\chi$.

Proof. The $(kj)^{th}$ entry of $A\chi$ is $(A\chi)_{kj} = \#$ neighbours of the vertex k in C_j. Now, if $k \in C_i$, then $(A\chi)_{kj} = b_{ij}$ and $(\chi B)_{kj} = b_{ij}$. Moreover, since $\chi^T\chi$ is a diagonal matrix with $(ii)^{th}$ entry $C_i \neq 0$, for all i, we have $\chi^T A\chi = \chi^T \chi B$, and (ii) follows. □

Lemma 4.6. *Let $X = (V, E)$ be a graph with adjacency matrix A, and let π be a partition of $V(X)$ with characteristic matrix χ . Then π is equitable if and only if the column space of χ is A-invariant.*

Proof. The column space of χ is A-invariant if and only if $A\chi = \chi B$ for some square matrix B. Now, if π is equitable partition we can set $B = A(X/\pi)$. On the other hand, if such a B exists then every vertex of C_i is adjacent to b_{ij} number of vertices in C_j. Hence, π is equitable. □

Theorem 4.7. *If π is an equitable partition of a graph X, then the characteristic polynomial of $A(X/\pi)$ divides the characteristic polynomial of $A(X)$.*

Proof. Let $A = A(X)$ and $B = A(X/\pi)$. Let $|X| = n$ vertices and the characteristic matrix of X with respect to partition π be χ. Moreover, let κ be a $n \times (n - |\pi|)$ matrix whose columns, together with the columns of χ, forms a basis of $\mathbb{R}^n$. Then,

$$A\kappa = \chi P + \kappa Q,$$

for some matrices P and Q. This can be also written as

$$\begin{aligned}
A\begin{bmatrix}\chi & \kappa\end{bmatrix} &= \begin{bmatrix}\chi & \kappa\end{bmatrix}\begin{bmatrix}B & P\\ 0 & Q\end{bmatrix}\ [\because\ A\chi = \chi B]\\
\implies A &= \begin{bmatrix}\chi & \kappa\end{bmatrix}\begin{bmatrix}B & P\\ 0 & Q\end{bmatrix}\begin{bmatrix}\chi & \kappa\end{bmatrix}^{-1}\ [\because \begin{bmatrix}\chi & \kappa\end{bmatrix} \text{ is invertible}]\\
\implies |A - \lambda I_n| &= \begin{vmatrix}B - \lambda I_{|\pi|} & P\\ 0 & Q - \lambda I_{(n-|\pi|)}\end{vmatrix}\\
\implies |A - \lambda I_n| &= |B - \lambda I_{|\pi|}||Q - \lambda I_{(n-|\pi|)}|.
\end{aligned}$$

Hence, the characteristic polynomial of B divides the characteristic polynomial of A. □

An immediate consequence is the following corollary.

Corollary 4.8. *Let π be an equitable partition of a graph X. If λ is an eigenvalue of X/π, then λ is also an eigenvalue of X.*

Corollary 4.9. *Let π be an equitable partition and let χ be the corresponding characteristic matrix. Then, y is an eigenvector with eigenvalue λ for X/π, if and only if χy is an eigenvector with eigenvalue λ for X.*

Proof. Since by assumption $A(X/\pi)y = \lambda y$, we have:

$$\begin{aligned} A(X)\chi y &= \chi A(X/\pi)y \\ &= \chi\lambda y \\ &= \lambda\chi y. \end{aligned}$$

Hence, χy is eigenvector of $A(X)$ with eigenvalue λ.
Conversely, if the column space of χ is $A(X)$-invariant, then it must have a basis consisting of eigenvectors of $A(X)$. Each of these eigenvectors is constant on the cells of χ, and hence has the form χy, where $y \neq 0$. Thus, if $A(X)\chi y = A(X/\pi)\chi y$, then it follows that $A(X/\pi)y = \lambda y$. □

Remark 4.10. Let π be an equitable partition of a graph X, and let χ be the corresponding characteristic matrix. Then $A(X)$ has two kinds of eigenvectors, namely:

(i) eigenvectors of the form χy, where y is an eigenvector of $A(X/\pi)$, and

(ii) eigenvectors whose coordinates sum to zero on each cell of π.

5. CAYLEY GRAPHS ON ALTERNATING GROUPS

Our central goal is to examine second largest eigenvalues of some Cayley graphs on alternating groups. To this end, we will first define alternating group graph AG_n, extended alternating group graph EAG_n, and the complete alternating group graph CAG_n. We will also see respective structures of those graphs for small n. This chapter is based on the work in [5].

Definition 5.1. Let A_n denote the alternating group of degree $n \geq 3$. The *alternating group graph* AG_n, *extended alternating group graph* EAG_n, and the *complete alternating group graph* CAG_n are defined respectively as follows:

1. $AG_n := \mathrm{Cay}(A_n, T_1)$ where $T_1 = \{(1,2,i),(1,i,2) \,|\, 3 \leq i \leq n\}$.
2. $EAG_n := \mathrm{Cay}(A_n, T_2)$ where $T_2 = \{(1,i,j),(1,j,i) \,|\, 2 \leq i < j \leq n\}$.
3. $CAG_n := \mathrm{Cay}(A_n, T_3)$ where $T_3 = \{(i,j,k),(i,k,j) \,|\, 1 \leq i < j < k \leq n\}$

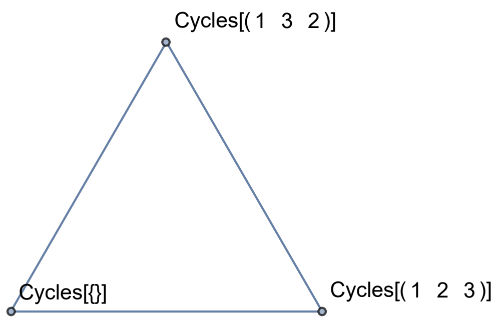


Fig. 5.1: The graph $AG_3 = EAG_3 = CAG_3$.

Example 5.2. For $n = 3$, $AG_n = EAG_n = CAG_n$, and the structure of the graph is shown in Figure 5.1.
For $n = 4$, AG_n has the structure in Figure 5.2, EAG_n has the structure shown in the Figure 5.3, and CAG_n is the graph in Figure 5.4.

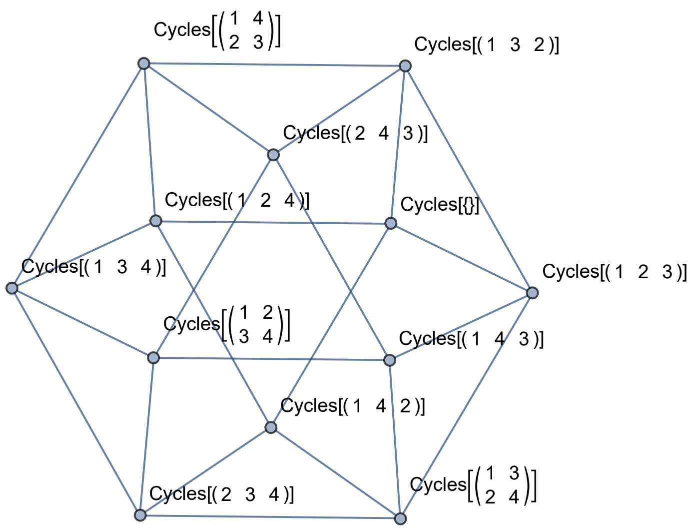


Fig. 5.2: The graph AG_4.

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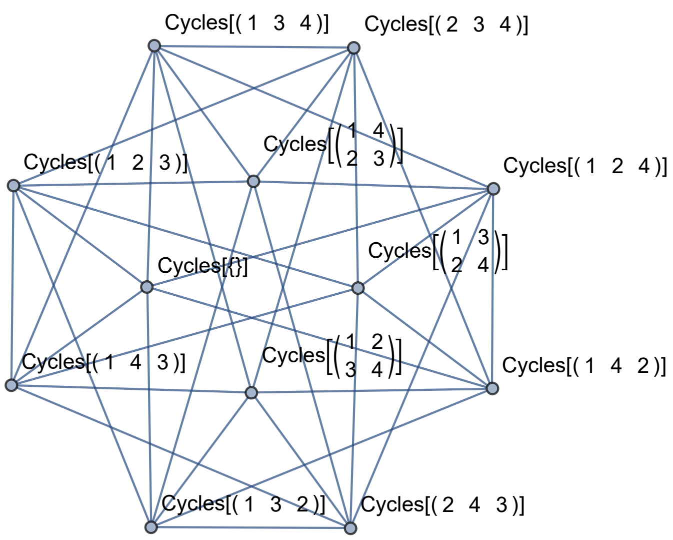

Fig. 5.3: The graph EAG_4.

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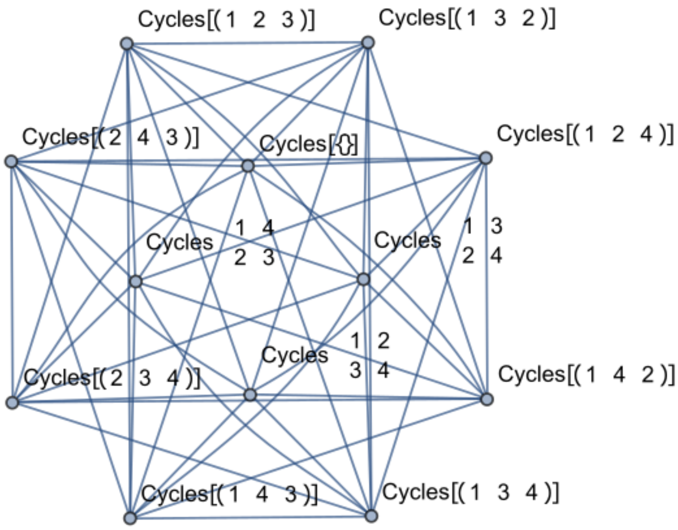

Fig. 5.4: The graph CAG_4.

Notation. Given an equitable partition π for a graph X, we will fix the following notation for the remainder of this chapter.

(a) The quotient matrix of a graph X over π by B_π.

(b) For $v \in V(X)$, we will define $N(v) := \{u \in V(X) \mid \{u, v\} \in E(X)\}$.

5.1 Second largest eigenvalue of AG_n

Let consider a partition π of AG_n for $n \geq 4$ defined as follows:

$$\begin{aligned}
X(i) &= \{\tau \in A_n \mid \tau(n) = i\}, \\
Y(i) &= \{\tau \in A_n \mid \tau(1) = i\}, \\
Z(i) &= \{\tau \in A_n \mid \tau(2) = i\}, \text{ and} \\
W(i) &= \{\tau \in A_n \mid \tau(1) \neq i, \tau(2) \neq i, \tau(n) \neq i\}.
\end{aligned}$$

For $\tau \in X(i)$, we have

$$\begin{aligned}
|N(\tau) \cap X(i)| &= |\{(1, 2, k)\tau, (1, k, 2)\tau \mid 3 \leq k \leq n-1\}| = 2n-6, \\
|N(\tau) \cap Y(i)| &= |\{(1, n, 2)\tau\}| = 1,
\end{aligned}$$

$$|N(\tau) \cap Z(i)| = |\{(1,2,n)\tau\}| = 1, \text{ and}$$
$$|N(\tau) \cap W(i)| = |\phi| = 0.$$

For $\tau \in Y(i)$,

$$|N(\tau) \cap X(i)| = |\{(1,2,n)\tau\}| = 1,$$
$$|N(\tau) \cap Y(i)| = |\phi| = 0,$$
$$|N(\tau) \cap Z(i)| = |\{(1,k,2)\tau \,|\, 3 \leq k \leq n\}| = n-2, \text{ and}$$
$$|N(\tau) \cap W(i)| = |\{(1,2,k)\tau \,|\, 3 \leq k \leq n-1\}| = n-3.$$

For $\tau \in Z(i)$, we have

$$|N(\tau) \cap X(i)| = |\{(1,n,2)\tau| = 1,$$
$$|N(\tau) \cap Y(i)| = |\{(1,2,k)\tau \,|\, 3 \leq k \leq n\}| = n-2,$$
$$|N(\tau) \cap Z(i)| = |\{\phi\}| = 0, \text{ and}$$
$$|N(\tau) \cap W(i)| = |\{(1,k,2)\tau \,|\, 3 \leq k \leq n-1\}| = n-3.$$

For $\tau \in W(i)$, we know that $\tau(\ell) = i$ for $3 \leq \ell \leq n-1$. Then we have:

$$|N(\tau) \cap X(i)| = |\phi| = 0,$$
$$|N(\tau) \cap Y(i)| = |\{(1,\ell,2)\tau\}| = 1,$$
$$|N(\tau) \cap Z(i)| = |\{(1,2,\ell)\tau\}| = 1, \text{ and}$$
$$|N(\tau) \cap W(i)| = |\{(1,2,k)\tau, (1,k,2)\tau \,|\, 3 \leq k \leq n, k \neq \ell\}| = 2n-6.$$

Hence, the quotient matrix of AG_n with respect to π is

$$B_\pi = \begin{bmatrix} 2n-6 & 1 & 1 & 0 \\ 1 & 0 & n-2 & n-3 \\ 1 & n-2 & 0 & n-3 \\ 0 & 1 & 1 & 2n-6 \end{bmatrix},$$

and the eigenvalues of B_π are $2n-4$, $2n-6$, $n-4$ and $2-n$.

Theorem 5.3. *For $n \geq 4$, we have $\lambda_1(AG_n) = 2n-6$.*

Proof. We establish this by induction on n. For $n = 3$, by direct computation we can show that the second largest eigenvalue is -1. Now suppose that $n \geq 4$, and assume that the result holds for $n-1$, that is, $\lambda_1(AG_{n-1}) = 2(n-1) - 6 = 2n - 8$. Let, λ be an eigenvalue of AG_n other than $2n-4$, $2n-6$, $n-4$, and $2-n$. We need to show $\lambda < 2n-6$. First, we make the following claim.

Claim: There exists an i such that

$$2 \sum_{x \in X(i)} f(x)^2 \geq \sum_{y \in Y(i)} f(y)^2 + \sum_{z \in Z(i)} f(z)^2,$$

and

$$\sum_{x \in X(i)} f(x)^2 > 0. \tag{5.1}$$

Proof to claim. To see this, first note that:

$$A_n = X(1) \sqcup X(2) \sqcup \ldots \sqcup X(n) = Y(1) \sqcup \ldots \sqcup Y(n) = Z(1) \sqcup \ldots \sqcup Z(n).$$

Then:

$$\sum_{j=1}^{n} \sum_{x \in X(j)} f(x)^2 = \sum_{j=1}^{n} \sum_{y \in Y(j)} f(y)^2 = \sum_{j=1}^{n} \sum_{z \in Z(j)} f(z)^2 > 0$$
$$\implies \sum_{j=1}^{n} 2 \sum_{x \in X(j)} f(x)^2 = \sum_{j=1}^{n} \Big(\sum_{y \in Y(j)} f(y)^2 + \sum_{z \in Z(j)} f(z)^2 \Big) > 0.$$

Therefore, there exists an index i such that

$$2 \sum_{x \in X(i)} f(x)^2 \geq \sum_{y \in Y(i)} f(y)^2 + \sum_{z \in Z(i)} f(z)^2, \text{ and}$$

$$\sum_{x \in X(i)} f(x)^2 > 0,$$

as $f \neq 0$. Hence, our claim follows.

Now observe that, induced subgraph of AG_n on $X(i) := AG_n[X(i)]$ is isomorphic to $AG_{n-1} = \mathrm{Cay}(A_{n-1}, T_1')$, where $T_1' = \{(1,2,k), (1,k,2) \,|\, 3 \leq$

$k \leq n-1\}$. To prove the isomorphism between $AG_n[X(i)]$ and AG_{n-1}, we define

$$\phi : X(i) \to A_{n-1}$$

by

$$\begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ \tau(1) & \tau(2) & \dots & \tau(n-1) & i \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & \dots & n-1 \\ \tau(1) & \tau(2) & \dots & \tau(n-1) \end{pmatrix}.$$

By a direct computation, we can check that for $\tau, \tau' \in X(i)$, we have $\tau'\tau^{-1} = \phi(\tau')\phi(\tau)^{-1}$. So, we have

$$\begin{aligned} &\{\tau, \tau'\} \in E(AG_n[X(i)]) \\ \iff &\tau'\tau^{-1} \in T_1' \\ \iff &\phi(\tau')\phi(\tau)^{-1} \in T_1' \\ \iff &\{\phi(\tau'), \phi(\tau)\} \in E(AG_{n-1}). \end{aligned}$$

Hence, ϕ is graph homomorphism between $AG_n[X(i)]$ and AG_{n-1}. Moreover, we can easily check that $\phi : X(i) \to A_{n-1}$ defined by

$$\begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ \tau(1) & \tau(2) & \dots & \tau(n-1) & i \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & \dots & n-1 \\ \tau(1) & \tau(2) & \dots & \tau(n-1) \end{pmatrix}$$

is bijective. Thus, ϕ is a isomorphism between $AG_n[X(i)]$ and AG_{n-1}.

Notice that any vertex $x \in X(i)$ has exactly one edge in $Y(i)$ and $Z(i)$. Let x' and x'' be two neighbours of x in $Y(i)$ and $Z(i)$ respectively. So,

$$\begin{aligned} \lambda f(x) &= \sum_{y \in N(x) \cap X(i)} f(y) + f(x') + f(x'') \\ \implies \lambda f(x)^2 &= \sum_{y \in N(x) \cap X(i)} f(x)f(y) + f(x)(f(x') + f(x'')). \end{aligned}$$

Now summing both sides over $x \in X(i)$ and dividing by $\sum_{x \in X(i)} f(x)^2$, we get:

$$\lambda = \frac{\sum_{x\in X(i)} \sum_{y\in N(x)\cap X(i)} f(x)f(y)}{\sum_{x\in X(i)} f(x)^2} + \frac{\sum_{x\in X(i)} f(x)(f(x') + f(x''))}{\sum_{x\in X(i)} f(x)^2}. \quad (5.2)$$

Setting $g = f|_{X(i)} \in L_0^2$, we see that $g \perp [1, 1, \dots, 1]^T$ since $\sum_{x\in X(i)} f(x) = 0$. Considering the first term (in the RHS) of Equation 5.2, we have:

$$\begin{aligned}
\frac{\sum_{x\in X(i)} \sum_{y\in N(x)\cap X(i)} f(x)f(y)}{\sum_{x\in X(i)} f(x)^2} &= \frac{g^T A(AG_n[X(i)])g}{g^T g} \\
&\leq \max_{h\perp \mathbf{1}} \frac{h^T A(AG_n[X(i)])h}{h^T gh} \quad \text{[by 3.14]} \\
&= \lambda_1(AG_n[X(i)]) \\
&= \lambda_1(AG_{n-1}) \\
&= 2n - 8.
\end{aligned}$$

For the second term, we have:

$$\begin{aligned}
\frac{\sum_{x\in X(i)} f(x)(f(x') + f(x''))}{\sum_{x\in X(i)} f(x)^2} &\leq \frac{\sqrt{(\sum_{x\in X(i)} f(x)^2)(\sum_{x\in X(i)} (f(x') + f(x''))^2)}}{\sum_{x\in X(i)} f(x)^2} \\
&= \sqrt{\frac{(\sum_{x\in X(i)} (f(x') + f(x''))^2)}{\sum_{x\in X(i)} f(x)^2}} \\
\text{[as we know, } (a+b)^2 \leq 2(a^2+b^2)] \\
&\leq \sqrt{\frac{(\sum_{x\in X(i)} 2(f(x')^2 + f(x'')^2))}{\sum_{x\in X(i)} f(x)^2}} \\
&= \sqrt{\frac{2(\sum_{y\in Y(i)} f(y)^2 + \sum_{z\in Z(i)} f(z)^2)}{\sum_{x\in X(i)} f(x)^2}} \\
\text{[by Equation 5.1]} \\
&\leq \sqrt{\frac{2.2 \sum_{x\in X(i)} f(x)^2}{\sum_{x\in X(i)} f(x)^2}} \\
&= 2.
\end{aligned}$$

Combining the last two inequalities we obtain

$$\lambda \leq 2n - 8 + 2 = 2n - 6.$$

Therefore, $\lambda_1(AG_n) = 2n - 6$. □

Corollary 5.4. $\{AG_n\}$ *is an expander family.*

Proof. $\lambda_0(AG_n) = 2n - 4$ since AG_n is a $2n-4$-regular graph. By Theorem 5.3, the spectral gap of AG_n is 2, and by Lemma 3.15, we get:

$$h(AG_n) \geq 1.$$

Hence, by Definition 3.11, the assertion follows. □

5.2 Second largest eigenvalue of EAG_n

For $1 \leq i, j \leq n$, we define:

$$X_i(j) = \{\tau \in A_n \,|\, \tau(j) = i\}.$$

We can see that,

$$\begin{cases} X_i(1), X_i(2), \ldots, X_i(n), & \text{for fixed } i, \text{ and} \\ X_1(j), X_2(j), \ldots, X_n(j), & \text{for fixed } j, \end{cases}$$

are partitions of A_n respectively. Moreover, we can easily verify that, $\pi : A_n = X_i(1) \cup X_i(2) \cup \ldots \cup X_i(n)$ is a equitable partition of EAG_n, for each i.

For $\tau \in X_i(1)$ we have:

$$|N(\tau) \cap X_i(1)| = |\phi| = 0, \text{ and}$$
$$|N(\tau) \cap X_i(j)| = |\{(1, k, j)\tau \,|\, 2 \leq k \leq n, k \neq j\}| = n - 2.$$

For fixed $2 \leq j \leq n$, let $\tau \in X_i(j)$, we have:

$|N(\tau) \cap X_i(1)| = |\{(i, j, k)\tau \,|\, 2 \leq k \leq n, k \neq j\}| = n - 2$,
$|N(\tau) \cap X_i(j)| = |\{(1, k, \ell)\tau, (1, \ell, k)\tau \,|\, 2 \leq k < \ell \leq n;\ k, \ell \neq j\}| = (n-2)(n-3)$, and
$|N(\tau) \cap X_i(\ell)| = |\{(1, \ell, j)\tau \,|\, \ell \notin \{1, j\}\}| = 1$.

With the partition π of EAG_n we get

$$B_\pi = \begin{bmatrix} 0 & n-2 & n-2 & \dots & n-2 \\ n-2 & (n-2)(n-3) & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-2 & 1 & 1 & \dots & (n-2)(n-3) \end{bmatrix}.$$

The characteristic polynomial of B_π is equal to

$$|\lambda I_n - B_\pi| = \begin{vmatrix} \lambda & -(n-2) & \dots & -(n-2) \\ -(n-2) & \lambda-(n-2)(n-3) & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -(n-2) & -1 & \dots & \lambda-(n-2)(n-3) \end{vmatrix}$$

$[C_1 = C_1 + C_2 + \ldots + C_n]$

$$= (\lambda - (n-1)(n-2)) \begin{vmatrix} 1 & -(n-2) & \dots & -(n-2) \\ 1 & \lambda-(n-2)(n-3) & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & \dots & \lambda-(n-2)(n-3) \end{vmatrix}$$

$[C_i = (n-2)C_1 + C_i \text{ for } i \neq 1]$

$$= (\lambda - (n-1)(n-2)) \begin{vmatrix} 1 & 0 & \dots & 0 \\ 1 & \lambda-(n-2)(n-4) & \dots & n-3 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & n-3 & \dots & \lambda-(n-2)(n-4) \end{vmatrix}$$

[Expanding along R_1]

$$= (\lambda - (n-1)(n-2)) \begin{vmatrix} \lambda - (n-2)(n-4) & \ldots & n-3 \\ n-3 & \ldots & n-3 \\ \vdots & \ddots & \vdots \\ n-3 & \ldots & \lambda - (n-2)(n-4) \end{vmatrix}$$

$[R_1 = R_1 + R_2 + \ldots + R_{n-1}]$

$$= (\lambda - (n-1)(n-2)) \begin{vmatrix} \lambda - (n-2) & \ldots & \lambda - (n-2) \\ n-3 & \ldots & n-3 \\ \vdots & \ddots & \vdots \\ n-3 & \ldots & \lambda - (n-2)(n-4) \end{vmatrix}$$

$$= (\lambda - (n-1)(n-2))(\lambda - (n-2)) \begin{vmatrix} 1 & \ldots & 1 \\ n-3 & \ldots & n-3 \\ \vdots & \ddots & \vdots \\ n-3 & \ldots & \lambda - (n-2)(n-4) \end{vmatrix}$$

[by $C_i = C_1 + C_i$ for $i \neq 1$ and expanding along C_1]

$$= (\lambda - (n-1)(n-2))(\lambda - (n^2 - 5n + 5))^{n-2}(\lambda + (n-2)).$$

Hence, B_π has the eigenvalues $(n-1)(n-2)$, $n^2 - 5n + 5$, and $2 - n$.

Theorem 5.5. *For $n \geq 3$, $\lambda_1(EAG_n) = n^2 - 5n + 5$.*

Proof. We will prove this result by induction on n. A direct computation yields the result for $n = 3$. Now, suppose that $n \geq 4$, and assume that the result holds for EAG_{n-1}, that is, $\lambda_1(EAG_{n-1}) = (n-1)^2 - 5(n-1) + 5$.
Let λ be an eigenvalue other than $(n-1)(n-2)$, $n^2 - 5n + 5$ and $2 - n$ and f be a eigenvector corresponding λ. Then $\sum_{x \in X_i(j)} f(x) = 0$, where $1 \leq i, j \leq n$.
Consider, a partition of the vertex set of EAG_n, $A_n = X_1(2) \cup X_2(2) \cup \ldots \cup X_n(2)$.
As in the proof of Theorem 5.3, we can show that the induced subgraph of EAG_n on $X_i(2) := EAG_n[X_i(2)]$ is isomorphic to EAG_{n-1}, for each $1 \leq i \leq n$, by defining a function

$$\phi : X_i(2) \to A_{n-1}.$$

Setting $E_1 := \cup_{i=1}^{n} E(EAG_n[X_i(2)])$ and $E_2 := E(EAG_n) \setminus E_1$, we make the following claim.

Claim: E_2 is the set of edges of AG_n.

Proof to claim. For any $\gamma \in X_i(2)$ and $\gamma' \in X_j(2)$ $(i \neq j)$, we have $\{\gamma, \gamma'\} \in E(EAG_n)$ iff $\gamma'\gamma^{-1} \in T_2$ iff $\gamma'\gamma^{-1} \in T_1 (\subseteq T_2)$. This implies that the edges in E_2 come from T_1. On the other hand, we see that T_1 can only be used to produce the edges in E_2 because each edge in E_1 comes from $T_2 \setminus T_1$, the set of three cycles in T_2 fixing 2. Thus, we have:

$$\begin{aligned}
\lambda &= \frac{f^T A(EAG_n) f}{f^T f} \\
&= \frac{2\sum_{\{x,y\}\in E(EAG_n)} f(x)f(y)}{\sum_{x\in A_n} f(x)^2} \\
&= \frac{2\sum_{\{x,y\}\in E_1} f(x)f(y)}{\sum_{x\in A_n} f(x)^2} + \frac{2\sum_{\{x,y\}\in E_2} f(x)f(y)}{\sum_{x\in A_n} f(x)^2}.
\end{aligned} \tag{5.3}$$

From the first term (on the RHS) of Equation 5.3 we have:

$$\begin{aligned}
\frac{2\sum_{\{x,y\}\in E_1} f(x)f(y)}{\sum_{x\in A_n} f(x)^2} &= \frac{\sum_{i=1}^{n} 2\sum_{\{x,y\}\in E(EAG_n[X_i(2)])} f(x)f(y)}{\sum_{i=1}^{n}\sum_{x\in X_i(2)} f(x)^2} \\
&\leq \max_{1\leq i\leq n} \frac{2\sum_{\{x,y\}\in E(EAG_n[X_i(2)])} f(x)f(y)}{\sum_{x\in X_i(2)} f(x)^2} \\
&\leq \lambda_1(EAG_{n-1}).
\end{aligned}$$

For the second term, we get:

$$\frac{2\sum_{\{x,y\}\in E_2} f(x)f(y)}{\sum_{x\in A_n} f(x)^2} \leq \max_{g\perp \mathbf{1}} \frac{g^T A(AG_n) g}{g^T g} = \lambda_1(AG_n),$$

since, f is orthogonal to the vector $[1, 1, \ldots, 1]^T$. Combining both bounds for λ, we get :

$$\begin{aligned}
\lambda &\leq \lambda_1(EAG_{n-1}) + \lambda_1(AG_n) \\
&= (n-1)^2 - 5(n-1) + 5 + 2n - 6 \\
&= n^2 - 5n + 5.
\end{aligned}$$

Hence, $\lambda_1(EAG_n) = n^2 - 5n + 5$. □

Corollary 5.6. *$\{EAG_n\}$ is an expander family.*

Proof. The largest eigenvalue of EAG_n is $(n-1)(n-2)$ because EAG_n is $(n-1)(n-2)$-regular. By Theorem 5.3, the spectral gap is $2n-3$, and by Lemma 3.15, we get:

$$h(EAG_n) \geq (n - \frac{3}{2}).$$

Hence, it follows that $h(EAG_n) \to \infty$, as $n \to \infty$, and the assertion follows. □

5.3 Second largest eigenvalue of CAG_n

In this case also we will partition the vertex set in the same way we did in the last one. For $1 \leq i, j \leq n$, we define

$$X_i(j) = \{\tau \in A_n \mid \tau(j) = i\}.$$

Now, we will verify that $\pi : A_n = X_i(1) \cup X_i(2) \cup \ldots \cup X_i(n)$ is also an equitable partition of CAG_n for fixed i. For each fixed $1 \leq j \leq n$, for $\tau \in X_i(j)$, we have

$$\begin{aligned} |N(\tau) \cap X_i(j)| &= |\{(k,\ell,m)\tau, (k,m,\ell)\tau \mid 1 \leq k < \ell < m \leq n; k,\ell,m \neq j\}| \\ &= 2\binom{n-1}{3}, \text{ and} \\ |N(\tau) \cap X_i(\ell)| &= |\{(\ell,j,k)\tau \mid 1 \leq k \leq n; k \neq \ell \neq j\}| = n-2. \end{aligned}$$

Thus, the quotient matrix becomes

$$B_\pi = \begin{bmatrix} 2\binom{n-1}{3} & n-2 & \ldots & n-2 \\ n-2 & 2\binom{n-1}{3} & \ldots & n-2 \\ \vdots & \vdots & \ddots & \vdots \\ n-2 & n-2 & \ldots & 2\binom{n-1}{3} \end{bmatrix}.$$

We can check that B_π has eigenvalues $2\binom{n}{3}$ and $\frac{1}{3}n(n-2)(n-4)$.

Theorem 5.7. *For $n \geq 3$, $\lambda_1(CAG_n) = \dfrac{1}{3}n(n-2)(n-4)$.*

Proof. We will once again use induction on n to prove the result. For $n = 3$ we have, $\lambda_1(CAG_3) = \lambda_1(K_3) = -1$ as required. Suppose that $n \geq 4$, and assume that the assertion holds for $n-1$, that is, $\lambda_1(CAG_{n-1}) = \frac{1}{3}(n-1)(n-3)(n-5)$.
Let λ be an eigenvalue other than those eigenvalues two mentioned previously.

Considering a partition the vertex set of CAG_n given by $A_n = X_1(1) \cup X_2(1) \cup \ldots \cup X_n(1)$, we define:

$$\begin{aligned} E_1 &: = \{\{\tau, \sigma\} \in E(CAG_n) \,|\, \tau(1) \neq \sigma(1)\}, \text{ and} \\ E_2^i &: = \{\{\tau, \sigma\} \in E(CAG_n) \,|\, \tau(1) = \sigma(1) = i\}, \text{ for } 1 \leq i \leq n. \end{aligned}$$

Then $E_1 \cup E_2^1 \cup \ldots \cup E_2^n$ is a partition of $E(CAG_n)$ and E_1 is exactly the set of edges of the extended alternating group graph EAG_n. For each i, E_2^i is exactly the set of edges of the induced subgraph of CAG_n on $X_i(1)$ which is also isomorphic to CAG_{n-1}. Then, we have

$$\begin{aligned} \lambda &= \frac{f^T A(CAG_n) f}{f^T f} \\ &= \frac{2\sum_{\{x,y\}\in E(CAG_n)} f(x)f(y)}{\sum_{x\in A_n} f(x)^2} \\ &= \frac{2\sum_{\{x,y\}\in E_1} f(x)f(y)}{\sum_{x\in A_n} f(x)^2} + \frac{\sum_{i=1}^n 2\sum_{\{x,y\}\in E_2^i} f(x)f(y)}{\sum_{x\in A_n} f(x)^2}. \end{aligned} \tag{5.4}$$

From the first term (on the RHS) of Equation 5.4, we get:

$$\frac{2\sum_{\{x,y\}\in E_1} f(x)f(y)}{\sum_{x\in A_n} f(x)^2} \leq \max_{g\perp\mathbf{1}} \frac{g^T A(EAG_n) g}{g^T g} = \lambda_1(EAG_n).$$

For the second term, since f is orthogonal to the vector $[1, 1, \ldots, 1]^T$, we have:

$$\frac{\sum_{i=1}^n 2\sum_{\{x,y\}\in E_2^i} f(x)f(y)}{\sum_{x\in A_n} f(x)^2} = \frac{\sum_{i=1}^n 2\sum_{\{x,y\}\in E(CAG_n[X_i(1)])} f(x)f(y)}{\sum_{i=1}^n \sum_{x\in X_i(1)} f(x)^2}$$

$$\leq \max_{1\leq i\leq n} \frac{2\sum_{\{x,y\}\in E(CAG_n[X_i(1)])} f(x)f(y)}{\sum_{x\in X_i(1)} f(x)^2}$$
$$\leq \lambda_1(CAG_{n-1}).$$

Combining both bounds for λ we get:

$$\begin{aligned}\lambda &\leq \lambda_1(EAG_n) + \lambda_1(CAG_{n-1})\\ &= n^2 - 5n + 5 + 1/3\ (n-1)(n-3)(n-5)\\ &= 1/3\ n(n-2)(n-4).\end{aligned}$$

Hence, $\lambda_1(CAG_n) = 1/3\ n(n-2)(n-4)$. □

Remark 5.8. The largest eigenvalue of CAG_n is $2\binom{n}{3}$ because CAG_n is $2\binom{n}{3}$-regular.

Corollary 5.9. $\{CAG_n\}$ *is an expander family.*

Proof. By Theorem 5.7, the spectral gap of CAG_n is

$$2\binom{n}{3} - \frac{1}{3}n(n-2)(n-4) = \frac{1}{3}n(n-1)(n-2) - \frac{1}{3}n(n-2)(n-4) = n(n-2).$$

By Lemma 3.15, we get:

$$\begin{aligned} h(CAG_n) &\geq \frac{n(n-2)}{2}\\ \Longrightarrow h(CAG_n) &\to \infty,\ \text{as } n \to \infty,\ \text{and}\end{aligned}$$

the assertion follows. □

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